A Postscript on Distances in Convex n -Gons

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Abstract. Let $g(n)$ be the largest integer k such that every convex polygon with n vertices and sides has a vertex x such that the next k vertices clockwise from x , or the next k vertices counterclockwise from x , are successively farther from x . We prove that $g(n) = \lfloor n/3 \rfloor + 1$ for $n \geq 4$. An example gives $g(n) \leq \lfloor n/3 \rfloor + 1$, and an extension of a 1952 construction of Leo Moser for a related planar problem shows that $g(n) \geq \lfloor n/3 \rfloor + 1$.

1. Introduction

Two old conjectures of the first author assert that, for every convex n -gon in the plane:

- C1. At least $\lfloor n/2 \rfloor$ different distances are determined by the n vertices [2].
- C2. Some vertex has at least $\lfloor n/2 \rfloor$ different distances to other vertices.

In both cases, $\lfloor n/2 \rfloor$ is attained by a regular polygon.

Altman [1] resolved C1 affirmatively, but C2 remains open. Let $f(n)$ be the minimum over all convex n -gons of the maximum over all vertices of a convex n -gon of the number of distinct distances to the other vertices. In view of regular polygons, C2 says that $f(n) = \lfloor n/2 \rfloor$. As far as we are aware, the best lower bound on f is still the one established by Moser [4] in 1952:

$$f(n) \geq \lfloor (n + 2)/3 \rfloor.$$

Difficulties in substantially improving Moser's bound or of resolving C2 are mentioned in Erdős and Fishburn [3].

Although Moser applied his bound to C1, others, including Altman [1], noted

that its proof applied to C2. One purpose of the present postscript is to observe that Moser's proof almost completely resolves one direction of the proof of a third problem, formulated as follows.

Let $d(x, y)$ be the Euclidean distance between points x and y in the plane. Define a *run* of a convex n -gon from a vertex x_0 as a sequence $x_0, x_1, \dots, x_k$ of successively adjacent vertices, going clockwise or counterclockwise from x_0 , such that

$$d(x_0, x_1) < d(x_0, x_2) < \dots < d(x_0, x_k).$$

The *length* of run $x_0, x_1, \dots, x_k$ is k . Let $g(n)$ be the minimum over all convex n -gons of the maximum run length of the n -gon. As observed shortly, Moser's proof implies that

$$g(n) \geq \lfloor (n + 2)/3 \rfloor$$

for all $n \geq 2$, and equality is easily seen to hold for $n \leq 5$. However $g(6) = 3$. In fact, as shown by the following theorem, Moser's bound is exact except when $n = 3t$ for $t \geq 2$.

Theorem. For all $n \geq 4$, $g(n) = \lfloor (n + 3)/3 \rfloor$.

Assume henceforth that $n \geq 4$. The example of the next section shows that $g(n) \leq \lfloor n/3 \rfloor + 1$. Section 3 recalls Moser's proof for $g(n) \geq \lfloor (n + 2)/3 \rfloor$, thus verifying the theorem for all n except $n = 6, 9, 12, \dots$. We then complete the proof that $g(3t) \geq t + 1$ by extending Moser's approach. The proof shows that x_0 for a run of length $\geq \lfloor (n + 3)/3 \rfloor$ can always be found among the vertices of P on the smallest circle enclosing P or among the vertices adjacent to these. Since the minimum spanning circle can be determined in $O(n)$ time, a similar result holds for determination of a run of length $\lfloor (n + 3)/3 \rfloor$.

The discussion in Section 4 comments on a few related problems, including the minimax run length for n when the angle at x_j between x_0x_j and x_jx_{j+1} is at least $\pi/2$ for $j = 1, \dots, k - 1$. In this case, Moser's bound of $\lfloor (n + 2)/3 \rfloor$ is exact for all n .

2. Example

Let a, b , and c be the vertices of an isosceles triangle with odd angle $\beta < \pi/4$ at b . For $n \geq 5$ form a convex n -gon with a, b, c , and $n - 3$ other vertices as follows. Vertices x and y are just above a and c and near lines ab and cb , respectively: see Fig. 1. Let A and B be nonnegative integers such that $2A + B = n - 5$. Place B vertices near line ac and its midpoint, symmetrically around the perpendicular bisector $\perp$ from b . The remaining $2A$ go near b , half near ba and the other half near bc symmetric to the first half around $\perp$. The $2A + B$ points are placed so the resulting n -gon is convex.

The construction allows the conclusion that if u is x or a or one of the B points, then $d(u, y) < d(u, c)$. Similarly, if v is y or c or one of the B points, $d(v, x) < d(v, a)$,

If P has only two vertices on C , their diagonal goes through C 's center. Moser's lemma then implies that

$$g(P) \geq \lfloor (n+1)/2 \rfloor \geq \lfloor n/3 \rfloor + 1.$$

Suppose henceforth that at least three vertices of P lie on C . As Moser observed, C 's minimality implies that P has three vertices on C which determine a triangle having no angle greater than $\pi/2$. Denote such vertices by a , b , and c and their respective interior angles by α , β , and γ with $\alpha + \beta + \gamma = \pi$. By convexity, all vertices of P lie in three subsemicircular sectors bounded by C and diagonals ab , bc , and ac : see Fig. 2. One of these sectors contains at least $\lfloor (n+5)/3 \rfloor$ vertices. Therefore, by Moser's lemma,

$$g(P) \geq \lfloor (n+5)/3 \rfloor - 1 = \lfloor (n+2)/3 \rfloor.$$

Since $\lfloor (n+2)/3 \rfloor = \lfloor (n+3)/3 \rfloor$ except when n is divisible by 3, we are done unless $n = 3t$. Assume henceforth that $n = 3t \geq 2$. We suppose that P has no run of length $t+1$ and derive a contradiction.

Given our supposition, Moser's lemma requires exactly $t-1$ vertices besides a , b , and c in each sector. Then each sector admits runs of length t . Assume for

Fig. 2. All vertices of P lie in three subsemicircular sectors bounded by C and diagonals ab , bc , and ac .

definiteness that $\alpha = \max\{\alpha, \beta, \gamma\}$ and align diagonal bc vertically as shown in Fig. 2. To prevent runs of length $t + 1$ from b or c , all vertices in the left two sectors besides a , b , and c must be in the interiors of triangles abp and acq determined by ab , ac and perpendiculars to the sides of triangle abc .

Let x and y be the vertices nearest to a in abp and acq , respectively. Thus x is clockwise adjacent to a , and y is counterclockwise adjacent to a . Because $\alpha = \max\{\alpha, \beta, \gamma\}$, we have $\alpha \geq \pi/3$, and therefore at least one of $d(x, y) > d(x, a)$ and $d(x, y) > d(y, a)$ holds. In fact, both must hold under our supposition of no $t + 1$ run.

To see this, observe first that if s is a point on ac and m is the midpoint of segment sa , then the vertical through s and the perpendicular to ac through m intersect on or below ab . The intersection point is on ab if and only if abc is isosceles with $\alpha = \gamma$. If $\alpha > \gamma$, the intersection point lies under ab .

Suppose not $\{d(x, y) > d(x, a), d(x, y) > d(y, a)\}$. Assume for definiteness that $d(x, y) > d(y, a)$ and $d(x, y) \leq d(x, a)$. Then:

- (i) To prevent a clockwise run of length $t + 1$ from y , there is a first clockwise vertex u from y at or after x and before b with clockwise adjacent vertex v such that $d(y, u) \geq d(y, v)$. This forces u to be to the left of y (but up in apb of course) since otherwise convexity is violated.
- (ii) Given $d(x, y) \leq d(x, a)$, let s be the point on ac nearest to y , and let m be the midpoint of sa . Then x must be to the right of the perpendicular to ac through m .

In view of the preceding paragraph, (i) and (ii) yield the absurdity that u is to the left of x . We conclude that $d(x, y) > d(y, a) \Rightarrow d(x, y) > d(x, a)$. By a similar proof, $d(x, y) > d(x, a) \Rightarrow d(x, y) > d(y, a)$.

With $d(x, y) > \max\{d(x, a), d(y, a)\}$ define u as in (i) above. Also define u' as the first counterclockwise vertex from x at or after y and before c with counterclockwise neighbor v' such that $d(x, u') \geq d(x, v')$. This is needed to prevent a counterclockwise run of length $t + 1$ from x . Similar to (i), convexity forces u' to be to the left of x .

Let $<_0$ mean *to the left of*. By the definitions of x and y and the observations in (i) and the preceding paragraph, we have

$$x \leq_0 u <_0 y \leq_0 u' <_0 x \quad (\leq_0 \text{ is } <_0 \cup =)$$

and hence $x <_0 x$ by transitivity, for the desired contradiction.

We conclude that when $n = 3t$, $t \geq 2$, P has a run of length $t + 1$. Hence $g(P) \geq \lfloor n/3 \rfloor + 1$ for every convex n -gon P .

4. Discussion

Our theorem carries Moser's 1952 approach to Erdős's conjecture C1 to one logical conclusion by showing that it almost precisely resolves the question of

maximum run lengths around the vertices of convex polygons. It also gives a tiny improvement on Moser's lower bound for C2:

$$f(n) \geq \lfloor (n+3)/3 \rfloor \quad \text{for } n \geq 4.$$

This leaves us a very long way from $\lfloor n/2 \rfloor$, and any substantial improvement to the preceding inequality would be welcome.

We conclude by commenting on three other problems for convex n -gons that are suggested by our minimax run problem.

1. Define a *wide run* as a run $x_0, x_1, \dots, x_k$ whose interior angle at x_j between x_0x_j and x_jx_{j+1} is at least $\pi/2$ for $j = 1, \dots, k-1$. Let $g_\pi(n)$ be the minimum over all convex n -gons of the maximum wide run length of the n -gon. Then $g_\pi(n) = \lfloor (n+2)/3 \rfloor$, so Moser's bound is exact for g_π . Moser's analysis in the initial paragraphs of the preceding section gives $g_\pi(n) \geq \lfloor (n+2)/3 \rfloor$; Fig. 1 yields the reverse inequality because the runs from x counterclockwise to c and from y clockwise to a are not wide runs.

2. Define a *skip run* from x_0 as a clockwise or counterclockwise sequence $x_0, x_1, \dots, x_k$ of not necessarily adjacent vertices that are progressively farther from x_0 . Let $h(n)$ be the minimum over all convex n -gons of the maximum skip run of the n -gon. Obviously

$$h(n) \geq g(n) = \lfloor n/3 \rfloor + 1.$$

Moreover, an example based on careful placements of vertices near the corners and sides of an equilateral triangle shows that

$$h(n) \leq \lceil n/3 \rceil + 2.$$

Hence $h(n)$ exceeds $g(n)$ by at most 2.

3. Fix a constant c_0 in $(0, \frac{1}{3}]$. Given a convex n -gon P , let $N_1(P, c_0)$ be the number of vertices from which there is a clockwise run of length $\lceil c_0 n \rceil$, and define $N_2(P, c_0)$ similarly for counterclockwise runs. Also let

$$N(n, c_0) = \min[N_1(P, c_0) + N_2(P, c_0)] \text{ over all convex } n\text{-gons}.$$

Figure 1 implies that

$$\begin{aligned} N(n, c_0) &\leq 4(A + 3 - \lceil c_0 n \rceil) + 2(B + 3 - \lceil c_0 n \rceil) \\ &= 2(n + 4 - 3\lceil c_0 n \rceil). \end{aligned}$$

Moser's approach, augmented as in the final part of Section 3, gives

$$2(n + 3 - 3\lceil c_0 n \rceil) + 1 \leq N(n, c_0).$$

If $c_0 = (1 - \varepsilon)/3$, then $N(n, c_0)$ is approximately $2n\varepsilon$.

References

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