On the Cayley Factorization of Calotte Conditions

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Abstract. The calotte condition is the projective condition that a cycle of lines radiating from the vertices of a plane n -gon be the correct projection of a ring of faces surrounding an n -gonal piece of plane in space, the spatial figure being not entirely coplanar. This condition can be expressed as a homogeneous bracket polynomial. In general, this polynomial is not Cayley factorable. Henry Crapo conjectured that it becomes so when multiplied by a product of $n - 4$ brackets. It is the purpose of this article to prove this conjecture.

1. Introduction

An important problem in computer-aided geometric reasoning is to find geometric interpretations for algebraic expressions automatically. For projective geometry, invariants can be expressed as homogeneous bracket polynomials. However, it is usually not easy to extract geometric information from a bracket polynomial. On the other hand, the two operations in Cayley algebra, *meet* and *join*, correspond to intersection and span. Thus the question reduces to the Cayley factorization problem—expressing a homogeneous bracket polynomial as a Cayley algebra expression (using only the join and meet operations).

White (1991) has an algorithm to do this whenever possible for the multilinear case. (Recall that a bracket polynomial is multilinear if and only if each variable point occurs exactly once in each monomial term which is a product of brackets.)

For the nonmultilinear case, no algorithm exists. However, Sturmfels and Whiteley (1991) proved that a homogeneous bracket polynomial is Cayley factorable after multiplication by a certain number of monomials. Crapo (1991) investigated the nonmultilinear case for several important examples drawn from scene analysis. One of them is the “calotte condition.” An n -ring is a cycle of n lines $a_1b_1, a_2b_2, \dots, a_nb_n$ radiating from the vertices $a_1, a_2, \dots, a_n$ of an n -gon in the

Fig. 1.1. A 3-ring.

plane. This n -ring is called an n -calotte if it is the correct projection of a ring of faces surrounding an n -gonal piece of plane in space, the spatial figure being not entirely coplanar. The projective condition for an n -ring to be an n -calotte is called the *calotte condition*. The calotte condition for the 3-ring in Fig. 1.1 is

$$[a_1a_2b_2][a_2a_3b_3][a_3a_1b_1] - [b_1a_1a_2][b_2a_2a_3][b_3a_3a_1] = 0,$$

which factorizes to

$$[a_1a_2a_3](a_1b_1 \wedge a_2b_2 \wedge a_3b_3) = 0.$$

From the latter condition, it is easily seen that the calotte condition is satisfied when either the triangle $a_1a_2a_3$ is collinear or the lines a_1b_1 , a_2b_2 , a_3b_3 are concurrent.

The calotte condition for the 4-ring in Fig. 1.2 is

$$[a_1a_2b_2][a_2a_3b_3][a_3a_4b_4][a_4a_1b_1] - [b_1a_1a_2][b_2a_2a_3][b_3a_3a_4][b_4a_4a_1] = 0.$$

This has the following two equivalent Cayley factorizations:

$$(a_1b_1 \wedge a_2b_2) \vee (a_1a_4 \wedge a_2a_3) \vee (a_3b_3 \wedge a_4b_4) = 0$$

Fig. 1.2. A 4-ring.

and

$$(a_1b_1 \wedge a_4b_4) \vee (a_2b_2 \wedge a_3b_3) \vee (a_1a_2 \wedge a_4a_3) = 0,$$

which can be interpreted as the three points $(a_1b_1 \wedge a_2b_2)$, $(a_1a_4 \wedge a_2a_3)$, $(a_3b_3 \wedge a_4b_4)$ and the three points $a_1b_1 \wedge a_4b_4$, $a_2b_2 \wedge a_3b_3$, $a_1a_2 \wedge a_4a_3$ being collinear, respectively.

The calotte condition for the 5-ring is not Cayley factorizable but becomes so when multiplied by a single bracket $[a_1a_3a_4]$.

The above Cayley factorizations can be found in [1]. Crapo also conjectured that the calotte condition for an n -ring, $n \geq 5$, is not Cayley factorizable but becomes so when multiplied by $n - 4$ brackets. It is the purpose of this paper to prove the second part of this conjecture.

Since we are concerned with Cayley factorization, the points are assumed to be in generic position. More precisely, we assume that the coordinates of the points are independent indeterminates over the reals. The bracket $[abc]$ is then a polynomial whose variables are the coordinates of a , b , and c .

We now give a brief description of Cayley algebra. Let V be a d -dimensional vector space over a field K . In our case $V = \mathbb{R}[X]^3$, where $\mathbb{R}[X]$ is the vector space of polynomials with real coefficients, whose variables are the homogeneous coordinates, (x_1, x_2, x_3) , of points of the calotte in question. Let $\bigwedge V$ denote the exterior algebra on V . The exterior product, known as the *join*, of a, b is denoted by $a \vee b$ or simply ab . Note that the join of $a_1, \dots, a_k$ is nonzero if and only if the a_i 's are distinct and linearly independent. The join of d points $a_1, \dots, a_d$, denoted by the *bracket* $[a_1, \dots, a_d]$, is the determinant of the vectors $a_1, \dots, a_d$.

The second operation *meet* ($\wedge$) is defined as follows. If $A = a_1 \cdots a_j$, $B = b_1 \cdots b_k$ with $j + k \geq d$, then

$$A \wedge B = \sum_{\sigma} \text{sign}(\sigma) [a_{\sigma(1)}, \dots, a_{\sigma(d-k)}, b_1, \dots, b_k] a_{\sigma(d-k+1)} \cdots a_{\sigma(j)}.$$

The sum is taken over all permutations σ of $\{1, 2, \dots, j\}$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(d-k) \quad \text{and} \quad \sigma(d-k+1) < \sigma(d-k+2) < \cdots < \sigma(j).$$

The meet is associative, distributive over addition and anticommutative in the sense that $A \wedge B = (-1)^{(d-k)(d-j)} B \wedge A$. In our case we work with the real projective plane, so $d = 3$. Thus $ab \wedge cd = [acd]b - [bcd]a = [abd]c - [abc]d$.

2. Pointing Algorithm

A closer examination of the condition for the 4-ring reveals the following. If we let $x = a_1a_4 \wedge a_2a_3$ and $y = a_3b_3 \wedge a_4b_4$ to obtain the 3-ring in Fig. 2.1, then the calotte condition for the 4-ring is equivalent to the calotte condition for this 3-ring.

Fig. 2.1. A 3-ring.

This procedure of reduction is called *pointing*. In general this procedure can be used to reduce an n -calotte to an $(n - 1)$ -calotte with an almost equivalent condition. In this section we look at one pointing sequence which allows us to get a proof of the main theorem.

For the n -ring shown in Fig. 2.2, the calotte condition is

$$C(n) = [a_1 a_2 b_2][a_2 a_3 b_3] \cdots [a_{n-1} a_n b_n][a_n a_1 b_1] \\ - [b_1 a_1 a_2][b_2 a_2 a_3] \cdots [b_{n-1} a_{n-1} a_n][b_n a_n a_1] = 0.$$

Let

$$\begin{aligned} x_{n-1} &= a_1 a_n \wedge a_{n-2} a_{n-1}, & y_{n-1} &= a_{n-1} b_{n-1} \wedge a_n b_n, \\ x_{n-2} &= a_1 x_{n-1} \wedge a_{n-3} a_{n-2}, & y_{n-2} &= a_{n-2} b_{n-2} \wedge x_{n-1} y_{n-1}, \\ &\vdots & &\vdots \\ x_m &= a_1 x_{m+1} \wedge a_{m-1} a_m, & y_m &= a_m b_m \wedge x_{m+1} y_{m+1}, \\ &\vdots & &\vdots \\ x_4 &= a_1 x_5 \wedge a_3 a_4, & y_4 &= a_4 b_4 \wedge x_5 y_5. \end{aligned}$$

$\{x_{n-1}, y_{n-1}, \dots, x_4, y_4\}$ is call a *pointing sequence*. Then the *derived m -ring* corresponding to this pointing sequence is the calotte with vertices $a_1, a_2, \dots, a_{m-1}, x_m$

Fig. 2.2. An n -ring.

Fig. 2.3. A 6-ring (a) and its derived 5-ring (b) and 4-ring (c).

and $b_1, b_2, \dots, b_{m-1}, y_m$. Figure 2.3 shows a 6-ring together with its derived 5- and 4-rings.

Let

$$C(m') = [a_1 a_2 b_2][a_2 a_3 b_3] \cdots [a_{m-2} a_{m-1} b_{m-1}][a_{m-1} x_m y_m][x_m a_1 b_1] \\ - [b_1 a_1 a_2][b_2 a_2 a_3] \cdots [b_{m-2} a_{m-2} a_{m-1}][b_{m-1} a_{m-1} x_m][y_m x_m a_1].$$

Then $C(m') = 0$ is the condition for the derived m -ring

Theorem 2.1. *If $n > 4$, then*

$$C((n-1)) = [a_1 a_{n-2} a_{n-1}][a_1 a_{n-2} a_n]C(n),$$

$$C(4') = [a_1 a_{n-2} a_{n-1}][a_1 a_{n-2} a_n][a_1 a_{n-3} a_{n-2}][a_1 a_{n-3} x_{n-1}] \\ \times [a_1 a_{n-4} a_{n-3}][a_1 a_{n-4} x_{n-2}] \cdots [a_1 a_3 a_4][a_1 a_3 x_5]C(n).$$

Proof. First note that

$$C(n) = [a_1 a_2 b_2][a_2 a_3 b_3] \cdots [a_{n-1} a_n b_n][a_n a_1 b_1] \\ - [b_1 a_1 a_2][b_2 a_2 a_3] \cdots [b_{n-1} a_{n-1} a_n][b_n a_n a_1]$$

and

$$C((n-1)') = [a_1 a_2 b_2][a_2 a_3 b_3] \cdots [a_{n-3} a_{n-2} b_{n-2}][a_{n-2} x_{n-1} y_{n-1}][x_{n-1} a_1 b_1] \\ - [b_1 a_1 a_2][b_2 a_2 a_3] \cdots [b_{n-3} a_{n-3} a_{n-2}][b_{n-2} a_{n-2} x_{n-1}][y_{n-1} x_{n-1} a_1].$$

So it suffices to prove that

$$[b_{n-2} a_{n-2} x_{n-1}][y_{n-1} x_{n-1} a_1] \\ = [b_{n-2} a_{n-2} a_{n-1}][b_{n-1} a_{n-1} a_n][b_n a_n a_1][a_1 a_{n-2} a_{n-1}][a_1 a_{n-2} a_n] \quad (2.1)$$

and

$$[a_{n-2} x_{n-1} y_{n-1}][x_{n-1} a_1 b_1] \\ = [a_{n-2} a_{n-1} b_{n-1}][a_{n-1} a_n b_n][a_n a_1 b_1][a_1 a_{n-2} a_{n-1}][a_1 a_{n-2} a_n]. \quad (2.2)$$

Since

$$x_{n-1} = a_1 a_n \wedge a_{n-2} a_{n-1},$$

$$y_{n-1} = a_{n-1} b_{n-1} \wedge a_n b_n,$$

we have

$$x_{n-1} = [a_1 a_n a_{n-1}]a_{n-2} - [a_1 a_n a_{n-2}]a_{n-1} \\ = [a_1 a_{n-2} a_n]a_{n-1} - [a_1 a_{n-1} a_n]a_{n-2}.$$

So

$$[b_{n-2} a_{n-2} x_{n-1}] = [a_1 a_{n-2} a_n][b_{n-2} a_{n-2} a_{n-1}].$$

Also

$$x_{n-1} = [a_1 a_{n-2} a_{n-1}]a_n - [a_{n-2} a_{n-1} a_n]a_1,$$

$$y_{n-1} = [a_{n-1} b_{n-1} b_n]a_n - [a_{n-1} b_{n-1} a_n]b_n.$$

Therefore

$$[y_{n-1} x_{n-1} a_1] = [a_{n-2} a_{n-1} a_1][b_{n-1} a_{n-1} a_n][b_n a_n a_1].$$

This proves (2.1). Equation (2.2) can be proved similarly.

The second equality follows by an inductive argument. $\square$

3. An Example

We now illustrate the proof of the main proof using the case $n = 6$. The proof of the general case works along the same reasoning and is presented in the next section.

According to Theorem 2.1,

$$C(4') = [a_1 a_4 a_5][a_1 a_4 a_6][a_1 a_3 a_4][a_1 a_3 x_5]C(6), \quad (3.1)$$

where

$$\begin{aligned} x_5 &= a_1 a_6 \wedge a_4 a_5, & y_5 &= a_5 b_5 \wedge a_6 b_6, \\ x_4 &= a_1 x_5 \wedge a_3 a_4, & y_4 &= a_4 b_4 \wedge x_5 y_5. \end{aligned} \quad (3.2)$$

However,

$$C(4') = (a_1 b_1 \wedge a_2 b_2) \vee (a_1 x_4 \wedge a_2 a_3) \vee (a_3 b_3 \wedge x_4 y_4). \quad (3.3)$$

The first step is to write x_4, x_5, y_4, y_5 in terms of the a_i 's and b_i 's. These are done inductively as follows. Let

$$\beta_4 = [a_1 a_4 a_5][a_1 a_3 a_4], \quad \beta_5 = [a_1 a_4 a_5]. \quad (3.4)$$

Thus, from (3.2),

$$x_5 = [a_1 a_4 a_5]a_6 - [a_6 a_4 a_5]a_1 = \alpha_5 a_1 + \beta_5 a_6, \quad (3.5)$$

$$\begin{aligned} x_4 &= [a_1 a_3 a_4]x_5 - [x_5 a_3 a_4]a_1 = [a_1 a_3 a_4]\beta_5 a_6 + \alpha_4 a_6 \\ &= \beta_4 a_6 + \alpha_4 a_1, \end{aligned} \quad (3.6)$$

where α_5, α_4 are bracket polynomials, and

$$[a_1 a_3 x_5] = \beta_5 [a_1 a_3 a_6]. \quad (3.7)$$

From (3.4) and (3.7), (3.1) can be rewritten as

$$C(4') = \beta_4 \beta_5 [a_1 a_4 a_6][a_1 a_3 a_6]C(6). \quad (3.8)$$

Next we have, from (3.5) and (3.6),

$$x_5 y_5 = (a_1 a_6 \wedge a_4 a_5) \vee (a_5 b_5 \wedge a_6 b_6), \quad (3.9)$$

$$\begin{aligned} x_4 y_4 &= (a_1 x_5 \wedge a_3 a_4) \vee (a_4 b_4 \wedge x_5 y_5) \\ &= \beta_5 (a_1 a_6 \wedge a_3 a_4) \vee (a_4 b_4 \wedge x_5 y_5), \end{aligned} \quad (3.10)$$

$$a_1 x_4 = \beta_4 a_1 a_6. \quad (3.11)$$

From (3.9), (3.10), and (3.11), (3.3) can be written as

$$\begin{aligned} C(4') &= \beta_4 \beta_5 (a_1 b_1 \wedge a_2 b_2) \vee (a_1 a_6 \wedge a_2 a_3) \\ &\quad \vee \{a_3 b_3 \wedge \{(a_1 a_6 \wedge a_3 a_4) \vee (a_4 b_4 \wedge ((a_1 a_6 \wedge a_4 a_5) \vee (a_5 b_5 \wedge a_6 b_6)))\}\}. \end{aligned} \quad (3.12)$$

Canceling β_4 and β_5 in the two expressions of $C(4')$ given in (3.8) and (3.12) yields the desired Cayley factorization of $C(6)$ multiplied by two brackets.

4. Proof of the Main Theorem

As stated in Section 1, we know that

$$C(4') = (a_1 b_1 \wedge a_2 b_2) \vee (a_1 x_4 \wedge a_2 a_3) \vee (a_3 b_3 \wedge x_4 y_4).$$

We now want to show that when x_4 and y_4 are written in terms of the a 's and b 's, this Cayley expression can be written as the product of two factors, one is a Cayley expression while the other is a product of exactly $(n - 4)$ brackets called $\beta_4, \dots, \beta_{n-1}$. Theorem 2.1 says that $C(4')$ is $C(n)$ times $2(n - 4)$ brackets. We will show that when x_4 and y_4 are substituted, these brackets are the product of the β_i 's and another $n - 4$ brackets. Cancellation will then give a Cayley factorization of $C(n)$ multiplied by $n - 4$ brackets.

For $4 \leq m < n$, define

$$\beta_m = [a_1 a_{n-2} a_{n-1}] [a_1 a_{n-3} a_{n-2}] \cdots [a_1 a_{m-1} a_m]$$

and

$$A_m = ((a_1 a_n \wedge a_{m-2} a_{m-1}) \vee (a_{m-1} b_{m-1}.$$

Note the unbalanced brackets in A_m . For convenience we assume that $\beta_n = 1$.

We need to prove three lemmas before presenting the proof of the main theorem.

Lemma 4.1. For each m such that $4 \leq m \leq n - 1$, $x_m = \alpha_m a_1 + \beta_m a_n$, where α_m is a bracket polynomial in the a_i 's.

Proof. Clearly, the assertion is true for $m = n - 1$ as

$$x_{n-1} = [a_1 a_{n-2} a_{n-1}] a_n - [a_{n-2} a_{n-1} a_n] a_1.$$

Suppose the result is true for m , then

$$\begin{aligned} x_{m-1} &= a_1 x_m \wedge a_{m-2} a_{m-1} \\ &= [a_1 a_{m-2} a_{m-1}] x_m - [x_m a_{m-2} a_{m-1}] a_1 \\ &= \beta_m [a_1 a_{m-2} a_{m-1}] a_n + (\alpha_m [a_1 a_{m-2} a_{m-1}] - [x_m a_{m-2} a_{m-1}]) a_1. \end{aligned}$$

Since $\beta_{m-1} = \beta_m [a_1 a_{m-2} a_{m-1}]$, the result follows by induction. $\square$

Lemma 4.2. For each m such that $4 \leq m \leq n - 1$,

$$[a_{m-2} x_m a_1] = \beta_m [a_{m-2} a_n a_1].$$

Lemma 4.3. For each m such that $4 \leq m \leq n - 1$,

$$(x_m y_m) = \beta_{m+1} \beta_{m+2} \cdots \beta_n A_{m+1} \wedge A_{m+2} \wedge \cdots \wedge A_n \wedge a_n b_n) \cdots).$$

Proof. We prove this assertion by induction. Clearly, it is true for $m = n - 1$, as

$$\begin{aligned} (x_{n-1} y_{n-1}) &= ((a_1 a_n \wedge a_{n-2} a_{n-1}) \vee (a_{n-1} b_{n-1} \wedge a_n b_n)) \\ &= \beta_n A_n \wedge a_n b_n). \end{aligned}$$

Now we assume that the assertion is true for $m + 1$ and prove that it is true for m . Thus

$$\begin{aligned} (x_m y_m) &= ((a_1 x_{m+1} \wedge a_{m-1} a_m) \vee (a_m b_m \wedge (x_{m+1} y_{m+1}))) \\ &= ((a_1 (\alpha_{m+1} a_1 + \beta_{m+1} a_n) \wedge a_{m-1} a_m) \vee (a_m b_m \wedge (x_{m+1} y_{m+1}))) \\ &= \beta_{m+1} A_{m+1} \wedge (x_{m+1} y_{m+1})) \\ &= \beta_{m+1} \beta_{m+2} \cdots \beta_n A_{m+1} \wedge A_{m+2} \wedge \cdots \wedge A_n \wedge a_n b_n) \cdots) \end{aligned}$$

and the proof is complete. $\square$

Lemma 4.4.

$$C(4') = \beta_4 \beta_5 \cdots \beta_{n-1} (a_1 b_1 \wedge a_2 b_2) \vee A_4 \wedge A_5 \wedge \cdots \wedge A_n \wedge a_n b_n) \cdots).$$

Proof. Since $a_1x_4 = \beta_4a_1a_n$ and

$$x_4y_4 = \beta_5\beta_6\cdots\beta_nA_5 \wedge A_6 \wedge \cdots \wedge A_n \wedge a_nb_n)\cdots)$$

we have

$$C(4') = \beta_4\beta_5\cdots\beta_{n-1}(a_1b_1 \wedge a_2b_2) \vee A_4 \wedge A_5 \wedge \cdots \wedge A_n \wedge a_nb_n)\cdots). \quad \square$$

Theorem 4.5. *The calotte condition $C(n) = 0$ for an n -ring, $n \geq 4$, is Cayley factorable after multiplication by $n - 4$ factors. In particular,*

$$\begin{aligned} & [a_1a_3a_n][a_1a_4a_n]\cdots[a_1a_{n-2}a_n]C(n) \\ &= (a_1b_1 \wedge a_2b_2) \vee ((a_1a_n \wedge a_2a_3) \vee (a_3b_3 \wedge ((a_1a_n \wedge a_3a_4) \vee (a_4b_4 \wedge \cdots \\ & \quad \wedge ((a_1a_n \wedge a_{n-2}a_{n-1}) \vee (a_{n-1}b_{n-1} \wedge a_nb_n)\cdots)). \end{aligned}$$

Proof. From Lemma 4.4 and the definition of the A_n 's, we have

$$\begin{aligned} C(4') &= \beta_4\beta_5\cdots\beta_{n-1}(a_1b_1 \wedge a_2b_2) \vee ((a_1a_n \wedge a_2a_3) \vee (a_3b_3 \wedge ((a_1a_n \wedge a_3a_4) \\ & \quad \vee (a_4b_4 \wedge \cdots \wedge ((a_1a_n \wedge a_{n-2}a_{n-1}) \vee (a_{n-1}b_{n-1} \wedge a_nb_n)\cdots)). \end{aligned}$$

From Lemma 4.1 we have, for $5 \leq m \leq n - 1$,

$$[a_1a_{m-2}x_m] = \beta_m[a_1a_{m-2}a_n]$$

and by the definition

$$\beta_4 = [a_1a_{n-2}a_{n-1}][a_1a_{n-3}a_{n-2}]\cdots[a_1a_3a_4].$$

Thus, from Theorem 2.1, we have

$$\begin{aligned} C(4') &= [a_1a_{n-2}a_{n-1}][a_1a_{n-2}a_n][a_1a_{n-3}a_{n-2}][a_1a_{n-3}x_{n-1}] \\ & \quad \times [a_1a_{n-4}a_{n-3}][a_1a_{n-4}x_{n-2}]\cdots[a_1a_3a_4][a_1a_3x_5]C(n) \\ &= ([a_1a_{n-2}a_{n-1}][a_1a_{n-3}a_{n-2}]\cdots[a_1a_3a_4])\beta_5\beta_6\cdots\beta_{n-1} \\ & \quad \times [a_1a_3a_n][a_1a_4a_n]\cdots[a_1a_{n-3}a_n][a_1a_{n-2}a_n]C(n) \\ &= \beta_4\beta_5\cdots\beta_{n-1}[a_1a_3a_n][a_1a_4a_n]\cdots[a_1a_{n-2}a_n]C(n). \end{aligned}$$

Since each bracket $[a_1a_2a_3]$ is a polynomial whose variables are the coordinates of the a_i 's, the β_i 's and $C(n)$ are all polynomials in the coordinates of the points. The same is true of the Cayley expression of $C(4')$ in Lemma 4.4. Moreover, the β_i 's are nonzero polynomials. Thus we can cancel the β_i 's from the two expressions of $C(4')$ to get the desired identity. $\square$

Remark 4.6. Note that the Cayley factorization gives a geometric condition which corresponds to the calotte condition for the derived calotte obtained from the pointing sequence we used. For example, the Cayley factorization of $C(4)$ gives the condition

$$\begin{aligned} 0 &= (a_1 b_1 \wedge a_2 b_2) \vee (a_1 a_4 \wedge a_2 a_3) \vee (a_3 b_3 \wedge a_4 b_4) \\ &= (a_1 b_1 \wedge a_2 b_2) \vee (x_3) \vee (y_3). \end{aligned}$$

This is precisely the geometric condition for the derived 3-ring.

5. Other Pointing Sequences

In the previous sections we have shown that pointing gives rise to an inductive procedure which reduces an n -calotte to a 4-calotte. Subject to the condition that certain brackets are not zero, we have

$$C(n) = 0 \Leftrightarrow C(4') = 0.$$

We also show that, for a particular pointing sequence, we have a particularly nice result.

However, there are other pointing sequences. For example, if we use the pointing sequence $\{x_5, y_5, x_4, y_4\}$ for the 6-ring in Fig. 5.1, then the calotte condition $C(4^*)$ of the derived 4-ring satisfies

$$[a_6 a_2 a_3][a_6 a_1 a_3][a_5 x_5 a_3][x_5 a_4 a_5]C(6) = C(4^*),$$

Fig. 5.1. A 6-ring.

where

$$x_5 = a_1 a_6 \wedge a_2 a_3 = [a_1 a_6 a_3] a_2 - [a_1 a_6 a_2] a_3 = [a_2 a_3 a_1] a_6 - [a_2 a_3 a_6] a_1,$$

$$x_4 = a_3 x_5 \wedge a_4 a_5 = [a_3 x_5 a_4] a_5 - [a_3 x_5 a_5] a_4 = [a_5 a_4 a_3] x_5 - [a_4 a_5 x_5] a_3,$$

$$y_5 = b_1 a_1 \wedge b_2 a_2, \quad y_4 = b_3 a_3 \wedge b_4 a_4.$$

Hence

$$\begin{aligned} x_4 y_4 &= (a_3 x_5 \wedge a_4 a_5) \vee (b_3 a_3 \wedge b_4 a_4) \\ &= [a_1 a_6 a_3] (a_3 a_2 \wedge a_4 a_5) \vee (b_3 a_3 \wedge b_4 a_4), \end{aligned}$$

$$a_6 x_5 = [a_2 a_3 a_6] a_1 a_6,$$

$$x_4 a_5 = -[a_3 x_5 a_5] a_4 a_5,$$

$$[x_5 a_4 a_5] = [a_2 a_3 a_1] [a_6 a_4 a_5] - [a_2 a_3 a_5] [a_1 a_4 a_5],$$

$$x_4 x_5 = -[x_5 a_4 a_5] a_3 x_5 = -[a_1 a_6 a_3] [x_5 a_4 a_5] a_3 a_2,$$

$$[a_5 x_5 a_3] = [a_1 a_6 a_3] [a_5 a_2 a_3].$$

However,

$$C(4^*) = (x_5 y_5 \wedge x_4 y_4) \vee (a_6 x_5 \wedge x_4 a_5) \vee (a_6 b_6 \wedge a_5 b_5).$$

Substitution then gives

$$\begin{aligned} &([a_2 a_3 a_6] [a_1 a_4 a_5] - [a_2 a_3 a_1] [a_6 a_4 a_5]) C(6) \\ &= \{((a_2 a_3 \wedge a_1 a_6) \vee (b_1 a_1 \wedge b_2 a_2)) \wedge ((a_3 a_2 \wedge a_4 a_5) \vee (b_3 a_3 \wedge b_4 a_4))\} \\ &\quad \vee (a_1 a_6 \wedge a_4 a_5) \vee (a_6 b_6 \wedge a_5 b_5). \end{aligned}$$

However, there is an equivalent expression for $C(4^*)$:

$$C(4^*) = (x_5 y_5 \wedge a_6 b_6) \vee (x_4 y_4 \wedge a_5 b_5) \vee (x_4 x_5 \wedge a_5 a_6).$$

From this we have

$$\begin{aligned} [a_6 a_2 a_3] [a_5 a_2 a_3] C(6) &= \{((a_1 a_6 \wedge a_2 a_3) \vee (b_1 a_1 \wedge b_2 a_2)) \wedge a_6 b_6\} \\ &\quad \vee \{((a_3 a_2 \wedge a_4 a_5) \vee (b_3 a_3 \wedge b_4 a_4)) \wedge a_5 b_5\} \\ &\quad \vee \{a_3 a_2 \wedge a_5 a_6\}. \end{aligned}$$

So we see that the calotte condition $C(n)$ can be factorized in many different ways, each corresponding to a pointing sequence. However, other than the particular pointing sequence used in the previous section, in general the multiplier need not be a monomial, i.e., a product of brackets. However, we believe that the monomial terms of the bracket polynomial used as the multiplier are products of $n - 4$ brackets.

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