The Reverse Isoperimetric Problem for Gaussian Measure*

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Abstract. It is shown that if g is the standard Gaussian density on $\mathbb{R}^n$ and C is a convex body in $\mathbb{R}^n$,

$$\int_{\partial C} g \leq 4n^{1/4}.$$

The arguments presented raise several questions in integral geometry.

1. Introduction

Borell [B] and Sudakov and Tsirel'son [ST] proved that if γ is the standard Gaussian measure on $\mathbb{R}^n$, $t \in (0, 1)$ and $\varepsilon > 0$, then among all measurable $A \subset \mathbb{R}^n$ with $\gamma(A) = t$, the sets for which A^ε (the ε -neighborhood of A) has smallest Gaussian measure, are half-spaces. Thus, half-spaces solve the isoperimetric problem for Gaussian measure on $\mathbb{R}^n$.

Motivated by several probabilistic problems, Mushtari and Kwapień asked about reverse inequalities: in particular, if g is the density of γ , how large can the integral

$$\int_{\partial C} g \tag{1}$$

be if C is a convex body in $\mathbb{R}^n$? The integral is taken with respect to the

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$(n - 1)$ -dimensional Hausdorff measure on ∂C . Let

$$v_n = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}$$

be the volume of the Euclidean unit ball in $\mathbb{R}^n$. Then, if C is a ball of radius r , centered at 0, expression (1) is

$$nv_n r^{n-1} \frac{1}{(\sqrt{2\pi})^n} e^{-r^2/2},$$

whose maximum value (attained when $r = \sqrt{n - 1}$) is

$$\frac{n}{2^{n/2}\Gamma(n/2 + 1)} (n - 1)^{(n-1)/2} e^{-(n-1)/2} \sim \frac{1}{\sqrt{\pi}}.$$

If C is a convex body which is very flat, and so approximately a “two-sided” one-codimensional subspace, then

$$\begin{aligned} \int_{\partial C} g &\approx 2 \int_{\mathbb{R}^{n-1}} \frac{1}{(\sqrt{2\pi})^n} e^{-|x|^2/2} dx \\ &= \sqrt{\frac{2}{\pi}}. \end{aligned}$$

However, neither of these examples can be extremal for the problem, since if Q is a centrally symmetric cube of appropriate size, in $\mathbb{R}^n$,

$$\int_{\partial Q} g \geq \frac{\sqrt{\log n}}{e}.$$

This observation is proved in Proposition 2 below.

It is very easy to obtain an upper estimate of $\sqrt{n}$ for expression (1) for general bodies. The purpose of this paper is to improve this trivial estimate. The argument used is a rather natural approach to the problem, suggested by Cauchy’s formula for the surface area of convex bodies. (See p. 89 of [E] for a statement and proof of Cauchy’s formula.) Unfortunately, this approach cannot give an estimate better than $Kn^{1/4}$ for some constant K , whereas vague analogies between this problem and others suggest that the correct estimate should be a multiple of $\log n$. It was noticed by Milman that, as a consequence of Pisier’s estimate for the K -convexity constant of a finite-dimensional normed space [P], every n -dimensional normed space has a representation $(\mathbb{R}^n, \|\cdot\|)$ with unit ball C of volume 1 and

$$\int_{\mathbb{R}^n} \|x\| g(x) dx \leq (\text{constant}) \log n.$$

On the other hand, the measure constructed in Theorem 4 answers a question which is itself quite natural, and it is the appearance of the “unusual” growth rate, $n^{1/4}$, in this context, which is perhaps the most interesting feature of this argument. The method used here raises several further geometric questions which are described in Section 4. Section 2 contains the simple upper and lower estimates mentioned above and Section 3 gives the main result.

2. Simple Bounds

Proposition 1. *If g is the standard Gaussian density on $\mathbb{R}^n$ and C is a convex body in $\mathbb{R}^n$,*

$$\int_{\partial C} g < \sqrt{n}.$$

Proof. For $x \in \mathbb{R}^n$,

$$e^{-|x|^2/2} = \int_0^\infty t e^{-t^2/2} \chi_{B(t)}(x) dt,$$

where $\chi_{B(t)}$ is the characteristic function of the n -dimensional Euclidean ball of radius t , centered at 0. Hence

$$\begin{aligned} \frac{1}{(\sqrt{2\pi})^n} \int_{\partial C} e^{-|x|^2/2} &= \frac{1}{(\sqrt{2\pi})^n} \int_0^\infty t e^{-t^2/2} |\partial C \cap B(t)| dt \\ &\leq \frac{1}{(\sqrt{2\pi})^n} \int_0^\infty t e^{-t^2/2} |\partial(C \cap B(t))| dt \\ &< \frac{1}{(\sqrt{2\pi})^n} \int_0^\infty t e^{-t^2/2} |\partial B(t)| dt \\ &= \frac{nv_n}{(\sqrt{2\pi})^n} \int_0^\infty t^n e^{-t^2/2} dt < \sqrt{n}. \end{aligned} \quad \square$$

Proposition 2. *If $Q \subset \mathbb{R}^n$ is a centrally symmetric cube of appropriate size, then*

$$\int_{\partial Q} g \geq \frac{\sqrt{\log n}}{e}.$$

Proof. Let Q be a symmetric cube of side $2r$. Then

$$\int_{\partial Q} g = \frac{2n}{(\sqrt{2\pi})^n} e^{-r^2/2} \left(\int_{-r}^r e^{-t^2/2} dt \right)^{n-1}.$$

Since

$$\begin{aligned} \frac{2}{\sqrt{2\pi}} \int_r^\infty e^{-t^2/2} dt &\leq \frac{2}{\sqrt{2\pi}} \frac{1}{r} \int_r^\infty t e^{-t^2/2} dt \\ &= \sqrt{\frac{2}{\pi}} \frac{e^{-r^2/2}}{r}, \\ \frac{1}{\sqrt{2\pi}} \int_{-r}^r e^{-t^2/2} dt &\geq 1 - \sqrt{\frac{2}{\pi}} \frac{e^{-r^2/2}}{r}. \end{aligned}$$

Hence

$$\int_{\partial Q} g \geq \frac{2n}{\sqrt{2\pi}} e^{-r^2/2} \left(1 - \sqrt{\frac{2}{\pi}} \frac{e^{-r^2/2}}{r} \right)^{n-1}.$$

Choose r so that

$$\sqrt{\frac{2}{\pi}} \frac{e^{-r^2/2}}{r} = \frac{1}{n}.$$

Then $r \geq \sqrt{\log n}$ and so

$$e^{-r^2/2} \geq \sqrt{\frac{\pi}{2}} \frac{\sqrt{\log n}}{n}.$$

So, for this r ,

$$\int_{\partial Q} g \geq \sqrt{\log n} \left(1 - \frac{1}{n} \right)^{n-1} > \frac{\sqrt{\log n}}{e}. \quad \square$$

3. The Main Result

The proof of Theorem 4, below, makes use of the following simple rearrangement inequality. Henceforth, S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ and σ_{n-1} denotes the rotation invariant probability on S^{n-1} .

Lemma 3. *Let f and g be nondecreasing, real-valued functions on $[0, 1]$. Then the integral*

$$\int_{S^{n-1}} f(|\langle \theta, \varphi \rangle|) g(|\langle \theta, \psi \rangle|) d\sigma_{n-1}(\theta)$$

is minimized over $\varphi, \psi \in S^{n-1}$ when $\langle \varphi, \psi \rangle = 0$.

Remark. The integral is obviously maximized when $\varphi = \pm\psi$ since in this case the functions $f(|\langle \cdot, \varphi \rangle|)$ and $g(|\langle \cdot, \psi \rangle|)$ are similarly arranged on S^{n-1} : i.e.,

$$f(|\langle \theta_1, \varphi \rangle|) > f(|\langle \theta_2, \varphi \rangle|)$$

if and only if

$$g(|\langle \theta_1, \psi \rangle|) > g(|\langle \theta_2, \psi \rangle|).$$

Proof of Lemma 3. Assume $\varphi \neq \pm\psi$ and let H be the two-dimensional subspace of $\mathbb{R}^n$ spanned by $\{\varphi, \psi\}$. For $u \in H^\perp$, $|u| < 1$, consider the circle

$$C_u = S^{n-1} \cap (H + u).$$

The restrictions of $f(|\langle \cdot, \varphi \rangle|)$ and $g(|\langle \cdot, \psi \rangle|)$ to C_u , are oppositely arranged if $\langle \varphi, \psi \rangle = 0$, and the distribution of each of these functions on C_u depends only upon $|u|$. So, if λ is the obvious measure on C_u ,

$$\int_{C_u} f(|\langle \theta, \varphi \rangle|)g(|\langle \theta, \psi \rangle|) d\lambda(\theta)$$

is minimized when $\langle \varphi, \psi \rangle = 0$. Integration over all $u \in H^\perp$ of norm at most 1, with respect to an appropriate measure, gives the desired inequality. $\square$

Remark. Presumably, analogous inequalities hold for n functions “on” S^{n-1} . For $n \geq 2$, the trivial argument used above does not seem to be of much use.

Theorem 4. Let g be the standard Gaussian density on $\mathbb{R}^n$, $n \geq 2$, and let C be a convex body in $\mathbb{R}^n$. Then

$$\int_{\partial C} g \leq 4n^{1/4}.$$

Proof. For a unit vector θ in $\mathbb{R}^n$, let P_θ be the orthogonal projection onto the one-codimensional subspace of $\mathbb{R}^n$, $\langle \theta \rangle^\perp$. The aim is to construct a measure μ on $\mathbb{R}^{n-1}$ which is rotation invariant, absolutely continuous with respect to Lebesgue measure and which satisfies

$$\int_S g \leq \int_{S^{n-1}} \mu(P_\theta S) d\sigma_{n-1}(\theta) \quad (2)$$

whenever S is a Borel subset of a hyperplane in $\mathbb{R}^n$. (The quantity $\mu(P_\theta S)$ is interpreted via some isometry between $P_\theta(\mathbb{R}^n)$ and $\mathbb{R}^{n-1}$.)

Now, if C is a convex body, then “ μ almost every” line in a given direction hits ∂C at most twice: i.e., almost all of $P_\theta(\mathbb{R}^n)$ is covered at most twice by $P_\theta(\partial C)$. Hence

$$\int_{\partial C} g \leq 2\mu(\mathbb{R}^{n-1}).$$

If μ is a measure on $\mathbb{R}^n$ with density F where $F(x) = f(|x|)$ for some

$$f: [0, \infty) \rightarrow [0, \infty)$$

then, in the limit for small pieces of hyperplane centered at $r\varphi$ ($r \geq 0$, $\varphi \in S^{n-1}$) with unit normal ψ , inequality (2) becomes

$$\tilde{g}(r) \leq \int_{S^{n-1}} f(r(1 - \langle \theta, \varphi \rangle^2)^{1/2}) \cdot |\langle \theta, \psi \rangle| d\sigma_{n-1}(\theta), \quad (3)$$

where $\tilde{g}(r) = (1/(\sqrt{2\pi})^n)e^{-r^2/2}$.

A function f will be constructed that is (nonnegative and) nonincreasing on $[0, \infty)$ and that satisfies (3) whenever φ and ψ are perpendicular. Since the function $t \mapsto f(r(1 - t^2)^{1/2})$ will then be increasing on $[0, 1]$, Lemma 3 will imply that the right-hand side of (3) is minimized when φ and ψ are perpendicular, and hence that (3) holds for all φ and ψ .

Fix $\varphi \in S^{n-1}$: the right-hand side of (3) is the same for all ψ perpendicular to φ . The average of $|\langle \theta, \psi \rangle|$ over all such ψ , as a function of θ , is proportional to the length of the projection $P_\varphi(\theta)$; i.e., to $(1 - \langle \theta, \varphi \rangle^2)^{1/2}$. The constant of proportionality is

$$\int_{S^{n-2}} |\psi_1| d\sigma_{n-2}(\psi),$$

where ψ_1 is the first coordinate of ψ . Hence if $\psi \perp \varphi$, the right-hand side of (3) is

$$\begin{aligned} & \int_{S^{n-2}} |\psi_1| d\sigma_{n-2}(\psi) \int_{S^{n-1}} f(r(1 - \langle \theta, \varphi \rangle^2)^{1/2})(1 - \langle \theta, \varphi \rangle^2)^{1/2} d\sigma_{n-1}(\theta) \\ &= \frac{\int_0^{\pi/2} \cos \theta \sin^{n-3} \theta d\theta}{\int_0^{\pi/2} \sin^{n-3} \theta d\theta} \frac{\int_0^{\pi/2} f(r \sin \theta) \sin^{n-1} \theta d\theta}{\int_0^{\pi/2} \sin^{n-2} \theta d\theta} \\ &= \frac{2}{\pi} \int_0^{\pi/2} f(r \sin \theta) \sin^{n-1} \theta d\theta. \end{aligned}$$

So the aim is to find a nonincreasing $f: [0, \infty) \rightarrow [0, \infty)$ satisfying

$$\tilde{g}(r) \leq \frac{2}{\pi} \int_0^{\pi/2} f(r \sin \theta) \sin^{n-1} \theta \, d\theta \quad \text{for all } r \geq 0 \quad (4)$$

for which $\mu(\mathbb{R}^{n-1})$ is as small as possible.

It is easy to check (for example by considering monomial functions) that if $h: [0, \infty) \rightarrow \mathbb{R}$ is defined by

$$\int_0^t h(r) r^{n-2} \, dr = t^{n-1} \int_0^{\pi/2} \tilde{g}(t \sin \theta) \sin^{n-2} \theta \, d\theta, \quad (5)$$

then

$$\frac{2}{\pi} \int_0^{\pi/2} h(r \sin \theta) \sin^{n-1} \theta \, d\theta = \tilde{g}(r), \quad 0 \leq r < \infty.$$

So the function $f = h^+$, the positive part of h , satisfies (4) for all r . It remains to show that this f is nonincreasing and that the corresponding measure μ satisfies $\mu(\mathbb{R}^{n-1}) \leq 2n^{1/4}$. Now, from (4)

$$\begin{aligned} h(t) &= \frac{1}{t^{n-2}} \frac{d}{dt} \left(t^{n-1} \int_0^{\pi/2} \tilde{g}(t \sin \theta) \sin^{n-2} \theta \, d\theta \right) \\ &= \frac{1}{(\sqrt{2\pi})^n} \int_0^{\pi/2} (n-1-t^2 \sin^2 \theta) \sin^{n-2} \theta \cdot e^{-(t^2/2) \sin^2 \theta} \, d\theta \\ &= \frac{e^{-t^2/2}}{(\sqrt{2\pi})^n} \int_0^{\pi/2} (n-1-t^2 \sin^2 \theta) \sin^{n-2} \theta \cdot e^{(t^2/2) \cos^2 \theta} \, d\theta. \end{aligned} \quad (6)$$

This expression is obviously positive if

$$0 \leq t \leq \sqrt{n-1}. \quad (7)$$

Also, it is evident that the integral has a power series expansion in t^2 , valid for all t ; for $k \geq 1$, the coefficient of t^{2k} is

$$\begin{aligned} \frac{n-1}{2^k k!} \int_0^{\pi/2} \cos^{2k} \theta \sin^{n-2} \theta \, d\theta - \frac{1}{2^{k-1} (k-1)!} \int_0^{\pi/2} \cos^{2k-2} \theta \sin^n \theta \, d\theta \\ = -\frac{1}{2^k k!} \int_0^{\pi/2} \cos^{2k-2} \theta \sin^n \theta \, d\theta < 0. \end{aligned}$$

This shows that the function $t \mapsto e^{t^2/2} h(t)$ is decreasing on $[0, \infty)$ so that h is decreasing where it is nonnegative and $f = h^+$ is nonincreasing.

Now, the coefficient of t^0 in the integral in (6) is

$$(n-1) \int_0^{\pi/2} \sin^{n-2} \theta \, d\theta = n \int_0^{\pi/2} \sin^n \theta \, d\theta$$

and that of t^2 is $-\frac{1}{2} \int_0^{\pi/2} \sin^n \theta \, d\theta$. So

$$h(t) = \frac{e^{-t^2/2}}{(\sqrt{2\pi})^n} n \int_0^{\pi/2} \sin^n \theta \, d\theta \left(1 - \frac{t^2}{2n} - \dots\right)$$

so the unique zero of h in $[0, \infty)$ occurs at some number $s < \sqrt{2n}$. Remark (7) shows that $s > \sqrt{n-1} \geq 1$. Observe that

$$\begin{aligned} \mu(\mathbb{R}^{n-1}) &= \int_{\mathbb{R}^{n-1}} f(|x|) \, dx \\ &= (n-1)v_{n-1} \int_0^\infty f(r)r^{n-2} \, dr \\ &= \frac{(n-1)\pi^{(n-1)/2}}{\Gamma((n+1)/2)} \int_0^s h(r)r^{n-2} \, dr \\ &= \frac{(n-1)}{2^{n/2}\Gamma((n+1)/2)\sqrt{\pi}} s^{n-1} \int_0^{\pi/2} e^{-(s^2/2)\sin^2\theta} \sin^{n-2} \theta \, d\theta \\ &= \frac{1}{2^{n/2-1}\Gamma((n-1)/2)\sqrt{\pi}} \int_0^s e^{-t^2/2} \frac{t^{n-2}}{\sqrt{1-(t/s)^2}} \, dt. \end{aligned}$$

It is simplest to estimate this integral in two parts. For $0 \leq t \leq s-1$,

$$\frac{1}{\sqrt{1-(t/s)^2}} \leq \frac{1}{\sqrt{2/s-1/s^2}} \leq \sqrt{s} \quad \text{since } s \geq 1.$$

Hence

$$\begin{aligned} &\frac{1}{2^{n/2-1}\Gamma((n-1)/2)\sqrt{\pi}} \int_0^{s-1} e^{-t^2/2} \frac{t^{n-2}}{\sqrt{1-(t/s)^2}} \, dt \\ &= \sqrt{s} \frac{1}{2^{n/2-1}\Gamma((n-1)/2)\sqrt{\pi}} \int_0^\infty e^{-t^2/2} t^{n-2} \, dt \\ &= \sqrt{\frac{s}{2\pi}} < n^{1/4} \quad \text{since } s < \sqrt{2n}. \end{aligned}$$

On the other hand,

$$\max_t e^{-t^2/2} t^{n-2} = e^{-(n-2)/2} (n-2)^{(n-2)/2}$$

and Stirling's formula shows that

$$\frac{1}{2^{n/2-1} \Gamma((n-1)/2) \sqrt{\pi}} e^{-(n-2)/2} (n-2)^{(n-2)/2} < \frac{1}{\pi}.$$

Hence

$$\begin{aligned} \frac{1}{2^{n/2-1} \Gamma((n-1)/2) \sqrt{\pi}} \int_{s-1}^s e^{-t^2/2} \frac{t^{n-2}}{\sqrt{1-(t/s)^2}} dt &\leq \frac{1}{\pi} \int_{s-1}^s \frac{dt}{\sqrt{1-(t/s)^2}} \\ &= \frac{s}{\pi} \left(\frac{\pi}{2} - \sin^{-1} \left(1 - \frac{1}{s} \right) \right) \\ &< \sqrt{\frac{s}{\pi}} < n^{1/4}. \quad \square \end{aligned}$$

4. Some Further Remarks

The proof of Theorem 4 shows something more general. Let $f: [0, \infty) \rightarrow [0, \infty)$ be nondecreasing and set

$$R = (n-1)v_{n-1} \int_0^\infty f(r) r^{n-2} dr.$$

Define $g: \mathbb{R}^n \rightarrow [0, \infty)$ by $g(x) = \tilde{g}(|x|)$ where

$$\tilde{g}(r) = \frac{2}{\pi} \int_0^{\pi/2} f(r \sin \theta) \sin^{n-1} \theta d\theta. \quad (8)$$

Then, for any rectifiable hypersurface $S \subset \mathbb{R}^n$, there is a line which meets S at least

$$\frac{1}{R} \int_S g \text{ times.} \quad (9)$$

The fact that $g(x) = e^{-|x|^2/2}$ cannot arise from a suitable f via (8), suggests that it is the wrong function to consider. It makes sense to ask whether there is a function

g satisfying (9) with $R = 1$ (say), and such that any rectifiable surface S , with small curvature, is hit at most about

$$\sup_s \int_s g$$

times, by any line (the sup being taken over translates $\tilde{S}$ of S).

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