Partitioning Euclidean Space

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Abstract. There is a partition of Euclidean n -space into countably many sets none of which contains the vertices of a regular n -simplex.

A simple consequence of van der Waerden's theorem on arithmetic progressions is that, for any triangle T and any partition of the plane $\mathbb{R}^2$ into finitely many pieces, there is a triangle similar to T whose vertices are in one of the pieces. This result, and its proof, easily extend to higher dimensions.

For an integer $n \geq 2$ we consider Euclidean space $\mathbb{R}^n$ as a vector space over $\mathbb{R}$ with the Euclidean norm $\|x\|$ and the usual inner product $x \cdot y$. We call a subset $T \subseteq \mathbb{R}^n$ an n -simplex if $|T| = n + 1$ and T is not contained in any hyperplane. An n -simplex T is *regular* if there is some positive $d \in \mathbb{R}$ such that $\|x - y\| = d$ whenever $x, y \in T$ are distinct. Equivalently, T is regular iff $(x - y) \cdot (x - z) = d^2/2$ whenever $x, y, z \in T$ are distinct. In particular, $T \subseteq \mathbb{R}^2$ is a regular 2-simplex iff T is the set of vertices of an equilateral triangle, and $T \subseteq \mathbb{R}^3$ is a regular 3-simplex iff T is the set of vertices of a regular tetrahedron. Two n -simplices S, T are *similar* provided there is a function $f: S \rightarrow T$ and some $r > 0$ such that $\|f(x) - f(y)\| = r\|x - y\|$ whenever $x, y \in S$.

The generalization of the aforementioned result can now be stated. If $S \subseteq \mathbb{R}^n$ is an n -simplex, then, for any partition of $\mathbb{R}^n$ into finitely many pieces, there is an n -simplex similar to S which is in one of the pieces.

Complementing this result, Ceder [1] proved that $\mathbb{R}^2$ can be partitioned into countably many pieces none of which contains all the vertices of an equilateral triangle. This was extended to $\mathbb{R}^3$ by Komjáth [2] who proved, using a different technique, that $\mathbb{R}^3$ can be partitioned into countably many pieces none of which contains the vertices of a regular tetrahedron. In this note we refine Komjáth's technique to extend this result simultaneously in two ways: to arbitrary dimension

n and to arbitrary n -simplices. We fix throughout an integer $n \geq 2$ which is to be the dimension of the space being considered.

Theorem. *Let $S \subseteq \mathbb{R}^n$ be any n -simplex. There is a partition of $\mathbb{R}^n$ into countably many pieces none of which contains an n -simplex similar to S .*

The proof of the theorem uses the Axiom of Choice. Some use of the Axiom of Choice is necessary since, as Steinhaus first noted, for any n -simplex $S \subseteq \mathbb{R}^n$, any $X \subseteq \mathbb{R}^n$ having positive Lebesgue measure contains an n -simplex similar to S . As a consequence we get that, for every n -simplex $S \subseteq \mathbb{R}^n$ and every partition of $\mathbb{R}^n$ into countably many measurable sets, one of the sets contains an n -simplex similar to S .

We identify the smallest infinite ordinal ω with the set $\{0, 1, 2, \dots\}$ of natural numbers, and any natural number m we identify with the set $\{0, 1, 2, \dots, m-1\}$ of smaller natural numbers. Let us fix some n -simplex $S = \{x_0, x_1, \dots, x_n\} \subseteq \mathbb{R}^n$. If $X \subseteq \mathbb{R}^n$ and $\psi: X \rightarrow \omega$, then we say that ψ is *proper* if there is no n -simplex $T \subseteq X$ which is similar to S such that $|\{\psi(x): x \in T\}| = 1$. The following is an obvious reformulation of the theorem.

For every infinite cardinal $\kappa \leq 2^{\aleph_0} = |\mathbb{R}|$ and every real-closed field $(*)$
 $F \subseteq \mathbb{R}$ such that $|F| = \kappa$, there is a proper $\psi: F^n \rightarrow \omega$.

To prove the theorem we prove $(*)$ by induction on the cardinal κ . For $\kappa = \aleph_0$, the result is trivial: just let $\psi: F^n \rightarrow \omega$ be any one-one function.

To proceed by induction, assume that $\aleph_0 < \kappa \leq 2^{\aleph_0}$ and that whenever $K \subseteq \mathbb{R}$ is a real-closed field and $|K| < \kappa$, then there is a proper function $\theta: K^n \rightarrow \omega$. Let $F \subseteq \mathbb{R}$ be real-closed such that $|F| = \kappa$. We easily obtain a sequence $\langle F_\alpha: \alpha < \kappa \rangle$ such that the following four conditions hold:

- (1) $S \subseteq F_0^n$;
- (2) if $\alpha < \beta < \kappa$, then F_α and F_β are real-closed fields, $F_\alpha \subseteq F_\beta \subseteq F$ and $|F_\beta| < \kappa$;
- (3) if $\beta < \kappa$ is a limit ordinal, then $F_\beta = \bigcup_{\alpha < \beta} F_\alpha$;
- (4) $F = \bigcup_{\alpha < \kappa} F_\alpha$.

We construct a sequence $\langle \psi_\alpha: \alpha < \kappa \rangle$ of proper functions $\psi_\alpha: F_\alpha^n \rightarrow \omega$ such that whenever $\alpha < \beta < \kappa$, then $\psi_\alpha = \psi_\beta|_{F_\alpha^n}$. Clearly, then $\psi = \bigcup_{\alpha < \kappa} \psi_\alpha$ will be a proper function $\psi: F^n \rightarrow \omega$.

The sequence $\langle \psi_\alpha: \alpha < \kappa \rangle$ is constructed by transfinite recursion. To that end, assume that $\gamma < \kappa$ and that we already have $\langle \psi_\alpha: \alpha < \gamma \rangle$. We wish to construct ψ_γ : If $\gamma = 0$, then let $\psi_0: F_0^n \rightarrow \omega$ be any proper function (the existence of which is guaranteed by the inductive hypothesis on κ). For γ a limit ordinal, we have no choice but to set $\psi_\gamma = \bigcup_{\alpha < \gamma} \psi_\alpha$. It remains to consider the case that $\gamma = \alpha + 1$.

We need two functions to define ψ_γ . The first can be any proper function $\theta: F_\gamma^n \rightarrow \omega$, the existence of which is guaranteed by the inductive hypothesis on κ . The second is a function φ as described in the next lemma.

Lemma 1. *There is $m \in \omega$ and a function $\varphi: F_\gamma^n \setminus F_\alpha^n \rightarrow m$ with the following property: whenever $T \subseteq F_\gamma^n$ is an n -simplex similar to S such that $T \cap F_\alpha^n \neq \emptyset \neq T \setminus F_\alpha^n$, then there are $x, y \in T \setminus F_\alpha^n$ such that $\varphi(x) \neq \varphi(y)$.*

Before proving Lemma 1, we show how to obtain ψ_γ . For $x \in F_\gamma^n$, let

$$\psi_\gamma(x) = \begin{cases} \psi_\alpha(x) & \text{if } x \in F_\alpha^n, \\ m\theta(x) + \varphi(x) & \text{if } x \notin F_\alpha^n. \end{cases}$$

It should be clear that $\psi_\gamma: F_\gamma^n \rightarrow \omega$ is proper and that $\psi_\alpha = \psi_\gamma|_{F_\alpha^n}$.

Several more lemmas are needed to prove Lemma 1. Let $S^{n-1} = \{x \in \mathbb{R}^n: \|x\| = 1\}$ be the unit $(n-1)$ -sphere. The next lemma is an immediate consequence of the compactness of S^{n-1} .

Lemma 2. *For every $\varepsilon > 0$ there is $m \in \omega$ and a function $\chi: S^{n-1} \rightarrow m$ such that whenever $x, y \in S^{n-1}$ and $\chi(x) = \chi(y)$, then $x \cdot y \geq 1 - \varepsilon$.*

The next lemma is suggested by [2], and the proof presented here is also influenced by [2]. Recall that we have a fixed n -simplex $S = \{x_0, x_1, \dots, x_n\}$.

Lemma 3. *There exists $\varepsilon_0 > 0$ such that whenever $T = \{y_0, y_1, \dots, y_n\} \subseteq \mathbb{R}^n$ is an n -simplex similar to S for which*

$$\begin{aligned} x_i &= y_i && \text{for some } i \leq n, \\ x_i &\neq y_i && \text{for some } i \leq n, \\ \|x_i - y_i\| &< \varepsilon_0 && \text{for all } i \leq n, \end{aligned}$$

then there are (necessarily distinct) $i, j \leq n$ such that $x_i \neq y_i$, $x_j \neq y_j$, and

$$\frac{y_i - x_i}{\|y_i - x_i\|} \cdot \frac{y_j - x_j}{\|y_j - x_j\|} < 1 - \varepsilon_0.$$

Proof. Intending to arrive at a contradiction, assume that there is no such ε_0 . Then we can assume, without loss of generality, that $x_0 = 0$ and for each integer $m \geq 1$ there is an n -simplex $\{y_{0m}, y_{1m}, \dots, y_{nm}\}$ and $r_m > 0$ such that, for all $i, j \leq n$,

$$\begin{aligned} y_{0m} &= 0, \\ y_{1m} &\neq x_1, \\ \|y_{im} - x_i\| &< \frac{1}{m}, \\ \|y_{im} - y_{jm}\| &= r_m \|x_i - x_j\|, \end{aligned}$$

and

$$(y_{im} - x_i) \cdot (y_{jm} - x_j) \geq \|y_{im} - x_i\| \|y_{jm} - x_j\| \left(1 - \frac{1}{m}\right).$$

Moreover, without loss of generality we can assume that $\|y_{1m} - x_1\| \geq \|y_{im} - x_i\|$ for all $i \leq n$.

For each $m \geq 1$, let $d_m = \|y_{1m} - x_1\|$ and $v_{im} = (1/d_m)(y_{im} - x_i)$. Observe that $\|v_{im}\| \leq \|v_{1m}\| = 1$ and $\lim_{m \rightarrow \infty} d_m = 0$. Thus, without loss of generality, we can assume that the limits $v_i = \lim_{m \rightarrow \infty} v_{im}$ exist, and that $v_i = b_i v_1$ where $0 \leq b_i \leq 1$. Obviously, $\|v_1\| = 1$.

For $i, j \leq n$, let $a_i = \|x_i\|$ and $c_{ij} = x_i \cdot x_j$. Clearly, for $m \geq 1$ and $i \neq j$ we have that

$$r_m^2 a_i^2 = \|y_{im}\|^2 = \|x_i + d_m v_{im}\|^2 = a_i^2 + 2d_m x_i \cdot v_{im} + d_m^2 \|v_{im}\|^2 \quad (1)$$

and

$$\begin{aligned} r_m^2 c_{ij} &= y_{im} \cdot y_{jm} = (x_i + d_m v_{im}) \cdot (x_j + d_m v_{jm}) \\ &= c_{ij} + d_m x_i \cdot v_{jm} + d_m x_j \cdot v_{im} + d_m^2 v_{im} \cdot v_{jm}. \end{aligned} \quad (2)$$

From (1) we get

$$r_m^2 a_i^2 a_1^2 = a_1^2 (a_i^2 + 2d_m x_i \cdot v_{im} + d_m^2 \|v_{im}\|^2) = a_i^2 (a_1^2 + 2d_m x_1 \cdot v_{1m} + d_m^2 \|v_{1m}\|^2),$$

from which we get

$$a_1^2 x_i \cdot v_{im} - a_i^2 x_1 \cdot v_{1m} = \frac{a_i^2}{2} d_m \|v_{1m}\|^2 - \frac{a_1^2}{2} d_m \|v_{im}\|^2.$$

Taking the limit as $m \rightarrow \infty$, we see that the right-hand side approaches 0, so we get

$$a_i^2 x_1 \cdot v_1 = a_1^2 x_i \cdot v_i. \quad (3)$$

Similarly, from (1) and (2) we get

$$\begin{aligned} r_m^2 c_{i1} a_1^2 &= c_{i1} (a_1^2 + 2d_m x_1 \cdot v_{1m} + d_m^2 \|v_{1m}\|^2) \\ &= a_1^2 (c_{i1} + d_m x_i \cdot v_{1m} + d_m x_1 \cdot v_{im} + d_m^2 v_{im} \cdot v_{1m}), \end{aligned}$$

from which we get

$$2c_{i1}(x_1 \cdot v_{1m}) - a_1^2(x_i \cdot v_{1m} + x_1 \cdot v_{im}) = a_1^2 d_m v_{im} \cdot v_{1m} - c_{i1} d_m \|v_{1m}\|^2.$$

Taking the limit as $m \rightarrow \infty$, we again see that the right-hand side approaches 0, so we get

$$a_i^2(x_i \cdot v_1 + x_1 \cdot v_i) = 2c_{i1}(x_1 \cdot v_1). \quad (4)$$

Replacing v_i by $b_i v_1$ in (3) and (4) yields respectively

$$a_i^2 x_1 \cdot v_1 = (a_i^2 x_i) \cdot (b_i v_1) \quad (5)$$

and

$$a_i^2 x_i \cdot v_1 = 2c_{i1} x_1 \cdot v_1 - a_i^2 x_1 \cdot b_i v_1. \quad (6)$$

Therefore, plugging (6) into the right-hand side of (5) yields

$$a_i^2 x_1 \cdot v_1 = b_i(2c_{i1} x_1 \cdot v_1 - a_i^2 b_i x_1 \cdot v_1)$$

or, equivalently,

$$(x_1 \cdot v_1)(a_i^2 b_i^2 - 2c_{i1} b_i + a_i^2) = 0.$$

We wish to conclude that $x_1 \cdot v_1 = 0$. If not, then b_i satisfies the equation

$$a_i^2 x^2 - 2c_{i1} x + a_i^2 = 0,$$

so that $4c_{i1}^2 - 4a_i^2 a_i^2 \geq 0$. However, since x_0, x_1, x_i are not collinear, the Cauchy-Schwarz inequality implies $c_{i1}^2 < a_i^2 a_i^2$, a contradiction. Thus, we have $x_1 \cdot v_1 = 0$.

Now from (6) we get that $x_i \cdot v_1 = 0$ for all i . Since $\{x_1, x_2, \dots, x_n\}$ spans $\mathbb{R}^n$, this implies $v_1 = 0$, contradicting that $\|v_1\| = 1$. $\square$

If $X \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we say that x is *algebraic* over X if each coordinate of x is algebraic over the set of all coordinates of elements in X .

Lemma 4. *Let $Y, Z \subseteq \mathbb{R}^n$ be finite sets such that $S \subseteq Z$ and no $x \in Y$ is algebraic over Z . Let $\varepsilon > 0$. Then there is a function $g: Y \cup Z \rightarrow \mathbb{R}^n$ such that whenever $T \subseteq Y \cup Z$ is an n -simplex similar to S , then so is $g[T]$; if $x \in Y$, then $0 < \|x - g(x)\| < \varepsilon$; and if $x \in Z$, then $g(x) = x$.*

Proof. This is just a special case of the following well known and easy consequence of Tarski's elimination of quantifiers for the first-order theory of $\mathbb{R}$ considered as an ordered field: If $m \in \omega$, $F \subseteq \mathbb{R}$ is a real-closed field, $X \subseteq \mathbb{R}^m$ is first-order definable using parameters in F , $\langle a_0, a_1, \dots, a_{m-1} \rangle \in X$, and $\varepsilon > 0$, then there is $\langle b_0, b_1, \dots, b_{m-1} \rangle \in X$ such that, for each $i < d$, $|b_i - a_i| < \varepsilon$ and if $b_i = a_i$, then $a_i \in F$. $\square$

We now finish the proof of the theorem by proving Lemma 1. Let m and χ be as in Lemma 2, using $\varepsilon = \varepsilon_0$ from Lemma 3. By compactness, it suffices to show that for each finite $X \subseteq F_\gamma^n$ there is a function $\varphi_0: X \setminus F_\alpha^n \rightarrow m$ with the following property: whenever $T \subseteq X$ is an n -simplex similar to S such that $T \cap F_\alpha^n \neq \emptyset \neq T \setminus F_\alpha^n$, then there are $x, y \in T \setminus F_\alpha^n$ such that $\varphi_0(x) \neq \varphi_0(y)$.

Consider some finite $X \subseteq F_\gamma^n$ such that $S \subseteq X$, and let $Y = X \setminus F_\alpha^n$ and $Z = X \cap F_\alpha^n$. Let $r > 0$ be small enough so that if $y_1, y_2 \in X$ are distinct and $0 \leq i < j \leq n$, then, $\|y_1 - y_2\| \geq r\|x_i - x_j\|$ and let $\varepsilon = r\varepsilon_0$. Now apply Lemma 4 to get $g: X \rightarrow \mathbb{R}^n$ satisfying the properties specified in that lemma. For each $x \in Y$, let

$$\varphi_0(x) = \chi\left(\frac{x - g(x)}{\|x - g(x)\|}\right).$$

To see that φ_0 works, consider some n -simplex $T \subseteq X$ which is similar to S such that $T \cap Z \neq \emptyset \neq T \cap Y$. Clearly, by Lemma 3, there are $x, y \in T \cap Y$ such that

$$\frac{x - g(x)}{\|x - g(x)\|} \cdot \frac{y - g(y)}{\|y - g(y)\|} < 1 - \varepsilon_0;$$

hence, by Lemma 2, $\varphi_0(x) \neq \varphi_0(y)$.

This completes the proof of Lemma 1, and, consequently, also of the theorem.

References

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