On the Euler Characteristic of Finite Unions of Convex Sets

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Abstract. The Euler characteristic plays an important role in many subjects of discrete and continuous mathematics. For noncompact spaces, its homological definition, being a homotopy invariant, seems not as important as its role for compact spaces. However, its combinatorial definition, as a finitely additive measure, proves to be more applicable in the study of singular spaces such as semialgebraic sets, finitely subanalytic sets, etc. We introduce an interesting integral by means of which the combinatorial Euler characteristic can be defined without the necessity of decomposition and extension as in the traditional treatment for polyhedra and finite unions of compact convex sets. Since finite unions of closed convex sets cannot be obtained by cutting convex sets as in the polyhedral case, a separate treatment of the Euler characteristic for functions generated by indicator functions of closed convex sets and relatively open convex sets is necessary, and this forms the content of this paper.

1. An Euler Integral

Given a function χ on the class of all compact convex sets of a Euclidean space $\mathbb{R}^n$ with constant value 1, Hadwiger [8] uniquely extended χ to the class of finite unions of compact convex sets by finite additivity. Groemer [3], using a similar idea, considered the vector space $V(\mathcal{K})$ of functions generated by indicator functions of compact convex sets of $\mathbb{R}^n$, and uniquely extended χ to a linear functional on $V(\mathcal{K})$. The *indicator function*, or simply indicator, of a subset A of $\mathbb{R}^n$ is the function 1_A such that $1_A(x) = 1$ for $x \in A$ and $1_A(x) = 0$ for $x \notin A$. Groemer's vector space obviously contains Hadwiger's convex ring of finite unions of compact convex sets. Moreover, it has been shown by Groemer that the Minkowski sum, i.e., vector addition, induces uniquely a commutative multiplication on $V(\mathcal{K})$ which plays an important role in combinatorial geometry,

where $\mathcal{K}$ is the class of all compact convex sets in Euclidean space. Both Hadwiger and Groemer used induction on the dimension of spaces by applying projection to derive the Euler characteristic for their objects. More recently, Chen [2] derived the duality between the Gram–Sommerville and Gauss–Bonnet formulas for polyhedra by using the Euler characteristic; and Schanuel [18] gave a classification of polyhedra up to polyhedral isomorphism by two invariants: an Euler characteristic in $\mathbb{Z} \times \mathbb{Z}$, and a “dimension” in order ideals of a certain partially ordered set. For unbounded polyhedra, the first component Euler characteristic is also shown to be well defined in [2] by cutting unbounded relatively open polytopes. In Schanuel’s classification of polyhedra, his Euler characteristic is naturally obtained for arbitrary closed convex sets and relatively open convex sets. The first component of $\mathbb{Z} \times \mathbb{Z}$ is of most interest to us because it is a topological invariant, while the second component is an approximation extension by compact convex sets, see [18]. Morelli [13] independently also obtained the second component Euler characteristic for unbounded polyhedra. The second Euler characteristic can be considered as a special case of the Euler characteristic of the Klee–Rota type [10], [15] for lattices, or, more generally, for partially ordered sets, see also [14]. For other indirectly related work, see the excellent references of [12] and [1] and [7]. We share the ideas, more or less, of Groemer [3], [5], Hadwiger [8], Rota [15], [16], and Schanuel [17], [18], from a different point of view by introducing an integral as follows.

Definition 1.1. Let r and s be two fixed real numbers such that $r + s = 1$. For a real-valued function $f(x)$ on $\mathbb{R}$, set

$$\int f(x) dx = \sum_{x \in \hat{\mathbb{R}}} (f(x) - rf(x^-) - sf(x^+)),$$

where $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, $f(\infty) = 0$, $f(x^-)$ and $f(x^+)$ denote the left and right limits of f near $x \in \hat{\mathbb{R}}$, respectively, $f(\infty^-) = \lim_{x \rightarrow +\infty} f(x)$, and $f(\infty^+) = \lim_{x \rightarrow -\infty} f(x)$. $\int f dx$ is called the Euler integral of f ; and f is called the Euler integrable if $f(x)$ is continuous except at finitely many points, and has both left and right limits at those discontinuous points.

Definition 1.2. Let $v = \{v_1, \dots, v_n\}$ be an ordered basis of a Euclidean n -space $\mathbb{R}^n$. The Euler integral of a function $f(x)$ on $\mathbb{R}^n$ is defined to be the formal iterated integral

$$\begin{aligned} \chi(f, u) &= \int \cdots \int f dv_1 \cdots dv_n \\ &= \int f(T_v) dx_1 \cdots dx_n, \end{aligned}$$

where T_v is the linear transformation of $\mathbb{R}^n$ which maps the standard basis

$e = \{e_1, \dots, e_n\}$ to v , all the components of e_i are zero except 1 at the i th position, $1 \leq i \leq n$. $f(x)$ is called the Euler integrable with respect to the basis v if each of its iterated Euler integrals is Euler integrable; and $f(x)$ is simply called Euler integrable if it is so for all bases.

The idea of the above definition is implicitly used by Hadwiger [8] for $r = 1$ and by Groemer [3] for $r = s = \frac{1}{2}$, but the use of the iterated integral process is due to Schanuel for $r = s = \frac{1}{2}$, via lectures at SUNY—Buffalo, and is actually a special case of his definition of curvature measures. The iterated integral process is also due to Rota [16] for the case $r = 1$.

Theorem 1.3. *For any fixed real numbers r and s such that $r + s = 1$, the set $\mathcal{E}_{r,s}(\mathbb{R}^n)$ of all Euler integrable functions on $\mathbb{R}^n$ with the chosen numbers r and s is a vector space over $\mathbb{R}$.*

Proof. $\mathcal{E}_{r,s}(\mathbb{R}^n)$ is obviously nonempty. For any Euler integrable functions f and g , i.e., all their iterated Euler integrals are continuous about each coordinate except at finitely many points and they have both left and right limits at those discontinuous points, it is obvious that the iterated integrals of $f + g$ and af , for a real number a , are continuous about each coordinate except at finitely many points and they have left and right limits at those discontinuous points. Then $f + g$ and af are also Euler integrable. So $\mathcal{E}_{r,s}(\mathbb{R}^n)$ is a vector space over $\mathbb{R}$. $\square$

It may be tedious to state the so simple result because it is a straightforward consequence of our definition. However, it is this vector property that shows the advantage of our definition in avoiding decomposition and extension as in the traditional treatment of the combinatorial Euler characteristic. It should be noticed that in general $\mathcal{E}_{r,s}(\mathbb{R}^n)$ is not closed under pointwise multiplication of functions.

2. The Euler Characteristic

In this section we show that the indicator functions of (unbounded) polyhedra, closed polyconvex sets, and relatively open polyconvex sets of $\mathbb{R}^n$ are all Euler integrable. There are many other sets of $\mathbb{R}^n$ whose indicator functions are Euler integrable, for example, semialgebraic sets and finitely subanalytic sets, and we treat them elsewhere by different techniques. For this reason, we give a definition of the Euler characteristic for sets of $\mathbb{R}^n$ whose indicator functions are Euler integrable.

Theorem 2.1. *For a subset E of $\mathbb{R}^n$, if its indicator function is Euler integrable, then the Euler integral of its indicator function with respect to any basis is always an integer, and it is independent of r and s such that $r + s = 1$.*

Proof. We apply induction on the dimension n . For $n = 1$, an Euler set E of $\mathbb{R}$ has to be a disjoint union of finitely many open intervals and singleton sets. For an open interval (a, b) and a singleton $\{c\}$, we have

$$\int 1_{(a,b)} dx = (0 - s) + (0 - r) = -1,$$

$$\int 1_{\{c\}} dx = 1.$$

So the Euler integral of 1_E is a sum of some 1 and -1 which is independent of r and s . Suppose it is true for Euler sets in a Euclidean space of dimension $n - 1$. For any basis $v = \{v_1, \dots, v_n\}$ of $\mathbb{R}^n$, consider 1_E as a function f_E on $\mathbb{R}^n$ under the coordinates of v , i.e.,

$$f_E(x_1, \dots, x_n) = 1_E(x_1 v_1 + \dots + x_n v_n).$$

For each x_n , $f_E(\cdot, x_n)$ defines a function on $\mathbb{R}^{n-1}$. Obviously, $f_E(\cdot, x_n)$ is Euler integrable over $\mathbb{R}^{n-1}$. It is Euler integrable especially with respect to the basis $\{v_1, \dots, v_{n-1}\}$. Then

$$\begin{aligned} f(x) &= \int f_E(\cdot, x) dx_1 \cdots dx_{n-1} \\ &= \int 1_E dv_1 \cdots dv_{n-1} \end{aligned}$$

defines a function on $\mathbb{R}$ which is also Euler integrable. By induction hypothesis, $f(x)$ is integer valued, so $f(x)$ can be written as a finite sum

$$f(x) = \sum \alpha_i 1_{(a_i, b_i)} + \sum \beta_j 1_{\{c_j\}},$$

where α_i and β_j are integers and independent of r and s . Thus

$$\int 1_E dv_1 \cdots dv_n = \int f(x) dx$$

is an integer, and it is independent of r and s such that $r + s = 1$. $\square$

Definition 2.2. A subset E of $\mathbb{R}^n$ is called an Euler set if its indicator function 1_E is Euler integrable. If the Euler integrals of 1_E with respect to all bases of $\mathbb{R}^n$ are the same, then the value is called the Euler characteristic of E , and is denoted by $\chi(E)$.

We hope, for any Euler set E , its Euler integral with respect to any basis is the same. Although we cannot prove this for all such sets, we can show it is true for some sets of this kind. As in the structure theorem of closed convex sets stated in Section 3, a closed convex set F has one of the following three forms:

- (1) F is compact,
- (2) $F = K \oplus V^k$, where K is a compact convex set and V^k is a nonzero k -dimensional subspace of $\mathbb{R}^n$, and
- (3) $F = M \oplus W \oplus L$, where W is a vector subspace, M is a topological manifold (with or without boundary) which is contained in the boundary $\partial\pi(F)$ of $\pi(F)$, where π is the canonical projection of $\mathbb{R}^n$ onto $\mathbb{R}^n/W$, and L is a closed half-line.

By this classification we obtain the following theorem.

Theorem 2.3. *Closed convex sets and relatively open convex sets of $\mathbb{R}^n$ are Euler sets. So the vector space spanned by indicator functions of those sets is a subspace of $\mathcal{E}_{r,s}(\mathbb{R}^n)$. Moreover, for a nonempty relatively open convex set U and a nonempty closed convex set F :*

1. $\chi(U) = (-1)^{\dim(U)}$.
2. $\chi(F) = 1$ whenever F is compact.
3. $\chi(F) = (-1)^k$ whenever $F = K \oplus V^k$.
4. $\chi(F) = 0$ whenever $F = M \oplus E \oplus L$.

Proof. As the proof of Theorem 2.1 shows, the Euler integrals of the indicator functions of an open interval and a singleton set of $\mathbb{R}$ are -1 and 1 , respectively. Therefore the Euler integral of an indicator function of any closed interval $[a, b]$ is 1 because $1_{[a,b]} = 1_{(a)} + 1_{(a,b)} + 1_{(b)}$. It can be easily checked that the Euler integral of any unbounded open interval is also -1 . The proof proceeds by induction on the dimension of spaces.

For dimension 1, it is true as stated above. Suppose it is true for dimension $n - 1$. Now, for dimension n , let $v = \{v_1, \dots, v_n\}$ be a basis of $\mathbb{R}^n$, let L_i denote the subspace spanned by v_i , $1 \leq i \leq n$, and let π be the projection of $\mathbb{R}^n$ onto $X = L_2 \oplus \dots \oplus L_n$ whose kernel is L_1 . We first consider a nonempty relatively open convex set U . Note that $\pi(U)$ is also a relative open convex set and, for every point $p \in \pi(U)$, the intersection $(p + L_1) \cap U$ is simultaneously an open interval or a singleton. Then we have

$$\int 1_U dv_1 \cdots dv_n = \delta \int 1_{\pi(U)} dv_2 \cdots dv_n,$$

where $\delta = -1$ if $\dim \pi(U) = \dim(U) - 1$ and $\delta = 1$ if $\dim \pi(U) = \dim(U)$. By induction we have

$$\chi(U) = \int 1_U dv_1 \cdots dv_n = (-1)^{\dim(U)}.$$

Let F be a nonempty closed convex set. If F is compact, then $(p + L_1) \cap F$ is a closed segment or a singleton for all $p \in \pi(F)$, and $\pi(F)$ is a compact convex set in X . So, by the induction hypothesis,

$$\chi(F) = \int 1_F dv_1 \cdots dv_n = 1.$$

If F is not compact, i.e., F is unbounded, we have two cases: $F = K \oplus V^k$ and $F = M \oplus W \oplus L$. For the case $F = K \oplus V^k$, if $v_1 \in V^k$, then $\pi(F) = \pi(K) \oplus V^{k-1}$, where V^{k-1} is a $(k-1)$ -dimensional subspace of X . Thus by induction

$$\begin{aligned} \int 1_F dv_1 \cdots dv_n &= - \int 1_{\pi(F)} dv_2 \cdots dv_n \\ &= -(-1)^{k-1} \\ &= (-1)^k. \end{aligned}$$

If $v_1 \notin V^k$, we have $\pi(F) = K_1 \oplus V^k$, where K_1 is a compact convex set of X , V^k_1 is a nonzero k -dimensional subspace of X , and $\pi^{-1}(p) \cap F$ is a bounded closed segment for all $p \in \pi(F)$. Then by induction

$$\int 1_F dv_1 \cdots dv_n = \int 1_{\pi(F)} dv_2 \cdots dv_n = (-1)^k.$$

For the case $F = M \oplus W \oplus L$, if $L_1 \subset W$, then $\pi(F) = M' \oplus W' \oplus L'$, where M' is a topological manifold with or without boundary, W' is a vector subspace of W such that $\dim(W') = \dim(W) - 1$, and L' is a closed half-line, thus by induction we have

$$\int 1_F dv_1 \cdots dv_n = - \int 1_{\pi(F)} dv_2 \cdots dv_n = 0.$$

If there exists a closed half-line $L'_1 \subset L_1$ such that $F = M_1 \oplus W \oplus L'_1$, then, for each $p \in \pi(F)$, $(p + L_1) \cap F$ is a closed half-line. Thus $\chi(F) = \int 1_F dv_1 \cdots dv_n = 0$. Otherwise, we claim that $\pi(F)$ is a closed unbounded convex subset of X , the fiber $\pi^{-1}(p) \cap F = (p + L_1) \cap F$ is a bounded closed segment for each $p \in \pi(F)$, and $\pi(F) = M' \oplus W' \oplus L'$ has the form (3).

The convexity of $\pi(F)$ is obvious. For boundedness of $\pi^{-1}(p) \cap F$, suppose there is one point $q \in \pi(F)$ such that the closed set $\pi^{-1}(q) \cap F$ is unbounded, that is, it is a closed half-line, say $\{q + tv_1 | t \geq t_0\}$ for some $t_0 \in \mathbb{R}$. Then, for any $p \in \pi(F)$, $p + s_0 v_1$ belongs to $\pi^{-1}(p) \cap F$ for some $s_0 \in \mathbb{R}$, here s_0 is chosen to be smallest. Thus the segment $\{s(p + s_0 v_1) + (1-s)(q + tv_1) | 0 \leq s \leq 1\}$ from $p + s_0 v_1$ to $q + tv_1$ is contained in F for all $t \geq t_0$. Let t tend to ∞ , then the segments tend to the closed half-line $\{p + tv_1 | t \geq s_0\}$ which should be $\pi^{-1}(p) \cap F$. This becomes the former case. Thus $\pi^{-1}(p) \cap F$ is a bounded segment for all $p \in \pi(F)$.

For closedness of $\pi(F)$, let $\{p_k\}$ be a sequence in $\pi(F)$ such that p_k converges to p . Let $\{q_k \in \pi^{-1}(p_k)\}$ be a corresponding sequence in F . It can be checked that $\{q_k\}$ is bounded. Then $\{q_k\}$ has a subsequence $\{q_{k_j}\}$ which converges to q . Thus $p_{k_j} = \pi(q_{k_j})$ converges to $\pi(q) = p$ as k_j tends to ∞ . So $\pi(F)$ is closed. Note that $\pi(F)$ is also unbounded, otherwise $\pi(F)$ will be compact, and this implies that F is compact, which is a contradiction. Since $\pi(F)$ contains the half-line $\pi(L)$, but not the whole line, then $\pi(F) = M' \oplus W' \oplus L'$ has the form (3). Therefore by induction

$$\int 1_F dv_1 \cdots dv_n = \int 1_{\pi(F)} dv_2 \cdots dv_n = 0. \quad \square$$

Let the vector space spanned by the indicator functions of closed convex sets and relatively open convex sets of $\mathbb{R}^n$ be denoted by

$$V(\mathcal{F} \cup \mathcal{O}) = \left\{ \sum_i a_i 1_{F_i} + \sum_j b_j 1_{U_j} \right\},$$

where $a_i, b_j \in \mathbb{R}$, F_i is a closed convex set, U_j is a relatively open convex set, and $\mathcal{F}$ and $\mathcal{O}$ denote the class of closed convex sets and the class of relatively open convex sets, respectively. If we identify a set of $\mathbb{R}^n$ with its indicator function, then $V(\mathcal{F} \cup \mathcal{O})$ certainly contains the sets which are finite unions of closed convex sets and the sets which are finite unions of relatively open convex sets because, for any subsets E_i ($i = 1, \dots, m$),

$$1_{E_1 \cup \dots \cup E_m} = \sum_{k=1}^m (-1)^{k-1} \sum_{i_1 < \dots < i_k} 1_{E_{i_1} \cap \dots \cap E_{i_k}}.$$

For example, any polyhedron is in it, and we immediately have Euler's formula: for a nonempty convex polytope P , let α_i be the number of i -faces of P , then $\sum_i (-1)^i \alpha_i = 1$. Furthermore, $V(\mathcal{F} \cup \mathcal{O})$ contains any convex surface ∂F of a closed convex set F and any polyhedron on ∂F since $\partial F = F - \text{Int}_r(F)$, where $\text{Int}_r(F)$ is the relative interior of F . A *convex surface polyhedron* on ∂F is a finite union of relatively open surface polytopes on ∂F . A *relatively open surface polytope* on ∂F is a nonempty intersection of ∂F and a relatively open convex polyhedral cone with apex $p \in \text{Int}_r(F)$ such that if two open rays $p + L_1$ and $p + L_2$ in the cone emanating from the point p meet ∂F , then any open ray in $p + L_1 \oplus L_2$ emanating from p meets ∂F . Indeed, let C be a closed convex polyhedral cone in the flat that F spans with apex $p \in \text{Int}_r(F)$. Note that the boundary ∂C is in $V(\mathcal{F} \cup \mathcal{O})$. If ∂C is written as $\partial C = C_1 \cup \dots \cup C_k$, where $C_1, \dots, C_k$ are facets of C , then $(\partial C) \cap F = (C_1 \cap F) \cup \dots \cup (C_k \cap F)$ is in $V(\mathcal{F} \cup \mathcal{O})$. Thus

$$F \cap C = F \cap \partial C - \text{Int}_r(F \cap C)$$

is in $V(\mathcal{F} \cup \mathcal{O})$, and it is a relatively open surface polytope on ∂F . So we have Euler's formula on convex surfaces.

Corollary 2.4. *Let P be a nonempty closed bounded polytope on a convex surface, and let α_i be the number of i -faces of P , then $\sum_i (-1)^i \alpha_i = 1$.*

The above corollary is a generalization of the spherical Euler relation of Hadwiger and Mani [9], it is also a generalization of Groemer's Euler relation for polytopes on convex surfaces of compact convex sets [4]. Our convex surface can be the boundary of any closed convex set. It should be noted that our unified treatment of Euler characteristic on $V(\mathcal{F} \cup \mathcal{O})$ includes all the Euler characteristics studied in [1]–[6], [8]–[10], and [12]–[18] and more geometric objects. It should be noticed that $V(\mathcal{F} \cup \mathcal{O})$ is not a ring under the pointwise multiplication because the intersection of two convex curves can be a union of infinitely many singletons.

3. The Structure of Closed Convex Sets

The set of finite unions of closed convex sets of a finite-dimensional vector space V over an ordered field is a distributive lattice under the intersection and union of sets. Similarly, the set of finite unions of relatively open convex sets is also a distributive lattice. The elements which are finite unions of compact convex sets in the first lattice are sometimes called polyconvex sets [16]. For this reason, we give the following definition.

Definition 3.1. Let V be a finite-dimensional vector space over $\mathbb{R}$. A subset E of V is called a polyconvex set if it is a finite union of convex sets of V . E is called a closed (relatively open) polyconvex set if the convex sets in the union are closed (relatively open).

Closed (relatively open) polyconvex sets are important geometric objects. In order to get a category for such objects, we need the notion of polyconvex maps.

Definition 3.2. Let A and B be closed (relatively open) polyconvex sets of finite-dimensional vector spaces V and W , respectively. A map f from A to B is called a closed (relatively open) polyconvex map if its graph

$$G(f) = \{(x, f(x)) | x \in A\}$$

is a closed (relatively open) polyconvex set of $V \times W$.

It can be seen that relatively open polyconvex maps need not be continuous. For example, for the relatively open polyconvex sets $A = [0, 2] \cup \{3\}$ and $B = [-1, 0] \cup [1, 2]$, define a map $f(x) = x - 1$ for $0 \leq x \leq 1$, $f(x) = x$ for $1 < x \leq 2$, and $f(3) = 1$. It can be checked that the graph of f is a finite union of

relatively open convex sets, so f is a relatively open polyconvex map, but it is not continuous.

Proposition 3.3. *The closed (relatively open) polyconvex sets and closed (relatively open) polyconvex maps form a category which is called the category of closed (relatively open) polyconvex sets.*

Proof. We just need to show that the composition of polyconvex maps is polyconvex. Let f be a closed (relatively open) polyconvex map from A to B and let g be a closed (relatively open) polyconvex map from B to C . Their graphs can be written as

$$G(f) = \bigcup_{i=1}^m E_i, \quad G(g) = \bigcup_{j=1}^n F_j,$$

where E_i and F_j are closed (relatively open) convex sets. Let the projection images of E_i and F_j into their first and second components be denoted by A_i, B_i, B'_j , and C_j , respectively, then A_i, B_i, B'_j , and C_j are closed (relatively open) convex sets. The convexity of E_i and F_j implies that f is an affine map from A_i to B_i , and g is an affine map from B'_j to C_j . Then $A_{ij} = (f|_{A_i})^{-1}(B_i \cap B'_j) = A_i \cap f^{-1}(B_i \cap B'_j)$ is convex. Thus the composition map gf is affine linear on each A_{ij} , $i = 1, \dots, m$, $j = 1, \dots, n$, and its graph

$$G(gf) = \bigcup_{i=1}^m \bigcup_{j=1}^n G_{ij}$$

is a closed (relatively open) polyconvex set because each $G_{ij} = \{(x, f(x)) | x \in A_{ij}\}$ is a closed (relatively open) convex set. $\square$

Proposition 3.4. *Let A and B be closed (relatively open) polyconvex sets. If there is a closed (relatively open) polyconvex isomorphism from A to B , then $\chi(A) = \chi(B)$.*

Proof. Let the graph of the isomorphism f from A to B be written as

$$G(f) = F_1 \cup \dots \cup F_n,$$

where each F_i is a closed (relatively open) convex set. Let A_i and B_i be the images of projections of $G(f)$ onto its first and second components. Then $A = A_1 \cup \dots \cup A_n$ and $B = B_1 \cup \dots \cup B_n$; A_i and B_i are closed (relatively open) convex sets. Thus for any $i_1 < \dots < i_k$, $A_{i_1} \cap \dots \cap A_{i_k}$ is nonempty if and only if

$B_{i_1} \cap \cdots \cap B_{i_k}$ is nonempty. Therefore

$$\begin{aligned}\chi(A) &= \sum_{k=1}^n (-1)^{k-1} \sum_{i_1 < \cdots < i_k} \chi(A_{i_1} \cap \cdots \cap A_{i_k}) \\ &= \sum_{k=1}^n (-1)^{k-1} \sum_{i_1 < \cdots < i_k} \chi(B_{i_1} \cap \cdots \cap B_{i_k}) = \chi(B)\end{aligned}$$

as desired. $\square$

By definition of the category of closed (relatively open) polyconvex sets, it will be noticed that this category has more objects than the maps. This is probably why the algebraic theory of convex sets is not as developed as other geometric objects whose categories have more maps. We prove the structure theorem which has been used in Section 2 for deriving the Euler characteristic of closed and relatively open convex sets.

Proposition 3.5 (Structure Theorem). *Let F be a closed convex set in a finite-dimensional vector space V . Then F has one of the following three forms:*

1. F is a compact convex set.
2. $F = K \oplus V^k$, where K is a compact convex set and V^k is a nonzero k -dimensional vector subspace of V .
3. $F = M \oplus W \oplus L$, where W is a vector subspace of V , M is a topological manifold (with or without boundary) which is contained in the boundary $\partial\pi(F)$ of $\pi(F)$, where π is the canonical projection from V to V/W , and L is a closed half-line.

Proof. Let us take an arbitrary point p of F and consider the set

$$C(p) = \{v \in V \mid p + tv \in F, \forall t \geq 0\}.$$

Then $C(p)$ is a closed convex cone in V with the apex at origin o . If $C(p)$ only contains the origin o , then F is bounded, so F is a compact convex set, which is the first case. Let W be the largest vector subspace of V which is contained in $C(p)$. If W is not the trivial space $\{o\}$, consider the canonical projection π of F onto a subspace W' , which is a direct complement of W in V . Let K be the image of F under the projection π in W' . Then $F = K \oplus W$, and K is a closed convex subset of W' . If K is compact, then F is the second case. Otherwise, K is unbounded, and, in this case, K does not contain any straight line. Note that there exists a nonzero vector v such that $q + tv \in K$ for all $q \in K$ and for all $t \geq 0$. Let $L = \{tv \mid t \in \mathbb{R}\}$ and let $L_q = q + L$ for $q \in K$. Let $M = \bigcup_{q \in K} (\partial L_q)$, i.e., M is the subset of ∂K whose points are exposed in the opposite direction of v . It can be shown that M is a topological manifold with or without boundary. In fact, let H_v be any subspace which is a direct complement of $\mathbb{R}v$ in the vector space $\langle K \rangle$ which is spanned by K , where v is any

nonzero vector such that $q + tv \in K$ for all $q \in K$ and $t \geq 0$, and let π_v be the canonical projection from $\langle K \rangle$ to H_v . Then π_v is a homeomorphism from M to $\pi_v(K)$, and $\pi_v(K)$ is a convex set with or without boundary in H_v . By our construction we have $F = M \oplus W \oplus L$, which is the desired case three. $\square$

Theorem 3.5 can be found in [11]. We do not know of any other references to the structure theorem.

4. Another Euler Characteristic

As is mentioned in Section 1, the second component of Schanuel's Euler characteristic for a closed (relatively open) polyconvex set E of $\mathbb{R}^n$ can be defined to be the following limit:

$$\bar{\chi}(E) = \lim_{t \rightarrow +\infty} \chi(E \cap I_t^n),$$

where I_t is the closed interval $[-t, t]$ with $t > 0$, I_t^n is the Cartesian product of n copies of I_t , and can be replaced by the closed ball

$$B_t = \{x \in \mathbb{R}^n \mid \|x\| \leq t\}$$

of radius t . Since the first component Euler characteristic χ satisfies the inclusion-exclusion principle, so does $\bar{\chi}$. Then $\bar{\chi}$ can be uniquely extended to a linear functional on $V(\mathcal{F})$ and $V(\mathcal{O})$ by Groemer's extension theorem [5]. We can also define an integral so that $\bar{\chi}$ is well defined on $V(\mathcal{F} \cup \mathcal{O})$ without extension, i.e., for a function $f(x)$ on $\mathbb{R}$ and real numbers r, s such that $r + s = 1$, define

$$\bar{\chi}(f) = \int f(x) dx = \sum_{x \in \mathbb{R}} (f(x) - rf(x^-) - sf(x^+)),$$

where $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, $f(-\infty) = \lim_{x \rightarrow -\infty} f(x)$, $f(+\infty) = \lim_{x \rightarrow +\infty} f(x)$, $f(-\infty^-) = 0$, $f(-\infty^+) = f(-\infty)$, $f(+\infty^-) = f(+\infty)$, and $f(+\infty^+) = 0$.

For a function $f(x)$ on $\mathbb{R}^n$ and an ordered basis $v = \{v_1, \dots, v_n\}$, $\bar{\chi}(f, v)$ is defined to be the iterated integral

$$\bar{\chi}(f, v) = \int f dv_1 \cdots dv_n.$$

$\bar{\chi}(f, v)$ is called the $\bar{\chi}$ -integral of $f(x)$ with respect to the basis v . f is called $\bar{\chi}$ -integrable if each of the iterated integrals of f with respect to any basis of $\mathbb{R}^n$ is continuous about each coordinate except at finitely many points, and has left and right limits at those discontinuous points. We immediately have the following theorems.

Theorem 4.1. For any fixed real numbers r and s such that $r + s = 1$, each member of the vector space $\mathcal{E}_{r,s}(\mathbb{R}^n)$ is also $\bar{\chi}$ -integrable and vice versa.

Theorem 4.2. For each set E of $\mathbb{R}^n$, if its indicator function 1_E is $\bar{\chi}$ -integrable, then $\bar{\chi}(1_E, v)$ is an integer for any basis, and is independent of r and s such that $r + s = 1$.

A similar structure theorem can occur for relatively open convex sets of $\mathbb{R}^n$. In fact, a relatively open convex set U has one of the following three types:

- (1) U is bounded,
- (2) $U = D \oplus V^k$, where D is a bounded relatively open convex set and V^k is a nonzero k -dimensional subspace of $\mathbb{R}^n$, and
- (3) $U = M \oplus W \oplus L$, where M is a topological manifold (without boundary) which is contained in ∂U , W is a subspace of $\mathbb{R}^n$, and L is an open half-line.

This classification induces the following theorem.

Theorem 4.3. The indicator functions of closed convex sets and relatively open convex sets of $\mathbb{R}^n$ are $\bar{\chi}$ -integrable. Moreover, for a nonempty closed convex set F and a nonempty relatively open convex set U :

1. $\bar{\chi}(F) = 1$.
2. $\bar{\chi}(U) = (-1)^{\dim(U)}$ whenever U is bounded.
3. $\bar{\chi}(U) = (-1)^{\dim(D)}$ whenever $U = D \oplus V^k$.
4. $\bar{\chi}(U) = 0$ whenever $U = M \oplus W \oplus L$.

The proofs of the above three theorems are omitted because they are essentially the same as the proofs of Theorems 1.3, 2.1, and 2.3. The interested reader can provide similar proofs without difficulty. It should be emphasized that $\bar{\chi}$ is not a topological invariant.

According to the definitions of the Euler integral and $\bar{\chi}$ -integral, the two integrals are the same for functions with bounded support. Next we give an example by showing that the compact smooth surface S_g of genus g with a standard shape in $\mathbb{R}^3$ is an Euler set. The standard means excluding some pathological cases.

We choose a special coordinate so that the handles of the surface face north (z -axis) and are parallel to the x -axis. When the indicator function 1_{S_g} is integrated with respect to z in the direction $\vec{k}$, then $\int 1_{S_g} dz$ becomes a function h on $\mathbb{R}^2$ whose support is a bounded closed region in $\mathbb{R}^2$ with g holes. The function h has a value of 2 in the interior of the closed region and a value of 1 on its boundary. When h is integrated with respect to y in the direction $\vec{j}$, the result is a function $f(x)$ on $\mathbb{R}$ given by

$$f(x) = \begin{cases} 1, & x = a \text{ or } b, \\ -1, & x = a_i \text{ or } b_i, \quad 1 \leq i \leq g, \\ 0, & \text{otherwise,} \end{cases}$$

where $a < a_1 < b_1 < a_2 < b_2 < \dots < a_g < b_g < b$, a_i and b_i correspond to the image of the boundary of the i th genus on $\mathbb{R}$. So

$$\chi(S_g) = 1 + \sum_{i=1}^g (-1 - 1) + 1 = 2 - 2g,$$

which is the ordinary Euler characteristic of S_g . The Euler integral of 1_{S_g} can be taken with respect to other coordinates, and after a slightly more complicate calculation the same value $2 - 2g$ is obtained. There are many other Euler sets in $\mathbb{R}^n$, but a general characteristic description is not yet clear. The following Proposition 4.4 is due to a conversation with Dan Klain.

Proposition 4.4. *For a function f on $\mathbb{R}^n$, if $\chi(f, v) = \chi(f, v_\sigma)$ for all bases $v = \{v_1, \dots, v_n\}$ of $\mathbb{R}^n$ and all permutations σ of $\{1, 2, \dots, n\}$, where $v_\sigma = \{v_{\sigma(1)}, \dots, v_{\sigma(n)}\}$, then $\chi(f, v)$ is independent of v .*

Proof. Fix a basis u , any basis v can be obtained by a nonsingular n by n matrix A by $v = uA$. Note that A is a product of some (column) elementary matrices. Then $\chi(f, v) = \chi(f, vA)$ for elementary matrices A and all bases v is equivalent to $\chi(f, v)$ being independent of v . It is trivial to check that $\chi(f, v) = \chi(f, vA)$ by definition whenever vA is $\{v_1, \dots, av_i, \dots, v_n\}$ or $\{v_1, \dots, v_i, \dots, v_j + av_i, \dots, v_n\}$ for a nonzero number a . Thus the proposition follows immediately. $\square$

In order to get a useful sufficient condition we need the notion of coordinately cellular sets which are inductively defined in the following.

Definition 4.5.

- (1) Every open interval or a singleton in a one-dimensional subspace L is a cell with respect to any basis of L .
- (2) Let $v = \{v_1, \dots, v_n\}$ be a basis of $\mathbb{R}^n$. Let H_i be the subspace spanned by $\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$ and let p_i be the canonical projection from $\mathbb{R}^n$ to H_i ($1 \leq i \leq n$). A subset C of $\mathbb{R}^n$ is a cell with respect to v if $p_i(C)$ is a cell with respect to the basis $\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$ of H_i and $p_i^{-1}(x) \cap C$ is simultaneously an open interval or a singleton for each $x \in p_i(C)$.
- (3) A subset E of $\mathbb{R}^n$ is called coordinately cellular if, for each basis v of $\mathbb{R}^n$, E can be decomposed into a disjoint union of cells with respect to v .

Theorem 4.6. *Every indicator function of a coordinately cellular set of $\mathbb{R}^n$ is Euler integrable, and its Euler integral is independent of the choice of bases.*

Proof. It is obvious that every coordinately cellular set E is Euler integrable. For each basis $v = \{v_1, \dots, v_n\}$ of $\mathbb{R}^n$, by definition, a cell structure $\{C_i\}$ of E with respect to v is also a cell structure with respect to the basis

$$v_\sigma = \{v_{\sigma(1)}, \dots, v_{\sigma(n)}\}$$

for any permutation σ of $\{1, 2, \dots, n\}$. Thus

$$\chi(E, v) = \sum_i (-1)^{\dim(C_i)} = \chi(E, v_\sigma).$$

The conclusion follows from Proposition 4.4. □

5. Remarks

The usual Euler integral can be referred to as the χ -integral. It is natural to ask whether the Euler integrals can be carried out on some more general spaces such as compact smooth manifolds by transversal foliations, Lie groups by independent left or right invariant vector fields, and projective spaces by projective lines. It will be very interesting and important to characterize the functions of $\mathcal{E}_{r,s}(\mathbb{R}^n)$ by some equivalent conditions, or even by some sufficient conditions. A possible condition for the characterization of Euler sets is asked in the following question.

Question 5.1. Is it true that a subset of $\mathbb{R}^n$ is an Euler set if and only if it is a coordinately cellular set?

It can be proved that closed convex sets, relatively open convex sets, semialgebraic sets, and finitely subanalytic sets are all coordinately cellular. By the calculation of Euler integrals of some simple functions, it will be noticed that the basis chosen is not essential. This suggests the following weak version of Question 5.1.

Question 5.2. If E is an Euler set of $\mathbb{R}^n$, is it true that the Euler integral of its indicator function 1_E with respect to any basis has same value?

Since curvature measures for finite unions of compact convex sets are constructed by the Euler characteristic, similar curvature measures for members of $V(\mathcal{F} \cup \mathcal{O})$ with compact support can be constructed by our Euler characteristic. Moreover, the Minkowski addition and related algebraic theory about $V(\mathcal{F})$ and $V(\mathcal{O})$ can also be obtained by the Euler characteristic.

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