Hamiltonian Simple Polytopes

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Abstract. We show that for every $n > n_0(d)$ (n even if d is odd) there exists a simple d -polytope with n vertices, whose graph admits a Hamiltonian circuit. The result sharpens an earlier lower bound due to Victor Klee.

1. Introduction

Let P be a simple d -polytope, let $h(P)$ be the length of the longest edge-cycle in P , and let $f_0(P)$ be the number of vertices in P . Let S_d be the class of all simple d -polytopes.

In his study of *paths on polyhedra* [5] Klee defines the following function,

$$K(n, d) = \max\{h(P) \mid P \in S_d; f_0(P) \leq n\},$$

and proves that $K(n, d) \geq (d - 1) \lfloor (n - 2)/(d - 1) \rfloor + 2$. That is, among all simple d -polytopes with at most n vertices a polytope can be found in which the longest edge-cycle has at least $(d - 1) \lfloor (n - 2)/(d - 1) \rfloor + 2$ edges.

We show that for large enough n ($n \geq n_0(d)$)¹ Hamiltonian simple d -polytopes with n vertices, i.e., simple polytopes whose graphs have Hamiltonian circuits, can be constructed. Our result is significant for three reasons:

1. The result closes the gap in Klee's lower bound mentioned above, by showing that, for large n , $K(n, d) = n$.
2. Equally important is the corollary of our result about the existence of simple

¹ Since, by definition, d edges are incident at every vertex of a simple d -polytope, a simple d -polytope with n vertices has $nd/2$ edges. Therefore if d is odd, n cannot be odd. The above restriction is implicit in the whole discussion.

d -polytopes with a given number of vertices. A number n is said to be d -realizable if a simple d -polytope with n vertices exists.

The celebrated Lower Bound Theorem [1] implies that no n in the range $d + 1 < n < 2d$ is d -realizable. For $d \leq 9$, [7] lists all n that are not d -realizable; this list shows that n that are not d -realizable occur beyond $2d$ as well. McMullen's conditions provide other restrictions on the d -realizable n [3].

By constructing simple d -polytopes with n vertices for all $n \geq n_0(d)$ our result shows that for each d there are only finitely many n that are not d -realizable.

3. Whether all *simple* polytopes of dimension higher than three have Hamiltonian circuits is still an open problem² [1].

Hamiltonian circuits in polytopes have been a subject of continuing investigation beginning with Hamilton's observations about circuits in the dodecahedron. The existence of Hamiltonian circuits in polytopes is intimately related to the four-color problem [2]. In contrast to Kirkman's non-Hamiltonian 3-polytopes and Tutte's non-Hamiltonian simple 3-polytopes, the following classes of 3-polytopes are known to have Hamiltonian circuits [4], [6]: 3-polytopes with 4-connected graphs; simple 3-polytopes with at most 36 vertices; simple 3-polytopes with at most two of three categories of faces: triangles, quadrilaterals, and hexagons; and simplicial 3-polytopes with maximum vertex degree six. Reference [2] contains other results and conjectures about Hamiltonian circuits in various classes of polytopes.

2. Result

Theorem 1. *For each $d \geq 2$ a constant $n_0(d)$ exists such that for all $n \geq n_0(d)$ a Hamiltonian simple d -polytope with n vertices can be constructed.*

Proof. Observe that if P is any Hamiltonian d -polytope, then a prism with basis P is a Hamiltonian $(d + 1)$ -polytope. The prism is the convex hull of P and its translate P' . If v is a vertex of P , then let v' be its neighbor in P' . If $v_0, v_1, \dots, v_m, v_0$ is the Hamiltonian circuit in P , then $v_0, v_1, \dots, v_m, v'_m, v'_{m-1}, \dots, v'_0, v_0$ is a Hamiltonian circuit in the prism.

Also observe that if a simple d -polytope P has a Hamiltonian circuit, then the d -polytope P'' obtained by truncating a vertex v_i of P ³ also has a Hamiltonian circuit. Truncating the vertex v_i of a simple d -polytope P creates a new facet F

² Non-Hamiltonian simple 3-polytopes were discovered by Tutte [8]; non-Hamiltonian polytopes are known to exist in every dimension higher than two [2].

³ Let H be a hyperplane that does not contain any vertex of P . Further let v_i lie in the closed half-space H^+ of H and the remaining vertices of P in the closed half-space H^- . If P'' is the d -polytope obtained by truncating the vertex v_i in P , then $P'' = P \cap H$ is a new facet in P'' .

which is a $(d - 1)$ -simplex. Each vertex of F comes from an edge incident on v_i in P . Let $v_0, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_n, v_0$ be a Hamiltonian circuit in P . In this circuit, replace v_i by a Hamiltonian path in F , between the vertices that are adjacent to v_{i-1} and v_{i+1} , respectively. The new circuit is Hamiltonian in P'' .

We proceed by induction on d and assuming the theorem for $d = k$ prove it for $d = k + 1$.

1. Let $k + 1$ be an odd number. Set $n_0(k + 1) = 2n_0(k)$. Let $n \geq n_0(k + 1)$ be an even number (see footnote 1). Then $n/2 \geq n_0(k)$ and hence we have a Hamiltonian simple k -polytope, say P , with $n/2$ vertices (inductive hypothesis). A prism with basis P has n vertices and is a Hamiltonian simple $(k + 1)$ -polytope.
2. Next assume $k + 1$ is even. Since k is odd, $n_0(k)$ is even. Inductively we have a Hamiltonian simple k -polytope with $2m$ vertices for all $2m \geq n_0(k)$. Thus we can construct Hamiltonian simple $(k + 1)$ -polytopes with $4m$ vertices for all $4m \geq 2n_0(k)$ (by taking prisms).

Observe that truncating a vertex of a simple $(k + 1)$ -polytope increases the number of vertices by k . Further since k is odd, either $k \equiv 1 \pmod{4}$ (i.e., $k = 4r + 1$) or $k \equiv 3 \pmod{4}$ (i.e., $k = 4r + 3$). If $k \equiv 1 \pmod{4}$, then $2k \equiv 2 \pmod{4}$ and $3k \equiv 3 \pmod{4}$. On the other hand if $k \equiv 3 \pmod{4}$, then $2k \equiv 2 \pmod{4}$ and $3k \equiv 1 \pmod{4}$.

Now we show that for all $n \geq 2n_0(k) + 4k$ we can construct a Hamiltonian simple $(k + 1)$ -polytope with n vertices.

Let $[n - (2n_0(k) + 4k)] \equiv q \pmod{k}$. To construct a Hamiltonian simple $(k + 1)$ -polytope with n vertices, it is sufficient to construct a Hamiltonian simple $(k + 1)$ -polytope P with $2n_0(k) + 4k + q$ vertices; we could then start with P and truncate vertices $(n - (2n_0(k) + 4k + q))/k$ times to obtain a Hamiltonian simple $(k + 1)$ -polytope with n vertices (each truncation preserves Hamiltonicity and increases the vertex-count by k).

Assume $k \equiv 1 \pmod{4}$. Let $2n_0(k) + 4k + q - k \equiv r \pmod{4}$. If $r = 0$, then starting with a Hamiltonian simple $(k + 1)$ -polytope with $2n_0(k) + 3k + q$ vertices and truncating a vertex we get a Hamiltonian simple polytope with $2n_0(k) + 4k + q$ vertices. If $r = 1$ we start with a Hamiltonian simple $(k + 1)$ -polytope with $2n_0(k) + 2k + q$ vertices and truncate two vertices to get a Hamiltonian simple $(k + 1)$ -polytope with $2n_0(k) + 4k + q$ vertices. If $r = 2$ we start with a Hamiltonian simple $(k + 1)$ -polytope with $2n_0(k) + k + q$ vertices and truncate three vertices. Finally if $r = 3$ we start with a Hamiltonian simple $(k + 1)$ -polytope with $2n_0(k) + q$ vertices and truncate four vertices to obtain a Hamiltonian simple $(k + 1)$ -polytope with $2n_0(k) + 4k + q$ vertices.

The arguments are similar if $k \equiv 3 \pmod{4}$ except that if $2n_0(k) + 4k + q - k \equiv r \pmod{4}$, for $r = 0, 1, 2, 3$ we start with Hamiltonian simple $(k + 1)$ -polytope with $2n_0(k) + 3k + q$, $2n_0(k) + q$, $2n_0(k) + k + q$, $2n_0(k) + 2k + q$ vertices, respectively, and truncating one, four, three, or two times respectively.

The above argument shows that a Hamiltonian simple $(k + 1)$ -polytope with n vertices can be constructed for all $n \geq 2n_0(k) + 4k$. $\square$

In the above construction

$$n_0(k+1) = \begin{cases} 2n_0(k) & \text{if } k+1 \text{ is odd,} \\ 2n_0(k) + 4k & \text{if } k+1 \text{ is even.} \end{cases}$$

Solving the recurrence equation with $n_0(1) = 2$ we see that $n_0(k) = O(k2^k)$.

3. Remark

It is not known if a Hamiltonian simple d -polytope with n vertices can be found for every n that is d -realizable.

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