Facet-Reducing Cuts of a Convex Polytope*

N. Prabhu*

Grissom Hall, Purdue University,
 West Lafayette, IN 47907, USA
 prabhu@ecn.purdue.edu

Abstract. We construct, for every $d \geq 4$, a d -polytope $P \subset \mathbb{R}^d$, containing two vertices v and w such that no hyperplane of $\mathbb{R}^d$ containing v and w has more than one facet of P in either of its closed half-spaces. The result shows that facet-reducing cuts containing a prespecified pair of vertices of a polytope, do not exist in general.

1. Introduction

An important class of problems about convex polytopes concerns studying the intersection of a hyperplane of $\mathbb{R}^d$ with the boundary complex of a d -polytope $P \subset \mathbb{R}^d$. A diverse body of results about this class of problems has been reported in the literature. For example, Bezdek *et al.* [1] show that a hyperplane can intersect the relative interiors of all the j -faces of a d -polytope only if $\lfloor d/2 \rfloor \leq j$. (See [2] for a generalization: *if a k -flat intersects the relative interiors of all the j -faces of a d -polytope, then $k \geq 2(d - j)$.*) O'Neil studied the popular problem of determining the *maximum number of edges of a d -cube that a hyperplane of $\mathbb{R}^d$ can cut* and proved a lower bound of $(d - \lfloor d/2 \rfloor) \binom{d}{\lfloor d/2 \rfloor}$ [3]; (see [4] for follow-up work on this problem). We make a special mention of Khovanskii's results [5]. He showed that if a hyperplane H intersects a d -polytope P (and H^+ and H^- are the two closed half-spaces of H), then two faces of P , $F_1 \in H^+$ and $F_2 \in H^-$, exist such that $\dim(F_1) + \dim(F_2) \geq d - 1$. For other results see [6]–[11].

In this paper we study the existence of *facet-reducing cuts* of a polytope. A hyperplane H that intersects the interior of a d -polytope P is called a *facet-reducing*

* This research was supported in part by an NSF Research Initiation Award.

cut if at least one of the two d -subpolytopes $P_1 = H^+ \cap P$ and $P_2 = H^- \cap P$ has fewer facets than P . It can be shown that P_1 (resp. P_2) has fewer facets than P if and only if H^- (resp. H^+) contains at least two facets of P ; see Theorem 11.1 in [12]. Facet-reducing cuts are useful in inductively reducing a d -polytope with $n > d + 1$ facets to a d -simplex.

As one would suspect, it is easy to show that every d -polytope (except a d -simplex) does admit a facet-reducing cut. However, when we seek facet-reducing cuts containing a subset of vertices of the polytope (as desired in [13]–[15]) we get a rather surprising result. The problem and the main results are stated below.

Let $k(d)$ be the largest integer such that *for every subset of vertices W of any d -polytope P with $|W| \leq k$, a facet-reducing cut of P containing W exists*. We determine $k(d)$ for all d .

Clearly, $k(d) \leq d$ for all d . In fact, it can easily be shown that $k(d) < d$ by observing that the hyperplane $x_1 + \cdots + x_d = 2$ intersects the relative interiors of all the facets of the d -cube $0 \leq x_i \leq 1$, $1 \leq i \leq d$, and contains $O(d^2)$ vertices. It is not surprising that a facet-reducing cut of a d -polytope P cannot contain an arbitrary set of d vertices of P , since d vertices can determine the cutting hyperplane completely.

In this paper we use Gale transforms first to prove Theorem 1 which could be of independent interest.

Theorem 1. *For every $d \geq 3$, a pair of d -simplices $\Delta_1, \Delta_2 \subset R^d$ exists which can be positioned such that:*

1. $0 \in \text{interior}(\Delta_1) \cap \text{interior}(\Delta_2)$.
2. *No hyperplane H of R^d passing through the origin contains more than one of the $2d + 2$ facets of Δ_1 and Δ_2 in either of its closed half-spaces H^- and H^+ .*

Using Theorem 1 we prove

Theorem 2. *For every $d \geq 4$, there is a d -polytope $P \subset R^d$, and two vertices v and w of P , such that no hyperplane of R^d containing v and w has more than one facet of P in either of its closed half-spaces.*

Thus for $d \geq 4$, $k(d) < 2$. It is easy to show that $k(d) \geq 1$ thus establishing $k(d) = 1$ for $d \geq 4$. It seems remarkable that, *for $d \geq 4$, $k(d)$ is independent of d* . Also, for $d = 3$ it can be shown that $k(d) = 3$. The drop in the value of $k(d)$ as d exceeds 3, as well as the dimension-independence of $k(d)$ for $d \geq 4$, have a unifying explanation in the curious fact that *2-neighborly d -polytopes with $n > d + 1$ vertices, exist only for $d \geq 4$* .

After recapitulating the relevant properties of Gale transforms in the next section, we prove the aforementioned results.

2. Gale Transforms

Let $v_1, \dots, v_n \in R^d$ be a set of $n > d + 1$ points that span R^d affinely. $\alpha =$

$\{\alpha_1, \dots, \alpha_n\}$ is said to be an *affine dependence* of $V = \{v_1, \dots, v_n\}$ if

$$\sum_{i=1}^n \alpha_i v_i = 0 \quad \text{and} \quad \sum_{i=1}^n \alpha_i = 0.$$

Since $v_1, \dots, v_n$ span R^d affinely, the affine dependences of V form an $(n - d - 1)$ -dimensional vector space; call it X . Let $x_1, \dots, x_{n-d-1}$ be any basis of X . If $x_1, \dots, x_{n-d-1}$ are arranged as the $n - d - 1$ rows of a matrix M , then the set of n columns of M is called a *Gale transform* of V . There is a natural correspondence between the vectors in V and the columns of the matrix M . (*Gale transforms are not uniquely defined in that any basis of X can be used to construct the matrix M .* However, any two Gale transforms of V are linearly equivalent.)

If F is a face of a polytope P and $V(F)$ and $V(P)$ are the vertex-sets of F and P , then the vertex-set $V(P) \setminus V(F)$ is called a *coface* of P .

For a proof of the following important result as well as a general discussion on Gale transforms see [16].

Lemma 1. *Let $P \subset R^d$ be a d -polytope with vertex-set $V(P) = \{v_1, \dots, v_n\}$ and let $Z = \{z_1, \dots, z_n\}$ be a Gale transform of $V(P)$. A subset of V forms a coface of P if and only if the convex hull of the corresponding subset of vectors from Z contains the origin in its relative interior.*

3. Results

Theorem 1. *For every $d \geq 3$, a pair of d -simplices $\Delta_1, \Delta_2 \subset R^d$ exists which can be positioned such that:*

1. $0 \in \text{interior}(\Delta_1) \cap \text{interior}(\Delta_2)$.
2. No hyperplane H of R^d passing through the origin contains more than one of the $2d + 2$ facets of Δ_1 and Δ_2 in either of its closed half-spaces H^- and H^+ .

Proof. Consider the cyclic $(d + 1)$ -polytope with $2d + 2$ vertices, $C(2d + 2, d + 1)$. Let $V = \{v_1, \dots, v_{2d+2}\}$ be the vertex-set of $C(2d + 2, d + 1)$ and let $\bar{V} = \{\bar{v}_1, \dots, \bar{v}_{2d+2}\}$ be a Gale transform of V .

Each vertex v_i of $C(2d + 2, d + 1)$ corresponds to a real number t_i (i.e., $v_i = (t_i, \dots, t_i^{d+1})$) and $t_1 < t_2 < \dots < t_{2d+2}$. It follows as an easy corollary of Proposition 18 on p. 84 in [16] that $\{v_1, \dots, v_{d+1}\}$ and $\{v_{d+2}, \dots, v_{2d+2}\}$ are the vertex-sets of two facets of $C(2d + 2, d + 1)$. In other words, both $\{v_1, \dots, v_{d+1}\}$ and $\{v_{d+2}, \dots, v_{2d+2}\}$ are cofaces of $C(2d + 2, d + 1)$. Therefore from Corollary 3 on p. 126 in [16], it follows that $\Delta_1 = \{\bar{v}_1, \dots, \bar{v}_{d+1}\}$ and $\Delta_2 = \{\bar{v}_{d+2}, \dots, \bar{v}_{2d+2}\}$ are the vertex-sets of two d -dimensional simplices. We use Δ_1 and Δ_2 to refer to the corresponding simplices. From Lemma 1, it follows that

$$0 \in \text{interior}(\Delta_1) \cap \text{interior}(\Delta_2),$$

which is property 1 of Theorem 1.

A cyclic m -polytope P is $\lfloor m/2 \rfloor$ -neighborly [16]; i.e., every subset of $\lfloor m/2 \rfloor$ vertices of P determines a face of the polytope. Equivalently, every subset of

$m - \lfloor m/2 \rfloor$ vertices of P is a coface. Thus in a Gale transform of P 's vertex-set every subset of $m - \lfloor m/2 \rfloor$ points (of the Gale transform) contains the origin in the relative interior of their convex hull. If a hyperplane through the origin has $\lfloor m/2 \rfloor$ or fewer points of the Gale transform in an open half-space, then $m - \lfloor m/2 \rfloor$ points from the opposite closed half-space cannot contain the origin in the relative interior of their convex hull. Hence if $G \subset R^l$ is a Gale transform of a cyclic m -polytope, then any hyperplane of R^l passing through the origin has at least $\lfloor m/2 \rfloor + 1$ points of G in each of its open half-spaces.

Setting $m = d + 1$ in the previous paragraph, and for $d \geq 3$ we conclude that if H is any hyperplane of R^d through the origin, then each of its open half-spaces contains at least three points from the set $\{\bar{v}_1, \dots, \bar{v}_{2d+2}\}$. Since the origin is in the interior of both Δ_1 and Δ_2 , each open half-space of any hyperplane through the origin has at least two vertices of Δ_1 or at least two vertices of Δ_2 . Hence no hyperplane of R^d through the origin can contain more than one of the $2d + 2$ facets of Δ_1 and Δ_2 in either of its closed half-spaces, which is property 2 stated in the theorem. $\square$

Theorem 2. For every $d \geq 4$, there is a d -polytope $P \subset R^d$, and two vertices v and w of P , such that no hyperplane of R^d containing v and w has more than one facet of P in either of its closed half-spaces.

Proof. Coordinatize R^d with $x_1, \dots, x_d$ axes and identify R^{d-1} with the hyperplane $x_d = 0$. Let $\hat{x}_i$ denote the unit vector along positive x_i -axis, $i = 1, \dots, d$.

There are two $(d - 1)$ -simplices $\Delta_1, \Delta_2 \subset R^{d-1}$ having the properties stated in Theorem 1. Let

$$\tilde{\Delta}_1 = \Delta_1 + \hat{x}_d = \{z \in R^d \mid z = \hat{x}_d + y; y \in \Delta_1\},$$

$$\tilde{\Delta}_2 = \Delta_2 - \hat{x}_d.$$

That is, $\tilde{\Delta}_1$ ($\tilde{\Delta}_2$) is obtained by translating Δ_1 (Δ_2) to the hyperplane $x_d = 1$ ($x_d = -1$). Let $\varepsilon > 0$ be small enough that every line containing the point $v = (1 + \varepsilon)\hat{x}_d$ and any point of $\tilde{\Delta}_2$ intersects $\tilde{\Delta}_1$ and every line containing the point $w = (-1 - \varepsilon)\hat{x}_d$ and any point of $\tilde{\Delta}_1$ intersects $\tilde{\Delta}_2$. (Such an ε exists because of condition 1 in Theorem 1.)

Let C_1 be the cone with vertex v and cross section $\tilde{\Delta}_1$ and let C_2 be the cone with vertex w and cross section $\tilde{\Delta}_2$. Each facet of C_1 is itself a cone with vertex v , having a facet of $\tilde{\Delta}_1$ as its cross section; similarly for C_2 . $P = C_1 \cap C_2$ is a convex d -polytope. P has exactly $2d$ facets; d of them in $\text{star}(v)$ (truncated facets of C_1) and the remaining d in $\text{star}(w)$ (truncated facets of C_2).

If possible let H be a hyperplane of R^d containing v and w , and having two facets of P , say F and F' , in one of its closed half-spaces, say H^+ . Observe that H intersects the interior of P and hence P cannot be contained in either half-space of H . If $F, F' \in \text{star}(v)$ the d edges of P incident at v and hence the entire P must lie in H^+ (see Theorem 11.6 in [16]). A similar argument for w implies that $F \in \text{star}(v)$ and $F' \in \text{star}(w)$.

$F \in \text{star}(v)$ implies that the corresponding facet of $\tilde{\Delta}_1$ and hence the corresponding facet of Δ_1 is contained in H^+ . Similarly, $F' \in \text{star}(w)$ implies the corresponding facet of Δ_2 is contained in H^+ . However, $H^+ \cap R^{d-1}$ is a half-space of R^{d-1}

containing a facet of Δ_1 and a facet of Δ_2 which is a contradiction of Theorem 1. Thus no H can contain v and w and have two facets of P in H^+ . $\square$

Theorem 2 shows $k(d) \leq 1$ for $d > 3$. Next we verify that 1 is a tight upper bound by showing $k(d) \geq 1$ for $d \geq 2$. For every d -polytope P having at least $d + 2$ facets, we need to show a facet-reducing cut containing any given vertex of P as well as a facet-reducing cut that does not contain any vertex of P .

If $G \subset R^d$ is a $(d - 2)$ -flat and $p \in R^d$ is a point not on G , then p^*G denotes the unique hyperplane of R^d containing G and p .

Theorem 3. $k(d) \geq 1$ for $d \geq 2$.

Proof. The theorem is obvious for $d = 2$. For $d > 2$ it is convenient to consider two cases.

The d -polytope P is a pyramid with base Q and apex v . Q being a $(d - 1)$ -polytope yields, by induction, a $(d - 2)$ -flat F that is a facet-reducing cut of Q . If $w \neq v$ is any point of R^d not in $\text{aff}(Q)$, w^*F is a facet-reducing cut of P . F can be assumed to either not contain any vertex of Q or contain any given vertex of Q . Accordingly we have a facet-reducing cut of P that either contains no vertex of P or contains any given vertex of Q . To obtain a facet-reducing cut containing the apex v we assume F does not contain any vertex of Q and take v^*F .

The d -polytope P is not a pyramid. Hence for every vertex v of P , there are at least two facets of P not in $\text{star}(v)$. Thus the vertex-figure (see [12]) of any vertex of P yields a facet-reducing cut containing no vertex of P . To obtain a facet-reducing cut containing a given vertex v consider some vertex w that is a neighbor of v . Let H be a hyperplane passing through v , containing w in its left open half-space and all the other vertices of P (except v and w) in its right open half-space. (It is easy to see that H exists.) H is a facet-reducing cut containing v . $\square$

Thus $k(d) = 1$ for $d \geq 4$. However, for 3-polytopes a facet-reducing cut containing any three given vertices can be found and hence $k(d) = 3$. Observe that Theorem 1 is obviously false for $d = 2$ from the fact that the 1-neighborliness of cyclic 3-polytopes is a trivial property as against the $\lfloor d/2 \rfloor$ -neighborliness of cyclic d -polytopes for $d \geq 4$ (which shows the graphs of cyclic d -polytopes for $d \geq 4$ are complete).

Theorem 4. Let $P \subset R^3$ be any convex 3-polytope with at least five facets. Let u , v , and w be any three vertices of P . Then there is a plane H containing u , v , and w and having at least two facets of P in one of its closed half-spaces.

Proof. Observe that any plane containing some vertex z of P contains at least one facet of P containing z in one of its closed half-spaces. Further if a plane contains an edge of P , then the two facets of P sharing the edge are contained in the opposite half-spaces of the plane.

The vertices u , v , and w of P must be noncollinear and hence determine a plane; call it H . From the previous paragraph, there are facets F_u , F_v , and F_w —containing

u , v , and w , respectively—which lie entirely in the left or right closed half-space of H . The nontrivial case arises when H is not a supporting hyperplane of P (i.e., when F_u , F_v , and F_w are not all identical). If F_u , F_v , and F_w are all distinct, then two of them must lie on the same side of H and the proof is complete. If two of the three facets are identical, say $F_u = F_v \neq F_w$, then $\overline{uv}$ must be an edge of P and hence we must have a facet containing $\overline{uv}$ on the same side of H as F_w . $\square$

4. Remarks

The d -polytopes constructed in Theorem 2 have $2d$ facets. It is not known if there are d -polytopes with arbitrarily large number of vertices (or facets), for which any hyperplane containing a pair of specified vertices has at most one facet of the polytope in each half-space.

Acknowledgment

I thank the referee for his/her suggestions, which have led to a substantial improvement in the clarity of the arguments. In particular, the question in the previous section was suggested by the referee.

References

1. Bezdek, K., Bisztriczky, T., and Connelly, R., On hyperplanes and polytopes, *Monatsh. Math.* **109** (1990), 39–48.
2. Prabhu, N., Properties of Convex Polytopes, Ph.D. Dissertation, Courant Institute of Mathematical Sciences, New York University, 1991.
3. O'Neil, P. E., Hyperplane cuts of an n -cube, *Discrete Math.* **1**(2) (1971), 193–195.
4. Emamy-Khansary, M. R., On the cut-number of the 5-cube, *Congr. Numer.* **72** (1990), 179–186.
5. Khovanskii, A. G., Hyperplane sections of polytopes, toric varieties and discrete groups in Lobachevskii space, *Funktsional. Anal. i prilozhen.* **20** (1986), 50–61.
6. Kincses, J., Convex hull representation of cut Polytopes. Preprint, 1988.
7. Bezdek, A., Bezdek, K., and Odor, T., On a Caratheodory-type theorem, Preprint, 1988.
8. Goodman, J. E., and Pach, J., Cell decomposition of polytopes by bending, *Israel J. Math.* **64**(2) (1988), 129–138.
9. Grünbaum, B., *Convex Polytopes*, Wiley Interscience, New York, 1966.
10. Kleinschmidt, P., and Pachner, U., Shadow-boundaries and cuts of convex polytopes, *Mathematika* **27** (1980), 58–63.
11. Shephard, G. C., Sections and projections of convex polytopes, *Mathematika* **19** (1972), 144–162.
12. Brøndsted, A., *An Introduction to Convex Polytopes*, Springer-Verlag, New York, 1983.
13. Gallivan, S., Lockeberg, E. R., and McMullen, P., Complete subgraphs of the graphs of convex polytopes, *Discrete Math.* **34** (1981), 25–29.
14. Tolke, J., and Wills, J. M., eds., Problem 57, *Contributions to Geometry—Proceedings of the Geometry Symposium*, Siegen, 1978, p. 269.
15. Problems presented at the DIMACS Workshop on Polytopes and Convex Sets, Rutgers University, New Brunswick, January 8–12, 1990.
16. McMullen, P., and Shephard, G. C., *Convex Polytopes and Upper Bound Conjecture*, London Mathematical Society Lecture Notes, Vol. 3, Cambridge University Press, London, 1971.

Received March 24, 1993, and in revised form March 23, 1994, and February 1, 1995.