On Polytopes Cut by Flats

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Abstract. We prove that if we cut a d -polytope by a k -flat, then the complex of the uncut faces has the same simple-homotopy type as the boundary complex of a $(d - k)$ -polytope. We also investigate the combinatorial properties of the complex of the uncut faces and, as a corollary, we answer negatively a question of Goodman and Pach.

1. Introduction

Bruggeser and Mani [1] proved that the boundary complex of a convex polytope is shellable. In the context of collapsibility this result yields that if we remove two facets of a convex d -polytope, then the complex of the remaining faces collapses to a subcomplex which is isomorphic to the boundary complex of a $(d - 1)$ -polytope. In the first part of this paper we investigate the generalization of the above result and we prove that if we cut a d -polytope by a k -flat, then the complex of the uncut faces has the same simple-homotopy type as the boundary complex of a convex $(d - k)$ -polytope. For $k = d - 1$ we can prove collapsibility. As an application of this result we give a topological proof of a weaker version of a theorem of Goodman and Pach [3]. In the second part of the paper we investigate the combinatorial structure of the complex of the uncut faces. We prove the counterintuitive fact that for the d -simplex Δ_d and for each k ($2 \leq k \leq d - 2$) there is a k -flat K intersecting Δ_d in general position such that any $(d - k)$ -flat intersecting Δ_d also meets a proper face of Δ_d which is cut by K . This result answers negatively a question of Goodman and Pach [3].

2. The Topological Structure of the Complex of Uncut Faces

By a *polyhedral complex* we understand a finite set of convex polytopes, all of which are contained in the same euclidean space $\mathbf{R}^d$ with the usual closedness and

intersection properties (see [4]). By $|K|$ we denote the corresponding polyhedron, $|K| = \bigcup_{P \in K} P$. Let K be a complex, and suppose that $F \in K$ is a proper face of exactly one polytope $P \in K$. Then the complex $K' = K \setminus \{F, P\}$ is obtained from K by an *elementary collapse* (and K is obtained from K' by an *elementary anticollapse*). We say that the complex K *collapses to the subcomplex* L iff L can be obtained from K by a finite sequence of elementary collapses. We say that the complexes K and L have the same *simple-homotopy type* if a sequence of elementary collapses and elementary anticollapses can transform K into L (see [2]).

Let P be a d -dimensional convex polytope and let L be a k -dimensional flat which intersects P in general position (that is, L intersects the interior of P but does not intersect any face of P with dimension at most $d - k - 1$). Denote by P^n the complex of faces of P which are not cut by L . Our main result is:

Theorem 2.1. *There is a subcomplex S of P^n which is isomorphic with the boundary complex of a $(d - k)$ -polytope and $|S|$ is a strong deformation retract of $|P^n|$. Moreover, the complexes S and P^n have the same simple-homotopy type.*

For two reasons we discuss first the special case when $k = d - 1$. On one hand, in this case we can prove a stronger statement, and, on the other hand, this is the key step in proving the general case.

Denote by H^u and H^l the closed half-spaces determined by the hyperplane H and we say that H^u (H^l) is the upper (lower) half-space of H . We call a face F of P an *upper* (*lower*) *face* of P or say it is *beyond* (*beneath*) H iff $F \subseteq H^u$ ($F \subseteq H^l$). The complex of uncut faces can be partitioned into two disjoint subcomplexes $P^n = P^u \cup P^l$ where P^u (P^l) is the set of upper (lower) faces.

Lemma 2.2. *The union of the upper (lower) faces of P are collapsible to a point.*

Proof. By symmetry, it is sufficient to prove the statement for P^u only. We prove by induction on the dimension of P . For $d = 1$ it is obviously true. Assume that it holds for every $(d - 1)$ -dimensional polytope and let P be a d -dimensional polytope and let H be a cutting hyperplane. We may suppose, by changing the position of H a little bit if necessary, that there is no hyperplane parallel to H containing more than one vertex of P . Let H_1 be the supporting hyperplane of P parallel to H and $H_1 \subseteq H^u$. Now we move H toward H_1 keeping parallel to H . Let H_t be the family of these hyperplanes, $0 \leq t \leq 1$, $H_0 = H$. The complex P_t^u of faces of P which are beyond H_t changes only when H_t passes through a vertex of P . Let $x_1, x_2, \dots, x_k$ be the upper vertices of P and let H_{t_i} be the hyperplane through x_i , $0 < t_1 < \dots < t_k$. $P_{t_i}^u$ differs from $P_{t_{i+1}}^u$ only in the complex C_{x_i} of those faces of P which are beyond H_{t_i} and x_i is a vertex of each of them. If V_{x_i} denotes the vertex figure (see [4]) of P at x_i , then the complex $V_{x_i}^u$ of those faces of V_{x_i} which are beyond H_{t_i} , by the induction hypothesis, is collapsible to a point. However, C_{x_i} is a cone over $B = C_{x_i} \cap P_{t_{i+1}}^u$ and, under the central projection from x_i , B is isomorphic with $V_{x_i}^u$ and so B is also collapsible to a point. To sum up, we get that $P_{t_i}^u = P_{t_{i+1}}^u \cup C_{x_i}$, where C_{x_i} is a cone over $B = C_{x_i} \cap P_{t_{i+1}}^u$ and B is collapsible to a point. Nevertheless, it is a well-known fact (see [2]) that a cone collapses to its base if the base is collapsible

to a point. This gives that $P_{t_i}^u$ collapses to $P_{t_{i+1}}^u$. Proceeding this way we finally have that $P_{t_1}^u = P^u$ collapses to $P_{t_k}^u = x_k$. $\square$

For the proof of Theorem 2.1 we need the following theorem which can be obtained as a combination of a result of Smale (see [8]) and Whitehead's theorem on the homotopy type of complexes.

Smale's Theorem. *Let K, L be complexes. If a map $f: |K| \rightarrow |L|$ is onto and for any point $x \in |L|$ the set $f^{-1}(x)$ is contractible to a point, then the map f is a homotopy equivalence.*

Proof of Theorem 2.1. We may suppose that the origin 0 is in $L \cap \text{int } P$. Let $L^\perp$ be the orthogonal complement of L and let $p_L: \mathbf{R}^d \rightarrow L^\perp$ be the orthogonal projection and $P^\perp = p_L(P)$. The shadow boundary of P with respect to L is defined as $p_L^{-1}(\text{bd } P^\perp) \cap P$, where $\text{bd } P^\perp$ denotes the boundary complex of $P^\perp$. Obviously, there is a subcomplex S_L of P^n such that $|S_L|$ is the shadow boundary. We may suppose, by changing the position of L a little bit if necessary, that the shadow boundary of P with respect to L is sharp, that is, for any point $x \in \text{bd } P^\perp$ the set $p_L^{-1}(x) \cap P$ consists of exactly one point (note that, because L cuts P in general position, a sufficiently small change of L does not change the set of uncut faces!). This gives that p_L induces isomorphism between S_L and $\text{bd } P^\perp$. Let $c: P^\perp \setminus 0 \rightarrow |\text{bd } P^\perp|$ be the central projection from the origin and consider the composite map $f = p_L^{-1}cp_L: |P^n| \rightarrow |S_L|$. Obviously, f is continuous and it is the identity on S_L , that is, f is a retraction of P^n onto S_L . For an arbitrary point $x \in |S_L|$ we have that $f^{-1}(x) = |P^n| \cap L_x^+$ where L_x^+ is the half $(k+1)$ -flat of the $(k+1)$ -flat L_x spanned by L and x . The convex $(k+1)$ -polytope $P \cap L_x$ is cut by the k -flat L in general position and one of the components of the union of the uncut faces is exactly $|P^n| \cap L_x^+$. This implies, using Lemma 2.2, that $f^{-1}(x)$ is contractible to a point. Applying Smale's theorem, we have that f is a homotopy equivalence. However, if a retraction is a homotopy equivalence, then it is a strong deformation retraction [9].

The second part of Theorem 2.1 is an easy consequence of the first part, by using the following result (see [2]):

If K is a complex and L is a subcomplex such that L is a strong deformation retract of K and the fundamental group of K is either trivial or the group of integers, then K and L have the same simple-homotopy type.

This completes the proof of theorem. $\square$

We believe that in Theorem 2.1 the term simple-homotopy type can be replaced by collapsibility:

Question. Is it true that the complex P^n collapses to the complex S_L ?

For later use we determine the homotopy type of the union of the relative interior of the faces cut by L (this union is denoted by P^c).

Lemma 2.3. *A continuous map $r: P^c \rightarrow L \cap P^c$ exists such that if x is on a face of P , then $r(x)$ is on the same face of P and $r(x) = x$ for $x \in P^c \cap L$, that is, $P \cap L$ is a strong deformation retract of P^c .*

Proof. We call a subset of P^c of the form $P^c \cap F \neq \emptyset$ an i -dimensional cell of P^c , where F is an i -dimensional proper face of P . We define r by induction on the dimension of the cells. If $P^c \cap F$ is a $(d - k)$ -cell (there are no $(d - k - 1)$ -cells), then we define the map r by sending each point of $P^c \cap F$ into the point $F \cap L$. Suppose that r has been defined on each cell $P^c \cap F'$ with dimension less than j such that $r(P^c \cap F') \subseteq F' \cap L$, $r(x) = x$ for $x \in F' \cap L$ and let $P^c \cap F$ be a j -cell. Then r is defined on the relative boundary of it and also on $L \cap F$ (identity) and r agrees on the common parts of different cells. However, $(P^c \cap \text{relbd } F) \cup (L \cap F)$ is a closed subset of the cell and r maps continuously this set into the $(j - d + k)$ -polytope $H \cap F$. By Tietze's theorem r can be extended to the whole j -cell. For different j -cells these extended maps are equal on the common part, thus these maps can be fitted together to yield a map on the union of j -cells with the desired properties. $\square$

In the special case when the cutting flat is a hyperplane H , we can map the intersection $P \cap H$ into the upper (lower) part of the uncut faces.

Lemma 2.4. *There are continuous maps*

$$p^u: H \cap P^c \rightarrow P^u, \quad p^l: H \cap P^c \rightarrow P^l,$$

such that p^u and p^l are homotopic to a constant and if x is on a face of P , then $p^u(x)$, $p^l(x)$ are on the same face of P .

Proof. Consider the polytope $P \cap H^u$. Each j -face F of $P \cap H$ is a face of a unique $(j + 1)$ -face F' of $P \cap H^u$, $F' \not\subseteq H$ (since H does not contain any vertices of P). For each j -face $(0 \leq j \leq d - 1)$ F of $P \cap H$ let ρ_F be the central projection in $\text{aff } F'$ from a point which is sufficiently close to the centroid of F and it is in $H^l \cap \text{aff } F'$. ρ_F is a retraction from F' onto $\text{relbd } F' \setminus \text{relint } F$. These retractions can be fitted together to yield a retraction ρ_j from the union of the $(j + 1)$ -faces of $P \cap H^u$ intersecting H . Then the map

$$\rho_0 \cdots \rho_{d-1} \rho_d: P \cap H^u \rightarrow P^u$$

is a retraction and let p^u be the restriction of it to $(\text{bd } P) \cap H$. Then $p^u(x)$ and x are on the same face and p^u is homotopic to a constant. For the latter we use the fact that a continuous map from the $(d - 1)$ -sphere into an arbitrary space is homotopic to a constant if and only if it can be extended to the d -ball.

The map p^l can be constructed in the same way as p^u . $\square$

As an application of the above results we prove a weaker version of a theorem of Goodman and Pach (see [3]).

Corollary 2.5. *If the hyperplane H cuts the convex polytope $P \subseteq \mathbb{R}^d$ but does not contain any vertices of P , then for each point x of P there is a line passing through the point x which meets the boundary of P in only uncut faces.*

Proof. Choose a point x of P . We may suppose that x is an interior point of P because if it is a boundary point, then we can work on the face of P containing x in its relative interior. Introducing the map $c_x: \text{bd } P \rightarrow \text{bd } P$ which corresponds to a point y of $\text{bd } P$, the unique intersection point of $\text{bd } P$, and the line through x and y which is different from y , we must prove that $c_x(P^l) \cap P^u \neq \emptyset$ or, equivalently, $P^l \cap c_x(P^u) \neq \emptyset$. We may suppose that $x \in H^u$. In this case $c_x(P^l)$ is disjoint from P^l and suppose, on the contrary, that $c_x(P^l)$ is disjoint from P^u . Then c_x maps P^l into P^c . We have, by Lemmas 2.3 and 2.4, that the map

$$rc_x p^l: P^c \cap H \rightarrow P^c \cap H$$

is continuous and homotopic to a constant. However, $P^c \cap H$ is the boundary of the $(d-1)$ -polytope $H \cap P$ and so $P^c \cap H$ is homeomorphic to the $(d-2)$ -sphere. This gives that $rc_x p^l$ has a fixed point x_0 . Let F_0 be the proper face of P containing $c_x p^l(x_0)$. By the properties of the maps we have that x_0 and $p^l(x_0)$ are also on F_0 which contradicts that x is an interior point of P .

Thus we proved the corollary. $\square$

3. On the Combinatorial Structure of the Complex of Uncut Faces

In this section we prove the counterintuitive fact that the union of the cut faces can be so “big” that it cannot be avoided by any $(d-k)$ -flat intersecting the interior of the polytope. Let Δ_d be the d -dimensional simplex.

Theorem 3.1. *If $d \geq 4$, then for each k ($2 \leq k \leq d-2$) there is a k -flat L intersecting Δ_d in general position such that any $(d-k)$ -flat intersecting Δ_d also meets a proper face of Δ_d which is cut by L .*

We prove this in two steps. The first one is the following lemma, which was proved by Prabhu and Perles (see [6]).

Lemma 3.2. *For each k ($2 \leq k \leq d-2$) there is a k -flat L intersecting Δ_d in general position such that L intersects the relative interior of every faces of Δ_d with dimension at least $d - [k/2]$ ($[\cdot]$ means the integer part).*

Proof. Let C be the k -dimensional cyclic polytope (see [4]) with $d+1$ vertices, the vertex set is $V = \{x_1, x_2, \dots, x_{d+1}\}$. Let $\bar{V} = \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{d+1}\}$ be a Gale diagram (see p. 100 [5]) of the set V . Now we take $\bar{V}$ to be the image under an affine mapping Φ of the ordered set of vertices of Δ_d . We state that the required flat is $L = \ker \Phi = \Phi^{-1} \{0\}$.

We use the *Gale diagram relationship* (see 3A1 on p. 100 of [5]):

If $\bar{X}$ is a Gale diagram of X , then $Y \subseteq X$ is a facial set if and only if $\tilde{Y} = \{\bar{x}_i \in \bar{X} \mid x_i \notin Y\} \subseteq \bar{X}$ is such that $0 \in \text{relint conv } \tilde{Y}$.

It is well known that the polytope C is $\lfloor k/2 \rfloor$ neighborly, that is, every subset of its vertices with cardinality at most $\lfloor k/2 \rfloor$ spans a face of C . Using the Gale diagram relationship we have that for each subset S of $\bar{V}$ with cardinality at least $d+1 - \lfloor k/2 \rfloor$ the relation $0 \in \text{relint conv } S$ holds which means that the face of Δ_d corresponding to S is cut by L in its relative interior.

The polytope C is simplicial, so, using again the Gale diagram relationship, we get that there is no nonempty subset H of $\bar{V}$ with cardinality at most $d-k$ such that $0 \in \text{relint conv } H$ which gives that L meets Δ_d in general position. $\square$

Lemma 3.3. *Let k be an integer $2 \leq k \leq d-2$ and $k \neq 3$ or $d \neq 5$. If a $(d-k)$ -flat L intersects the interior of Δ_d , then L also intersects the relative interior of a proper face of Δ_d with dimension at least $d - \lfloor k/2 \rfloor$.*

Proof. Obviously, we may suppose that L contains the origin and the barycenter of Δ_d is the origin. Project Δ_d along the flat L into the k -flat $L^\perp$ orthogonal to L . Denote the images of the vertices of Δ_d by $\bar{X} = \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{d+1}\}$. We have (see 2B4 on p. 97 of [5]) that there is a $(d-k)$ -dimensional point set X such that the Gale transform of X is $\bar{X}$. Now for each vertex v of $\text{conv } X$ let S_v be the subset of X consisting of points equal to v . Obviously, there is a vertex such that

$$|S_v| \leq \left\lceil \frac{d+1}{d-k+1} \right\rceil.$$

The set S_v is clearly facial, so, using the Gale diagram relationship, it is enough to prove that

$$\left\lceil \frac{d+1}{d-k+1} \right\rceil \leq \left\lceil \frac{k}{2} \right\rceil. \quad (*)$$

Now let $d+1 = l(d-k+1) + r$ ($0 \leq r \leq d-k$). Then $(*)$ can be reduced to

$$l \leq \frac{k}{2} = \frac{l(d-k+1) + r - (d-k+1)}{2}.$$

Simple calculation gives that it is enough to prove that

$$1 + \frac{2}{d-k-1} \leq l,$$

which is true if $l \geq 3$ because $d-k-1 \geq 1$. If $l = 1$, then $(*)$ is reduced to

$$1 \leq \left\lceil \frac{k}{2} \right\rceil,$$

which is certainly true since $k \geq 2$. If $l = 2$, then $d + 1 = 2(d - k + 1) + r$, that is, $k = d - k + 1 + r$ and, since $d - k + 1 \geq 3$, thus $k \geq 3$. This gives that

$$2 \leq \frac{k}{2} = \frac{d - k + 1 + r}{2}$$

except in the case when $k = 3$, $r = 0$, and $d = 5$, which was excluded. $\square$

Proof of Theorem 3.1. Using Lemmas 3.2 and 3.3 we should only discuss the exceptional case $d = 5$ and $k = 3$.

Let $H = \{a, b, c, -a, -b, -c\}$ be the set of vertices of a regular hexagon centered at the origin such that $\text{conv}\{a, b\}$ and $\text{conv}\{b, c\}$ are edges of the hexagon. Now we take H to be the image under an affine mapping Φ of the ordered set of the vertices of Δ_5 . Pick a point x in the interior of the triangle $\text{conv}\{a, b, 0\}$ close enough to 0. Let L be the 3-flat $\Phi^{-1}\{x\}$ and take an arbitrary 2-flat T intersecting the interior of Δ_5 .

If T cuts the relative interior of a 4-face of Δ_5 , then we are ready because L meets each 4-face of Δ_5 .

If however T does not cut any 4-face of Δ_5 , then projecting Δ_5 along T into the three-dimensional space orthogonal to T , the image of the vertex set of Δ_5 must be the following type (by a theorem of Steinitz, see Theorem 1.3 [7]):

$$\{x_1, -\lambda_1 x_1, x_2, -\lambda_2 x_2, x_3, -\lambda_3 x_3\},$$

where x_1, x_2, x_3 is a basis and the λ_i 's are positive. In this case T meets only three 3-faces of Δ_5 such that the intersection of every two of them is an edge of Δ_5 . However, L does not cut only three 3-faces of Δ_5 and there are two out of these three faces which have a common 2-face. This gives that T cuts the relative interior of a 3-face of Δ_5 which is cut by L also. $\square$

Remark. Theorem 3.1 answers negatively the following question of Goodman and Pach [3]:

Let L be a k -dimensional flat intersecting a d -polytope P in general position (that is, L does not intersect any faces of P with dimension at most $d - k - 1$). Then, given any point $x \in P$, can we find a flat T of dimension $d - k$ which passes through x and meets the boundary of P in only uncut faces?

References

1. H. Bruggesser and P. Mani, Shellable decompositions of cells and spheres, *Math. Scand.* **29** (1971), 197–205.
2. M. M. Cohen, *A Course in Simple-Homotopy Type*, Springer-Verlag, New York, 1973.
3. J. E. Goodman and J. Pach, Cell decomposition of polytopes by bending, *Israel J. Math.* **64** (1988), 129–138.

4. B. Grünbaum, *Convex Polytopes*, Interscience, London, 1967.
5. P. McMullan, Transforms, diagrams and representations, in *Contributions to Geometry* (Proc. Geom. Symp., Siegen, 1978), Birkhäuser, Basel, 1979, pp. 92–130.
6. N. Prabhu, Properties of Convex Polytopes, Ph.D. Dissertation, New York University, 1991.
7. J. R. Reay, *Generalizations of a Theorem of Carathéodory*, Memoirs of the American Mathematical Society, vol. 54, American Mathematical Society, Providence, RI, 1965.
8. S. Smale, A Vietoris mapping theorem for homotopy, *Proc. Amer. Math Soc.* **8** (1957), 604–610.
9. E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

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