The Order of Points on the Second Convex Hull of a Simple Polygon

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Abstract. Given a set V of points on the plane, if $\{q_1, \dots, q_n\}$ is the set of points on the second convex hull of V , the order in which these points are visited in any V -gon is characterized. This order must verify two similar conditions to those of Kuratowski's theorem for planar graphs. Moreover, the number of possible orders that verify these conditions is obtained. It is $O(5^n)$.

1. Introduction and Notation

Given a set V of points on the plane, a simple V -gon (or crossing-free line-segment tour) is a cycle which uses the points as vertices and line segments as edges, so that no two edges cross. Henceforth, we assume that all the cycles and curves are simple; in addition, the cycles are visited in the clockwise direction, that is to say, in the neighborhood of an edge, the infinite face lies to the left and the finite face to the right.

Let P be the convex hull of V , that is, the smallest convex set that contains V . ∂P denotes the boundary of P , and $V_P = \{p_1, \dots, p_m\}$ is the set $V \cap \partial P$, where the vertices are numbered clockwise with respect to ∂P . The convex hull of $V \setminus V_P$, which we call Q , is the second convex hull of V ; ∂Q is its boundary, and $V_Q = \{q_1, \dots, q_n\}$ is the set $V \cap \partial Q$, where the vertices are numbered clockwise with respect to ∂Q .

It is well known that, for any simple V -gon, the points $\{p_1, \dots, p_m\}$ are visited in this same order. In this paper we characterize the order in which the points $\{q_1, \dots, q_n\}$ must be visited in any simple V -gon. Hence, we are interested in cyclic permutations of the set $(1, 2, \dots, n)$. Thus, we assume that two permutations of the set $(1, 2, \dots, n)$ are the same if one can be obtained from the other by shifting its elements. So, we can suppose that the permutations always begin with 1, and that

the first and last elements of a permutation are adjacent. Equally, we assume that indices 1 and n are consecutive.

A subset of V_Q formed by points with consecutive subindices is called a connected zone. Lastly, T denotes any simple cycle which uses the points of $V_P \cup V_Q$ as vertices, and polygonal lines on the remaining points of V as edges; v_i denotes a generic vertex of T .

In the next section we give a necessary condition that must satisfy the order sought. In Section 3 this order is completely characterized. Finally, Section 4 is dedicated to calculating the number of possible orders.

2. The Subdivision Property

Definition 1. Let C_1 be a simple closed curve with F and F' as its finite and infinite faces. Given $q_i, q_j, q_k, q_l \in C_1$ and a simple curve C_2 joining q_i to q_j , disjoint with q_k and q_l , we say that this curve (or link) is internal with respect to q_k and q_l , if these last two vertices each lie in two different connected components of $(F \cup C_1) \setminus C_2$. If they lie in two different components of $(F' \cup C_1) \setminus C_2$, we say that this link is external.

Figure 1 shows several kinds of links.

Lemma 1. Let C_1 be a simple closed curve and $q_1, q_2, q_3, q_4 \in C_1$, clockwise numbered. Let C_2 and C_3 be two simple disjoint curves joining q_1 to q_3 and q_2 to q_4 , respectively. Then one of them is an internal link and the other is an external link.

Internal, not external

External, not internal

Internal, external

Not internal, not external

Fig. 1. Various kinds of links.

This lemma is a consequence of the Jordan curve theorem. For a proof of this lemma and the Jordan curve theorem see [6] or [4].

For a given T , we say that the vertices q_i and q_j are internally (externally) joined, if q_k and q_l exist such that the link from q_i to q_j is internal (external) with respect to q_k and q_l . If q_i and q_j are internally joined, one edge of T must exist in this link separating q_k from q_l . Besides, this edge holds at least one extreme in V_Q and divides V_Q into exactly two nonempty connected zones, which we call the left zone, L , and the right zone, R , depending on which side of the edge their vertices lie.

Theorem 1. *Let v_1 and v_2 be two consecutive points in T with an internal link that divides V_Q into two nonempty connected zones L and R . Then, starting at v_1 , the vertices of V_Q are visited in the following order:*

- (i) *If $v_1 \in V_P \wedge v_2 \in V_Q$ the order is v_1, v_2, L, R .*
- (ii) *If $v_1 \in V_Q \wedge v_2 \in V_P$ the order is v_1, v_2, R, L .*
- (iii) *If $v_1, v_2 \in V_Q$ and the next visited vertex of V_Q lies in R the order is v_1, v_2, R, L . If it lies in L , the order is v_1, v_2, L_1, r, L_2 , where L_1 and L_2 are connected zones of L , v_2 is the next vertex to L_1 , and v_1 is the previous vertex to L_2 .*

Proof. Only (iii) is proved, since the proof of (i) and (ii) is similar. The key idea of the proof is that, at most, one single visit from L to R is allowed in T , and it is through an external link that “jumps” over v_2 . Any external link from v_1 to a point in R , or from a point in L to v_1 , is done in the same way.

Likewise, to join points in R to points in L , or v_2 to a point in L , or a point in R to v_2 , at most one external link that “jumps” over v_1 can exist (see Fig. 2).

If, after v_2 , the first visited point lies in R , and besides not all points of this zone are visited, the tour will necessarily pass through a vertex $v'_1 \in V_P$ to go to zone L . From v'_1 , visiting points in L and points in V_P , a vertex $v'_2 \in V_P$ will be reached, which should be joined to v_1 , using vertices of the zone R . v'_1 (and v'_2) can be chosen so that it can be joined to v_1 (to v_2) with a curve disjoint to T , and contained in $P \setminus Q$. In the right half-plane of the curve v'_1, v_1, v_2, v'_2 , there must be a link

Fig. 2. Links $v_2v'_1, v'_2v_1$ cannot be both in the right zone.

Fig. 3. The order v_4, v_3, v_2, v_1 in ∂Q is not possible.

between v_2 and v'_1 and another link between v'_2 with v_1 which contradicts Lemma 1 (see Fig. 2).

If, on the contrary, after v_2 , the first visited point lies in L , the order for the points in ∂Q is v_1, v_2, L_1, R, L_2 . Now, we need only prove that $L_1 \cup v_2$ and $L_2 \cup v_1$ are connected zones. If $L_2 = \emptyset$, it is obvious; if $L_2 \neq \emptyset$ there is a “jump” from L to R and another from R to L . If v_4 is the nearest point in L_1 to v_1 , and v_3 is the nearest point in L_2 to v_2 , then $L_1 \cup v_2$ and $L_2 \cup v_1$ will not be connected zones if and only if v_4, v_3, v_2, v_1 appear in ∂Q in just this order. In this case either the link between v_2 and v_4 or that between v_3 and v_1 will be external by Lemma 1. On applying Lemma 1 again, it is not possible to have one external link between v_2 and v_4 and another from L to R ; and neither is it possible to have an external link between v_1 and v_3 , and another from R to L (see Fig. 3). $\square$

Notice that, if we allow $L_1 = \emptyset$ or $L_2 = \emptyset$, the order of visiting the points of V_Q is always $\{v_1, v_2\} \cap V_Q, L_1, R, L_2$.

Theorem 2. *Given a cycle T , if the points in V_Q are visited in the order L, R where L is an arbitrary connected zone with at least two vertices, then L can be subdivided into two nonempty, connected, and disjoint zones L_1 and L_2 , such that the order is L_1, L_2, R .*

Proof. If $R = \emptyset$, it is obvious. Therefore, we assume that $R \neq \emptyset$, $L = \{q_1, q_2, \dots, q_k\}$ with $k \geq 2$, and that these vertices are visited in the order $q_{i_1}, q_{i_2}, \dots, q_{i_k}$.

If $i_1 = 1$, or $i_1 = k$, or $i_k = 1$, or $i_k = k$, the result is obtained choosing $L_1 = q_1$, or $L_1 = q_k$, or $L_2 = q_1$, or $L_2 = q_k$, respectively. So we can assume that $q_{i_1} \neq q_1$, $q_{i_1} \neq q_k$, $q_{i_k} \neq q_1$, and $q_{i_k} \neq q_k$.

Let $q_{j_1} \in R$ be the preceding vertex to q_{i_1} in T . Given the relative position of $q_1, q_{i_1}, q_k, q_{j_1}$, one of the links in T , either that going from q_{j_1} to q_{i_1} or that going from q_1 to q_k (or from q_k to q_1), must be internal by Lemma 1. Hence, we can have the two following cases:

Fig. 4. Case (a): the link $q_{j_1}q_{i_1}$ is internal.

- (a) The link between q_{j_1} and q_{i_1} is internal. Then in this link the edge $q_{j_1}q_{i_1}$ appears or an edge of type $q_{j_1}p_j$ or $p_iq_{i_1}$ which separates q_1 from q_k . In any case, by Theorem 1, the order of the points of V_Q is $q_{j_1}, q_{i_1}, L_1, R', L_2$, and, besides, $q_1 \in L'$ and $q_k \in R'$ (see Fig. 4). Since a point of L lies in R' , then $L'_1 \subset L$ to keep the order R, L . Then we choose $L_1 = \{q_{i_1} \cup L'_1\} \cap L$ and $L_2 = R' \cap L$. If $L_1 \cup L_2 = L$ the result is obtained; if $L_1 \cup L_2 \neq L$, then $L'_2 \cap L \neq \emptyset$, $R' \subset L$, and we can take $L_1 = \{q_{i_1} \cup L'_1 \cup R'\} \cap L$ and $L_2 = L'_2 \cap L$.
- (b) The link between q_{j_1} with q_{i_1} is not internal. Then the link between q_1 and q_k not containing q_{j_1} and q_{i_1} must be internal. Hence, an edge v_1v_2 is in T , which separates q_{j_1} from q_{i_1} , and, as the vertices of L are visited consecutively, at least one of v_1, v_2 belongs to L and the other belongs to $L \cup V_P$. Among the above edges, we choose one such that the number of points in the connected zone in which q_{i_1} lies is maximum.

Let us assume that q_{i_1} lies on the left side of the chosen edge (see Fig. 5). Then, by Theorem 1, the order of visit is v_1, v_2, L'_1, R', L'_2 with $q_{j_1} \in R'$, $q_{i_1} \in L'_2$, $L'_1 \subset L$ (if $v_2 \notin L$, then $L'_1 = \emptyset$). Moreover, to keep the order L, R then $L'_2 \subset L$, and $v_1 \in L$ because $L'_2 \neq \emptyset$.

This order can be written as $v_1, v_2, L'_1, R' \cap L, R' \cap R, L'_2$. Now, $R' \cap L$ consists, at most, of two connected zones, one, A , composed of the vertices from q_1 to v_1 (v_1 excluded) and another, B , composed of the vertices from the intersection point of ∂Q with the edge v_1v_2 to q_k (previous intersection point excluded). However, an edge between a point in A and another in B can be neither internal (because L' is maximum) nor external (because the link between q_{j_1} with q_{i_1} is external). Therefore, either A or B is empty. If $q_1 = v_1$, we can take $L_1 = L'_2 \cup q_1$ and $L_2 = L \setminus L_1$. If $v_2 = q_k$, we can take $L_1 = L'_2 \cup v_1 \cup q_k \cup L'_1$ and $L_2 = L \setminus L_1$.

Finally, let us assume that q_{i_1} lies on the right side of the edge v_1v_2 (see Fig. 5). Then the order is v_1, v_2, L'_1, R', L'_2 with $q_{j_1} \in L'_1$, $q_{i_1} \in R'$. As $L'_1 \neq \emptyset$, necessarily, $v_2 \in L$. To keep the order L, R , then $R' \subset L$ and $L'_2 \subset L$; therefore the order is $v_1, v_2, L'_1 \cap L, L'_1 \cap R, R', L'_2$ (if $v_1 \notin L$, then $L'_2 = \emptyset$). Now $L'_1 \cap L$ may consist, at most, of two connected zones, one

Fig. 5. Case (b): the link $q_{j_1}q_{i_1}$ is not internal.

composed of the vertices from q_1 to v_2 (v_2 excluded) and the other composed of the vertices from q_k to the extreme point of L'_2 (this point excluded). As in the previous case, it follows that one of these zones is empty. (Note that if $L'_2 = \emptyset$, the extreme point is v_1 .) We can take $L_1 = v_2 \cup \{v_1 \cap V_Q\} \cup L'_2 \cup R'$ and $L_2 = L'_1 \cap L$, independently if L'_2 is empty or not. $\square$

We note that Theorem 2 implies that if R contains at least two points, it may also be subdivided as L is. Likewise, every obtained connected zone $L_1, L_2, R_1, R_2, \dots$ may also be subdivided. We say that a permutation of $(1, 2, \dots, n)$ has “the subdivision property” if it is formed by, first, a permutation σ_1 of consecutive indices and later a permutation σ_2 of the remaining indices and, besides, this division process can be repeated again for σ_1 and σ_2 . For example, the permutation 13254 has this property but 13524 has not. The above theorem states that the order of $\{q_1, \dots, q_n\}$ in any V -gon holds the subdivision property.

Before examining additional conditions about the order of $\{q_1, \dots, q_n\}$, we show several equivalences of the subdivision property.

We say that a permutation of the set $(1, 2, \dots, n)$ is g -pyramidal, when it belongs to the family $\mathcal{S}_n$, defined recursively as follows:

- (i) $\mathcal{S}_1 = (1)$.
- (ii) The permutation π_n belongs to $\mathcal{S}_n$ if and only if two adjacent and consecutive indices exist in π_n , and, by contracting them, the new permutation obtained, π_{n-1} , belongs to $\mathcal{S}_{n-1}$.

We use this name because these permutations are a generalization of pyramidal permutations, see [3]. For example, the permutation (124867593) is g -pyramidal because we are able to do the following contractions:

$$\begin{aligned}
 (124867593) &\rightarrow (13756482) \rightarrow (1365472) \rightarrow (135462) \\
 &\rightarrow (13452) \rightarrow (1342) \rightarrow (132) \rightarrow (12) \rightarrow (1).
 \end{aligned}$$

Lemma 2. *Given a g-pyramidal permutation π_n with n indices, by contracting any pair of adjacent and consecutive indices to one index, we obtain a g-pyramidal permutation with $n - 1$ indices.*

Proof. By hypothesis, there is one pair $(i, i + 1)$ such that by shrinking this pair the resultant permutation π_{n-1} is g-pyramidal. Let $(j, j + 1)$ be any other pair of π_n ; it is also transformed into one pair of π_{n-1} , therefore, by induction, if we reduce this pair we obtain a g-pyramidal permutation π_{n-2} . Finally, the permutation obtained by duplicating the index i of π_{n-2} is g-pyramidal, and is the same as the one obtained from π_n when contracting the pair $(j, j + 1)$. $\square$

Theorem 3. *A permutation is g-pyramidal if and only if the subdivision property holds.*

Proof. Let us assume that the permutation π_n verifies the subdivision property. When splitting repeatedly the set of indices from π_n , we find, necessarily, a set with only two consecutive indices. By contracting them we obtain a permutation π_{n-1} that continues holding the subdivision property, and hence we can continue this process of contractions until the permutation (1) is obtained. Therefore, π_n is g-pyramidal.

Reciprocally, let π_n be a g-pyramidal permutation. π_n has two adjacent and consecutive indices, therefore when we reduce them, by induction, we can get a g-pyramidal permutation which satisfies the subdivision property. Then the initial permutation can be subdivided in the same way so holding the property. $\square$

From the previous theorem it is easily deduced by induction that:

Corollary 1.

- (i) *If $n \geq 3$ there are at least two pairs of adjacent and consecutive indices in any g-pyramidal permutation.*
- (ii) *Any g-pyramidal permutation can be written in the way $(1, i_2, \dots, i_n)$ where $(i_2, \dots, i_n)$ holds the subdivision property.*

Another interesting characterization of the g-pyramidal permutations appears when observing permutations with five indices. All of them are g-pyramidal except $(1, 3, 5, 2, 4)$ and $(1, 4, 2, 5, 3)$. In these two permutations is not possible to find two adjacent and consecutive indices, so they are not g-pyramidal permutations.

Definition 2. Given a permutation of n indices, we say that five indices $i_1 < i_2 < i_3 < i_4 < i_5$ form a five-point star if they appear in the order $(\dots, i_1, \dots, i_3, \dots, i_5, \dots, i_2, \dots, i_4, \dots)$ or in the order $(\dots, i_1, \dots, i_4, \dots, i_2, \dots, i_5, \dots, i_3, \dots)$ (see Fig. 14(a)).

Theorem 4. *A permutation is g-pyramidal if and only if it does not contain a five-point star.*

Proof. Notice that if a permutation holds a five-point star, and it has two consecutive indices, by contracting them the resultant permutation continues holding a five-point star. Hence, it is not possible to reach the permutation (1) by contracting.

Reciprocally, let $\pi_n = (i_1, \dots, i_n)$ be a permutation containing no five-point stars. Let us prove, by way of the induction principle, that this permutation must be g-pyramidal. When $n \leq 5$ it is trivial. Let $\pi_{n-1} = (i'_1, \dots, i'_{n-1})$ be the permutation obtained by removing the index n . As π_n does not contain a five-point star neither does π_{n-1} ; hence, π_{n-1} would be a g-pyramidal permutation, and must contain at least two pairs of consecutive indices.

If, when we add the index n , π_n maintains a pair of consecutive indices, then they can be contracted and the resulting permutation does not contain a five-point star, hence it will be g-pyramidal. So π_n must be g-pyramidal too.

If, when we add the index n , π_n does not have a pair of consecutive indices, then it must have one of the following forms:

1. $(1, n-1, \dots, i, n, i+1, \dots)$,
2. $(1, \dots, i+1, n, i, \dots, n-1)$,
3. $(1, n-1, \dots, i+1, n, i, \dots)$,
4. $(1, \dots, i, n, i+1, \dots, n-1)$,

because only in these cases do two pairs of consecutive indices disappear when the index n is included in π_{n-1} .

However, none of them is possible. This is obvious for the first and second cases because they contain a five-point star.

In the third case if the index n is removed, we obtain one g-pyramidal permutation that holds only the two pairs of consecutive indices $(1, n-1)$ and $(i+1, i)$. By reducing one of these pairs, for example the $(1, n-1)$ pair, the new permutation of $n-2$ indices must necessarily have the pairs $(n-2, 1)$ (or $(1, 2)$) and $(i+1, i)$, because any g-pyramidal permutation must have at least two pairs of consecutive indices. Hence, the initial permutation π_n must have the indices in the order $(1, n-1, \dots, i+1, n, i, \dots, n-2)$ (or in the order $(1, n-1, 2, \dots, i+1, n, i, \dots)$) which contains the five-point star formed by the indices $1, i, i+1, n-2, n$ (or $1, 2, i, n-1, n$). So this third case never occurs.

Analogously, it can be proved that the fourth case is never possible. $\square$

3. Six-Point Star

Given a permutation π_n of $(1, \dots, n)$, we say that this permutation is feasible if there is a set of points V , with n vertices in ∂Q , and one simple V -gon that visits the vertices in V_Q in only the order given for the permutation. For a drawing of π_n we mean a drawing on the plane of one such V -gon. Figure 6 shows a drawing of $(1, 2, 4, 5, 3)$.

From the previous section we know that, in order to be feasible, a permutation must be g-pyramidal. However, this property is not sufficient; the permutation $(1, 2, 5, 6, 3, 4)$ is g-pyramidal but is not feasible. Precisely this order $(1, 2, 5, 6, 3, 4)$ and the equivalent $(1, 4, 5, 2, 3, 6)$ give the full characterization.

 Fig. 6. A drawing of $(1, 2, 4, 5, 3)$.

Definition 3. Given a permutation of n indices, we say that six indices $i_1 < i_2 < i_3 < i_4 < i_5 < i_6$ form a six-point “star” if they appear in the order $(\dots, i_1, \dots, i_2, \dots, i_5, \dots, i_6, \dots, i_3, \dots, i_4, \dots)$ or in the order $(\dots, i_1, \dots, i_4, \dots, i_5, \dots, i_2, \dots, i_3, \dots, i_6, \dots)$ (see Fig. 7).

Before seeing the full characterization, we prove the following:

Lemma 3. Let $q_i, q_j, q_k \in V_Q$ be three points situated in this order in ∂Q and visited in T in this same order. Then q_j can be joined to the infinite point through a simple curve without crossing either ∂Q or the link $q_k q_i$.

Proof. We need to prove that it is not possible for the link $q_k q_i$ to jump over q_j using points of V_P . We will see that if there is such a jump, T contains a five-point star. Let $x, x' \in \partial Q$ be the extremes of the jump (see Fig. 8).

Fig. 7. The six-point “star.”

Fig. 8. Points q_k, y, x, x', q_i form a five-point star.

If x and x' are in ∂Q in the order q_i, x, q_j, x', q_k , then the five-point star q_i, q_j, q_k, x, x' appears in T .

If x and x' are in ∂Q in the order q_i, q_j, q_k, x', x , then the points q_i, q_k, x' , and x are in the conditions of Lemma 1. However, in this case neither the link $q_k x$ nor $x' q_i$ can be external because none of these four points can be jumped over.

If the position of x and x' in ∂Q is q_i, x, q_j, q_k, x' (see Fig. 8), then the points q_i, q_j, q_k , and x are in the hypothesis of Lemma 1. The link $q_i q_j$ cannot be external because q_k is jumped over by the link xx' and so the link $q_k x$ must be external and jump over q_i (q_j is jumped over by xx'). Let y and y' be the extremes of the jump. In T the order of points is q_k, y, y', x, x', q_i , and then q_k, y, x, x' , and q_i form a five-point star.

Finally, if the position of x and x' in ∂Q is q_i, q_j, x', q_k, x , the same result is achieved. $\square$

Lemma 4. *If a permutation of n points has a six-point star, then it is not feasible.*

Proof. Let us take a six-point star in which the order is $q_{i_1}, q_{i_2}, q_{i_5}, q_{i_6}, q_{i_3}, q_{i_4}$ (the reasoning is the same for the other order). By applying Lemma 1 to the three links $q_{i_4} q_{i_1}$, $q_{i_2} q_{i_5}$, and $q_{i_6} q_{i_3}$, there is necessarily one of them, let us call it C , such that another link is external with respect to C and C is external with respect to the remaining link. Without loss of generality, we assume that this link C is $q_{i_4} q_{i_1}$.

Then, if $q_{i_6} q_{i_3}$ is external with respect to $q_{i_4} q_{i_1}$ and $q_{i_4} q_{i_1}$ is external with respect to $q_{i_2} q_{i_5}$, by the previous lemma, we have the jumps shown in Fig. 9. In any drawing the vertices q_{i_1}, y', q_{i_5}, x' , and y are in this order in ∂Q , and they are visited in T in the order $q_{i_1}, q_{i_5}, y, y', x'$, forming a five-point star.

The same reasoning is used when $q_{i_2} q_{i_5}$ is external with respect to $q_{i_4} q_{i_1}$ and $q_{i_4} q_{i_1}$ is external with respect to $q_{i_6} q_{i_3}$. $\square$

In order to prove the opposite result, i.e., if a g -pyramidal permutation is not feasible, then it has got a six-point star, we need the following definition.

Fig. 9. Points $q_{i_1}, q_{i_5}, y, y', x'$ form a five-point star.

Definition 4. Suppose that we have drawn a feasible g-pyramidal permutation of n points. If $q_i \in V_Q$ can be joined to the infinite point through a simple curve without crossing either ∂Q or the drawing, we say that q_i is outside the drawing. Otherwise, we say that q_i is inside the drawing.

If q_i is outside the drawing, then it can be duplicated in two different ways giving rise to two g-pyramidal feasible permutations with $n + 1$ points. If q_i is inside, then it can only be duplicated in one way, putting a new point in ∂Q close to q_i in the counterclockwise direction, because the other possibility is forbidden by Lemma 1 (see Fig. 10).

Theorem 5. Let π_n be a g-pyramidal permutation which is not feasible. Then in π_n there is a six-point star.

Fig. 10. Ways of duplicating an outside or inside point.

Fig. 11. The position of the points in any drawing of π_k .

Proof. If $n = 2, 3, 4$, or 5 , any g-pyramidal permutation is feasible. If $n > 5$ let us reduce π_n by shrinking until we find a feasible g-pyramidal permutation π_k . Then we will have the sequence

$$\pi_n \rightarrow \pi_{n-1} \rightarrow \cdots \rightarrow \pi_{k+1} \rightarrow \pi_k,$$

where only π_k is feasible. If π_{k+1} contains a six-point star, π_n does too. So, we only need to prove that π_{k+1} contains such a star.

Given a drawing of π_k , on duplicating one appropriate point q_{i_1} of π_k we will obtain π_{k+1} , on duplicating one appropriate point of π_{k+1} we will obtain π_{k+2} , and so on. If q_{i_1} is outside the drawing, then π_{k+1} is feasible. Therefore, q_{i_1} must be inside any drawing of π_k and, moreover, in order to obtain π_{k+1} , the duplication of q_{i_1} has to be done in the way forbidden by Lemma 1.

For this drawing of π_k , let q_{i_6} and q_{i_3} be two vertices in V_Q , such that their link causes q_{i_1} to be inside and q_{i_3} is the following vertex of V_Q visited after q_{i_6} . Let x and x' be the extremes of the jump of the link $q_{i_6}q_{i_3}$ over q_{i_1} (see Fig. 11). Due to Lemma 3, the vertices q_{i_1} , q_{i_3} , and q_{i_6} are in this order in ∂Q , and they are visited in T in the order $q_{i_1}, q_{i_6}, q_{i_3}$.

After visiting q_{i_1} and before reaching q_{i_6} , it is not possible to visit any point q_i situated in ∂Q between q_{i_1} and q_{i_3} , because if q_i is between q_{i_1} and x' , then q_{i_1}, q_i , and q_{i_6} contradicts Lemma 3, and if q_i is between x' and q_{i_3} , then $q_{i_6}, q_{i_1}, x', q_i$, and q_{i_3} forms a five-point star. In addition, if only points between q_{i_6} and q_{i_1} are visited, it is not possible to visit all of them in counterclockwise order, because in this case it is possible to draw π_k with q_{i_1} outside (see Fig. 12).

Fig. 12. Obtaining a drawing of π_k with q_{i_1} outside.

Fig. 13. Obtaining a drawing of π_k with q_{i_1} outside.

Therefore, either q_{i_5} exists between q_{i_3} and q_{i_6} , visited between q_{i_1} and q_{i_6} , or there are two points $q_{j_5}, q_{j_6} \in V_Q$, in the order $q_{i_6}, q_{j_5}, q_{j_6}, q_{i_1}$ in ∂Q , visited in T in the order $q_{i_1}, q_{j_5}, q_{j_6}, q_{i_6}$.

Similarly, it can be proved that, after visiting q_{i_3} and before reaching q_{i_1} , it is not possible to visit any point between q_{i_6} and q_{i_1} . Also, if only points between q_{i_1} and q_{i_3} are visited, it is not possible to visit all of them in counterclockwise order (see Fig. 13). Hence, either q_{i_4} exists between q_{i_3} and q_{i_6} , visited between q_{i_3} and q_{i_1} , or there are two points $q_{j_3}, q_{j_4} \in V_Q$ in the order $q_{i_1}, q_{j_3}, q_{j_4}, q_{i_3}$ in ∂Q , visited in T in the order $q_{i_3}, q_{j_3}, q_{j_4}, q_{i_1}$.

When vertices q_{i_5} and q_{i_4} exist, they must be in ∂Q in the order $q_{i_3}, q_{i_4}, q_{i_5}, q_{i_6}$ because in the order $q_{i_3}, q_{i_5}, q_{i_4}, q_{i_6}$ the cycle T contains a five-point star. Hence, when vertex q_{i_1} is duplicated in the forbidden way, vertices $q_{i_1}, q_{i_2}, q_{i_3}, q_{i_4}, q_{i_5}$, and q_{i_6} are in this order and visited in the order $q_{i_1}, q_{i_2}, q_{i_5}, q_{i_6}, q_{i_3}, q_{i_4}$, so π_{k+1} contains a six-point star. If q_{i_5} or q_{i_4} do not exist, a six-point star can be formed by substituting the pair q_{i_5}, q_{i_6} for q_{j_5}, q_{j_6} or the pair q_{i_3}, q_{i_4} for q_{j_3}, q_{j_4} . $\square$

Theorems 2 and 5 and Lemma 4 imply:

Theorem 6. A permutation of $(1, \dots, n)$ is feasible if and only if it does not contain either the five-point or the six-point stars.

We can observe the similarity between these conditions and those which must verify a planar graph (Kuratowski's theorem): a graph is planar if and only if it contains no subgraph homeomorphic with K_5 (five-point star) or $K_{3,3}$ (six-point star). Geometrically, the configurations shown in Fig. 14 are forbidden. In the six-point star only the tour in the marked direction is forbidden.

Fig. 14. The two forbidden configurations.

Note that Theorem 5 gives a method for finding a drawing for a feasible permutation. Suppose that π_n is feasible and that it is obtained from π_{n-1} by duplicating a point q_{i_1} . To achieve a drawing of π_n , we first get one of π_{n-1} . If in this drawing q_{i_1} is outside, or it is inside but to obtain π_n , q_{i_1} is duplicated in the allowed direction, then we can obtain a drawing of π_n , according to the scheme of Fig. 10. On the contrary, if q_{i_1} is inside, and it must be duplicated in the forbidden direction, then it is possible to get a drawing of π_{n-1} with q_{i_1} outside according to the scheme of Figs. 12 and 13, because π_n does not hold a six-point star.

4. The Number of Feasible Permutations

According to Corollary 1 each g-pyramidal permutation can be divided as (1) (a permutation from $2, \dots, n$). This second permutation, which we call σ , is formed by one permutation, σ_1 , of $(2, \dots, k)$ and another, σ_2 , of $(k+1, \dots, n)$, in such a way that σ is $\sigma_1\sigma_2$ or $\sigma_2\sigma_1$. If the order followed by σ_1 and σ_2 is the former, let us express σ as $(\sigma_1) + (\sigma_2)$, and as $(\sigma_1) - (\sigma_2)$ if the order is the latter. If σ_1 and σ_2 have at least two indices, the same type of expressions can be used for σ_1 and σ_2 .

So, each algebraic expression from $(2, \dots, n)$ using brackets and two operations $+$ and $-$ is equivalent to one way of subdividing the set $(2, \dots, n)$, and, hence, one g-pyramidal permutation is obtained by interpreting $+$ as the operation concatenation and $-$ as inverted concatenation. However, as $+$ and $-$ operations hold the associative property, the same permutation can be obtained by different expressions. For example, the expressions $((2-3) + (4+5))$ and $((((2-3) + 4) + 5))$ represent two possible ways of subdividing the g-pyramidal permutation $(1, 3, 2, 4, 5)$. We can represent every one of these algebraic expressions like a binary tree with $n-2$ internal nodes containing a $+$ sign or a $-$ sign, and $n-1$ external nodes containing the numbers $2, 3, \dots, n$ (see Fig. 15).

The following lemma gives the relation between these binary trees and g-pyramidal permutations.

Fig. 15. Two binary trees for permutation $(1, 3, 2, 4, 5)$.

Fig. 16. Obtaining another tree for π_n verifying the conditions of Lemma 5.

Lemma 5. *There is a one-to-one correspondence between g-pyramidal permutations of $(1, 2, \dots, n)$ and binary trees with $n - 2$ internal nodes, where each internal node contains a $+$ or a $-$ sign, and satisfies the property that its left child, when it exists, must contain the opposite sign.*

Proof. As we have seen, every g-pyramidal permutation π_n can be associated with several binary trees with internal nodes having a $+$ or a $-$ sign.

If one tree like that shown in Fig. 16(a) represents a permutation, it can be modified to the one of Fig. 16(b), which represents the same permutation. Identical modification can be made if there are two $-$ signs instead of two $+$ signs. Hence, it is possible to modify one tree until another tree is obtained representing the same permutation and verifying the conditions of the lemma.

We still have to prove that two different trees satisfying this property will be associated to different permutations.

When these trees have different roots, they will also be different, because a $+$ sign means that the very first index is found before the final one, and for the $-$ sign the meaning is the opposite. When they have equal roots and both are $+$, if the left subtree from the first tree has i internal nodes and the left of the second one has j internal nodes, with $i < j$, then the permutation corresponding to the first tree is $(1, \dots, i + 2, \dots, 2, \dots, j + 2, \dots)$ and the permutation corresponding to the second one is $(1, \dots, j + 2, \dots, 2, \dots)$; so they are different permutations. The same happens when they have equal roots and both have a $-$ sign. Finally, when $i = j$, the same result is reached by way of induction. $\square$

Among all the binary trees of this kind, we are interested in those which represent feasible permutations.

Theorem 7. *A binary tree of the above kind, with n internal nodes, represents a g-pyramidal permutation that does not contain a six-point star, if and only if, for each node of the tree with a $-$ sign, any of these three conditions are verified (see Fig. 17):*

- (i) *The left subtree of this node is formed only by one external node.*
- (ii) *All the internal nodes of the right subtree of this node contain a $-$ sign.*
- (iii) *All the internal nodes neither in the left nor in the right subtree of this node contain a $-$ sign.*

Fig. 17. The three possibilities for a node with a $-$ sign.

Proof. If the permutation contains a six-point star, for example in the order $i_1, i_2, i_5, i_6, i_3, i_4$, then, necessarily, there is a $-$ sign because i_5 and i_6 appear before i_3 and i_4 . Besides, the left branch of this node must contain the vertices i_3, i_4 , and because they appear in this order, a $+$ sign must exist in this subtree. The same argument serves for the two remaining branches of the node, where the vertices i_1, i_2 and i_5, i_6 must stay (even the case when $i_1 = 1$).

Reciprocally, if there is a node with a $-$ sign, verifying none of the three conditions, then a $+$ sign must exist in each of the branches of this node. Thus, we can choose six indices i_1, i_2 in the ascendent branch, i_3, i_4 in the left branch, and i_5, i_6 in the right branch forming a six-point star. $\square$

Consider trees representing feasible permutations; we say that a tree T is conditioned if another tree T' exists, such that T is a subtree of T' and there is a $+$ sign in a node of T' which is not in T . Figure 18 shows a conditioned tree of three nodes.

Let h_n be the number of trees with n internal nodes representing feasible permutations. The number of feasible permutations of $n + 2$ points is precisely h_n . Let h_n^+ denote the number of trees whose root is $+$ and h_n^- the number of those whose root is $-$. Equally, let h_n^c be the number of conditioned trees of n internal nodes and h_n^{c+} and h_n^{c-} the number of conditioned trees, whose roots contain a $+$ or a $-$ sign, respectively.

Fig. 18. A conditioned subtree with three internal nodes.

Obviously,

$$h_n = h_n^+ + h_n^- \quad (1)$$

except for $n = 0$. By convention we choose $h_0 = h_0^+ = h_0^- = 1$.

Equally

$$h_n^c = h_n^{c+} + h_n^{c-} \quad (2)$$

except for $n = 0$. By convention we choose $h_0^c = h_0^{c+} = h_0^{c-} = 1$.

Also, we have

$$h_n^+ = h_n^{c+}, \quad n \geq 0, \quad (3)$$

and

$$h_n^+ = \sum_{i=0}^{n-1} h_i^{c-} h_{n-i-1}^c, \quad n \geq 1, \quad (4)$$

because if the root has a $+$ sign, we can put a conditioned subtree with i nodes whose root has a $-$ sign on the left branch, and another conditioned subtree with $n - i - 1$ nodes on the right branch.

For h_n^- we have

$$h_n^- = \sum_{i=1}^{n-1} h_i^+ h_{n-i-1}^c + h_{n-1}, \quad n \geq 1, \quad (5)$$

because if the root has a $-$ sign, we can put a subtree whose root has a $+$ sign on the left branch, and a conditioned subtree with $n - i - 1$ nodes on the right branch. If there are no nodes on the left, a generic subtree (h_{n-1}) appears on the right.

Finally, we have

$$h_n^{c-} = h_{n-1}^c + \sum_{i=1}^{n-1} h_i^{c+}, \quad n \geq 1, \quad (6)$$

because if there are no nodes on the left branch, a conditioned subtree with $n - 1$ nodes appears on the right branch. If there are some nodes on the left subtree, this must be conditioned and have a $+$ sign in the root, and in the right subtree all the internal nodes must contain a $-$ sign.

By applying (2) and (3) to expression (6) we have

$$h_n^{c-} = h_{n-1}^+ + \sum_{i=1}^{n-1} h_i^+ + h_{n-1}^{c-}, \quad n \geq 1,$$

which, by recurrence becomes

$$h_n^{c-} = 2h_{n-1}^+ + 3h_{n-2}^+ + \cdots + nh_1^+ + h_1^{c-}, \quad n \geq 1. \quad (7)$$

From (2) we have

$$h_n^c = h_n^+ + 2h_{n-1}^+ + \cdots + nh_1^+ + 1, \quad n \geq 1. \quad (8)$$

By adding (4) and (5) we obtain

$$h_n = h_{n-1} + \sum_{i=0}^{n-1} h_i^c h_{n-i-1}^c, \quad n \geq 1. \quad (9)$$

Formulas (1), (5), (8), and (9), allow us to calculate h_n by recurrence, and hence know the number of feasible permutations.

Theorem 8. *The generating function of h_n is*

$$H(z) = \frac{1}{2} \left(\frac{(2z-1)\sqrt{(1-5z)(1-z)}}{z^3(2-z)^2} + \frac{1-6z+9z^2+2z^3-6z^4+2z^5}{z^3(2-z)^2(1-z)} \right).$$

Asymptotically h_n is

$$h_n = \frac{125\sqrt{5}}{54\sqrt{\pi}} n^{-3/2} 5^n + o(5^n n^{-3/2}).$$

Proof. Let $H^+(z)$ be the generating function of h_n^+ , i.e., $H^+(z) = \sum_{n \geq 0} h_n^+ z^n$. Analogously, we define $H^c(z) = \sum_{n \geq 0} h_n^c z^n$, $H^{c-}(z) = \sum_{n \geq 0} h_n^{c-} z^n$, and $H(z) = \sum_{n \geq 0} h_n z^n$. We are interested in calculating the value of $H(z)$.

As the characteristic function of $1/(1-z)^2$ is

$$1 + 2z + 3z^2 + \cdots + (n-1)z^n + \cdots,$$

then the coefficient of z^n for the function

$$\frac{H^+(z) - z}{(1 - z)^2}$$

will be

$$h_n^+ + 2h_{n-1}^+ + \cdots + (n-1)h_2^+ + n(h_1^+ - 1) + (n+1) = h_n^c.$$

We obtain a first equality

$$H^c(z) = \frac{H^+(z) - z}{(1 - z)^2}. \quad (10)$$

Besides, as $h_n^{c-} = h_n^c - h_n^{c+} = h_n^c - h_n^+$, $n \geq 1$, we have

$$H^{c-}(z) = \frac{H^+(z) - z}{(1 - z)^2} - H^+(z) + 1 = \frac{H^+(z)(2z - z^2) - z}{(1 - z)^2} + 1. \quad (11)$$

Since $\sum_{i=0}^n h_i^{c-} h_{n-i}^c$ is the coefficient of z^n in the product $H^{c-}(z)H^c(z)$ and we know that $h_n^+ = \sum_{i=0}^{n-1} h_i^{c-} h_{n-i-1}^c$ and $h_{n+1}^+ = \sum_{i=0}^n h_i^{c-} h_{n-i}^c$, it follows that

$$\begin{aligned} H^+(z) &= H^{c-}(z)H^c(z)z + 1 \\ &= \frac{H^+(z) - z}{(1 - z)^2} \left(\frac{H^+(z)(2z - z^2) - z}{(1 - z)^2} - 1 \right) z + 1. \end{aligned} \quad (12)$$

Of this equation we can obtain $H^+(z)$:

$$H^+(z) = \frac{-3z^3 + 9z^2 - 5z + 1 - (1 - z)^2 \sqrt{(1 - 5z)(1 - z)}}{2z^2(2 - z)}. \quad (13)$$

From (10) we obtain, for $H^c(z)$,

$$H^c(z) = \frac{(1 - z)(1 - 2z) - \sqrt{(1 - 5z)(1 - z)}}{2z^2(2 - z)}.$$

Finally, from (9) it is easy to see that

$$H(z) = \frac{zH^c(z)^2 + 1}{1 - z}.$$

So, we get

$$H(z) = \frac{1}{2} \left(\frac{(2z-1)\sqrt{(1-5z)(1-z)}}{z^3(2-z)^2} + \frac{1-6z+9z^2+2z^3-6z^4+2z^5}{z^3(2-z)^2(1-z)} \right). \quad (14)$$

We can write this expression as

$$H(z) = \left(1 - \frac{z}{\frac{1}{5}}\right)^{1/2} g(z) + f(z) \quad \text{with} \quad g(z) = \frac{1}{2} \frac{\sqrt{1-z}(2z-1)}{(2-z)^2 z^3}, \quad (15)$$

where $f(z)$ and $g(z)$ are analytic in a neighborhood of $z = \frac{1}{5}$. So, we are in the condition to apply Darboux's theorem (see [1]). It must be

$$h_n = \frac{1}{n} \frac{g(\frac{1}{5})n^{-1/2}}{\Gamma(-\frac{1}{2})(\frac{1}{5})^n} + o\left((\frac{1}{5})^{-n} n^{-3/2}\right), \quad (16)$$

that is,

$$h_n = \frac{125\sqrt{5}}{54\sqrt{\pi}} n^{-3/2} 5^n + o(5^n n^{-3/2}). \quad \square \quad (17)$$

5. Open Questions

In this paper we have described the necessary and sufficient conditions that must be verified by the order of visit of the points of V_Q in any V -gon. A first open question is to find the conditions that must be verified by the order of visit of the points of the k th convex hull.

A second question is how many feasible cycles are in a given configuration of n points in two convex hulls? We believe we have solved this question, but finding a configuration with the maximum number of feasible cycles remains an open question.

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