Extremal Polynomials for Obtaining Bounds for Spherical Codes and Designs*

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Abstract. We investigate two extremal problems for polynomials giving upper bounds for spherical codes and for polynomials giving lower bounds for spherical designs, respectively. We consider two basic properties of the solutions of these problems. Namely, we estimate from below the number of double zeros and find zero Gegenbauer coefficients of extremal polynomials. Our results allow us to search effectively for such solutions using a computer. The best polynomials we have obtained give substantial improvements in some cases on the previously known bounds for spherical codes and designs. Some examples are given in Section 6.

1. Introduction

A nonempty finite set $W \subset \mathbb{S}^{n-1}$ is called a spherical code. If W is a spherical code, then the number

$$d(W) = \min\{d(x, y) \mid x, y \in W, x \neq y\}$$

is called a minimum distance of code W . The maximal scalar product is

$$s(W) = \max\{(x, y) \mid x, y \in W, x \neq y\} = 1 - \frac{d(W)^2}{2}.$$

Problem 1. Given a dimension $n \geq 3$ and a maximal scalar product s (or a minimal distance $d = \sqrt{2(1-s)}$), one wishes to find the maximal possible cardinality of a spherical code $W \subset \mathbb{S}^{n-1}$ with $s(W) \leq s$ (or $d(W) \geq d$).

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We denote this maximal cardinality by $\tau(n, s)$, i.e.,

$$\tau(n, s) = \max\{|W|: W \subset S^{n-1}, (x, y) \leq s, x, y \in W, x \neq y\}.$$

The number $\tau(n, s)$ is equal to the maximum number of nonoverlapping spheres of radius $r = d/(2 - d)$ ($d = \sqrt{2(1 - s)}$ as above) that can simultaneously touch the unit sphere S^{n-1} .

Problem 1 was solved for $-1 \leq s \leq 0$ by Rankin [33] (see also [18]). There are only particular solutions for other values of n and s (see [20]). General upper bounds for $\tau(n, s)$ were given by Rankin [33] and Levenshtein [27] (see also [29]). Better upper bounds were found in some cases ($s(W) = \frac{1}{2}$, $d(W) = 1$, the so-called kissing numbers) by Odlyzko and Sloane [32] and the author [9], [13], [14].

In general, the Levenshtein bounds [27], [29] state

$$\tau(n, s) \leq \begin{cases} \binom{k+n-3}{k-1} \left[\frac{2k+n-3}{n-1} - \frac{P_{k-1}^{(n)}(s) - P_k^{(n)}(s)}{(1-s)P_k^{(n)}(s)} \right] & \text{for } \xi_k \leq s < \eta_k, \\ \binom{k+n-2}{k} \left[\frac{2k+n-1}{n-1} - \frac{(1+s)(P_k^{(n)}(s) - P_{k+1}^{(n)}(s))}{(1-s)(P_k^{(n)}(s) + P_{k+1}^{(n)}(s))} \right] & \text{for } \eta_k \leq s < \xi_{k+1}, \end{cases} \quad (1)$$

where ξ_k is the greatest zero of the Jacobi polynomial $P_k^{((n-1)/2, (n-1)/2)}(t)$ [1, Chapter 22], [36], η_k is the greatest zero of $P_k^{((n-1)/2, (n-3)/2)}(t)$, and $P_i^{(n)}(t)$ are the Gegenbauer polynomials [37, Chapter 9]

$$P_i^{(n)}(t) = \frac{P_i^{((n-3)/2, (n-3)/2)}(t)}{\binom{i + (n-3)/2}{i}}.$$

The following equality [20, Corollary 3.8], [27, equation 1.7] has been shown to be fundamental for estimating sizes of spherical codes and designs:

$$|W|f(1) + \sum_{x, y \in W, x \neq y} f((x, y)) = |W|^2 f_0 + \sum_{i=1}^k \frac{f_i}{r_i} \sum_{j=1}^{r_i} \left(\sum_{x \in W} v_{ij}(x) \right)^2. \quad (2)$$

Here $W \subset S^{n-1}$ is a spherical code, $f(t)$ is a real polynomial of degree k , $f(t) = \sum_{i=0}^k f_i P_i^{(n)}(t)$ is the Gegenbauer expansion of $f(t)$, the system $\{v_{ij}(x), j = 1, \dots, r_i\}$, $i = 1, 2, \dots, k$, is an orthonormal basis for the space $V^{(i)} \subset L_2(S^{n-1})$ of homogeneous harmonic polynomials in n variables (on S^{n-1}) of total degree i , and

$$r_i = \dim(V^{(i)}) = \binom{n+i-1}{n-1} - \binom{n+i-3}{n-1}.$$

The next theorem [20, Theorem 4.3], [23] gives the so-called linear programming bound for spherical codes. It follows easily from (2).

Theorem 1. Let $f(t)$ ($n \geq 3$) be a real polynomial such that:

(A1) $f(t) \leq 0$ for $-1 \leq t \leq s$.

(A2) The coefficients in the Gegenbauer expansion $f(t) = \sum_{i=0}^k f_i P_i^{(n)}(t)$ satisfy $f_0 > 0$, $f_i \geq 0$ for $i = 1, \dots, k$.

Then $\tau(n, s) \leq f(1)/f_0$.

This theorem is the basic tool used in [20], [27], [29], [32], [13], [14], [19], and [9] for obtaining upper bounds for cardinalities of spherical codes.

A spherical code $W \subset \mathbf{S}^{n-1}$ is called a spherical τ -design if and only if $\sum_{x \in W} f(x) = 0$ for all homogeneous harmonic polynomials $f(x) = f(x_1, x_2, \dots, x_n)$ of degree $1, 2, \dots, \tau$. These designs were introduced in 1977 by Delsarte *et al.* [20].

An equivalent definition is the following [20]. A spherical code $W \subset \mathbf{S}^{n-1}$ is a spherical τ -design if and only if

$$\int_{\mathbf{S}^{n-1}} f(x) d\mu(x) = \frac{1}{|W|} \sum_{x \in W} f(x)$$

(where $\mu(\mathbf{S}^{n-1}) = 1$) holds for any polynomial $f(x) = f(x_1, x_2, \dots, x_n)$ of degree at most τ . Thus spherical designs can be used for numerical integration.

Seymour and Zaslavsky [34] have shown the existence of spherical τ -designs $W \subset \mathbf{S}^{n-1}$ for any τ and n provided sufficiently large $|W|$. Explicit constructions of spherical designs can be found, for example, in [2], [4], [31], [22], and [3]. We are interested in lower bounds for their cardinality.

Problem 2. Given a dimension $n \geq 3$ and an integer number $\tau \geq 4$ (the remaining cases are trivial), one wishes to find the minimal possible cardinality of a spherical τ -design on $\mathbf{S}^{n-1}$.

This problem is solved only in several cases (see [20]). More precisely, for $n \geq 3$, $\tau \geq 4$, exactly eight designs (listed in [20]) are known to attain the following Fisher-type lower bound for their cardinality [20, Theorems 5.11 and 5.12]:

$$|W| \geq \begin{cases} \binom{n+e-1}{n-1} + \binom{n+e-2}{n-1} & \text{if } \tau = 2e, \\ 2\binom{n+e-1}{n-1} & \text{if } \tau = 2e+1. \end{cases} \quad (3)$$

Such designs are called tight [20, Definition 5.13]. Tight designs were investigated by Bannai and Damerell [5], [6]. For $n \geq 3$, they show that an equality in (3) is impossible if $\tau = 2e \geq 6$ or $\tau = 2e+1 \geq 9$ except for the case $\tau = 11$, $n = 24$ (see [7]; this design is formed by vectors with a minimum norm in the Leech lattice, after rescaling). Classification of the tight spherical designs is still unknown for $\tau = 4, 5, 7$ (see [5], [6], and [11]). Further improvements of the bound (3) for $\tau \geq 6$ were obtained by Nikova and the author [16], [15], [9].

The bound (3) and the improvements mentioned [16], [15], [9] are obtained by the next theorem [20, Theorem 5.10] (the linear programming bound for spherical designs).

Theorem 2. *Let $f(t)$ be a real polynomial such that:*

(B1) $f(t) \geq 0$ for $-1 \leq t \leq 1$.

(B2) *The coefficients in the Gegenbauer expansion $f(t) = \sum_{i=0}^k f_i P_i^{(\tau)}(t)$ satisfy $f_i \leq 0$, $i = \tau + 1, \dots, k$.*

Then the cardinality of a spherical τ -design $W \subset S^{n-1}$ is bounded below by $|W| \geq f(1)/f_0$.

More general theorems about the cardinality of codes and designs in polynomial metric spaces (finite and infinite) can be found in [29], [19], [23], [28], [21], and [30].

Definition 1. We denote

$$A_{n,s} = \{f(t) \in \mathbf{R}[t]: f(t) \text{ satisfies (A1) and (2)}\},$$

$$B_{n,\tau} = \{f(t) \in \mathbf{R}[t]: f(t) \text{ satisfies (B1) and (B2)}\},$$

and

$$L(f) = \frac{f(1)}{f_0}.$$

As remarked, the upper bounds (1) for spherical codes and the lower bounds (3) for spherical designs were obtained by suitable polynomials for Theorems 1 and 2, respectively. Therefore, the following two problems arise as an attempt to solve Problems 1 and 2.

Problem 3. Given $n \geq 3$, $s \in (0, 1)$, and integer $k \geq 1$, one wishes to find a polynomial $f(t) \in A_{n,s}$ of degree k (if it exists) which gives the best upper bound for $\tau(n, s)$ among the polynomials in $A_{n,s}$ having the same or lower degree.

Problem 4. Given $n \geq 3$, $\tau \geq 4$, and integer $k \geq 1$, one wishes to find a polynomial $f(t) \in B_{n,\tau}$ of degree k (if such exists) which gives the best lower bound for the cardinalities of the spherical τ -designs on S^{n-1} among the polynomials in $B_{n,\tau}$ having the same or lower degree.

Definition 2. Any solution of Problem 3 (resp. Problem 4) is called an $A_{n,s}$ -extremal (resp. $B_{n,\tau}$ -extremal) polynomial.

Thus we are interested in finding extremal polynomials in order to obtain better (than (1) and (3)) bounds for spherical codes and designs.

Levenshtein [27], [29] obtains bounds (1) for $\tau(n, s)$ using the following polynomials:

$$f_{2k-1}^{(s)}(t) = (t-s) \left(\sum_{i=0}^{k-1} r_i \begin{vmatrix} P_i(s) & P_k(s) \\ P_i(1) & P_k(1) \end{vmatrix} P_i(t) \right)^2, \quad (4)$$

$$f_{2k}^{(s)}(t) = (t-s)(t+1) \left(\sum_{i=0}^{k-1} r_i \begin{vmatrix} P_i(s) & P_k(s) & P_{k+1}(s) \\ P_i(1) & P_k(1) & P_{k+1}(1) \\ P_i(-1) & P_k(-1) & P_{k+1}(-1) \end{vmatrix} P_i(t) \right)^2, \quad (5)$$

with suitable chosen degrees $h(n, s)$ depending on n and s . In 1980 Sidel'nikov [35] proved that polynomials (4) and (5) are $A_{n,s}$ -extremal (for another, more general proof, see Section 4 of [30]). We denote by $h(n, s)$ and $r(n, s)$ the degree and the number of double zeros, respectively, for the Levenshtein $A_{n,s}$ -extremal polynomials (4) or (5).

The bound (3) has been obtained by polynomials of degree τ that are $B_{n,\tau}$ -extremal. These have the form $G^2(t)$ or $G^2(t)(t+1)$ for τ even and odd, respectively [20].

Any positive linear combination of polynomials in $A_{n,s}$ ($B_{n,\tau}$) is a polynomial from $A_{n,s}$ ($B_{n,\tau}$) as well. Moreover, this polynomial gives an upper bound (a lower bound) in the interval between the combined better and the weaker bounds [8, Lemma 1.1.], [12], [9]. We use this argument very often in order to investigate the properties of extremal polynomials.

We search for extremal polynomials of higher degree than previously known from [20], [27], and [29]. Such polynomials (if they exist) would give better bounds than (1) and (3). The first polynomials of higher degree improving the Levenshtein bounds for the kissing numbers $\tau_n = \tau(n, \frac{1}{2})$ (in the range $4 \leq n \leq 23$, $n \neq 8$) were proposed by Odlyzko and Sloane in [32] (see also [8], [9], and [12]–[14]).

We investigate the following two basic properties of extremal polynomials. First, we give a lower bound on the number of double zeros of extremal polynomials of high enough degree. This bound is equal to the number $r(n, s) + 1$ for spherical codes (Theorems 2.2 and 2.3) and to $\lceil \tau/2 \rceil + 1$ for spherical designs (Theorem 2.7 [16, Theorem 2]). Next, we describe (Theorems 3.1, 3.2, and 3.4) a method for finding zero coefficients (if such exist) in the Gegenbauer expansion $f(t) = \sum_{i=0}^k f_i P_i^{(n)}(t)$ of an extremal polynomial. This method, although sometimes carried out by a computer, works when the corresponding extremal polynomial exists and has zero Gegenbauer coefficients.

These facts (about double zeros and zero Gegenbauer coefficients) allow us to restrict the number of the parameters on which putative extremal polynomials depend. In particular, for degrees $h(n, s) + 3$ or $h(n, s) + 4$ for spherical codes and $\tau + 3$ for spherical τ -designs, the remaining parameters can be found effectively on a PC using a Monte Carlo method. Of course, the final results can be verified without a computer. Extremal polynomials (and new bounds, respectively) or nonexistence of extremal polynomials having the considered form are obtained.

The two basic properties of the extremal polynomials are considered in Sections 2 and 3. Investigations on spherical codes and designs are made separately. In Section 4 we describe our method for finding extremal polynomials. In Section 5 we prove that our method gives polynomials that are extremal provided they give better bounds than the bounds given by polynomials on the boundary of some bounded domain (where $f(t) \in A_{n,s}$ or $B_{n,\tau}$, respectively). Some examples with new upper bounds for spherical codes are presented in Section 6. We also list the cases where we have obtained in [9], [15], and [16] new lower bounds on spherical designs. In Section 7 we list some open problems.

2. Double Zeros of Extremal Polynomials

2.1. Spherical Codes

We assume that $n \geq 3$ and $s \in (0, 1)$ are fixed. A real polynomial $f(t)$ which satisfies (A1) can be written as follows:

$$f(t) = G(t)(t - \xi), \quad (6)$$

where $s \leq \xi < 1$, and $G(t) \geq 0$ for $-1 \leq t \leq s$. If the polynomial $G(t)$ has $\deg(G)$ zeros in the interval $[-1, s]$, then these zeros are double or equal -1 or s . Therefore, $G(t)$ equals $A_m^2(t)$, $A_m^2(t)(t+1)$, $A_m^2(t)(s-t)$, or $A_m^2(t)(t+1)(s-t)$. In the last two cases the leading coefficient of $f(t)$ is negative. Thus $f(t)$ does not satisfy (A2) (because $f_k < 0$). In the first two cases the function $F = f(1)/f_0$ can be investigated. This gives (see [26]) exactly the Levenshtein polynomials (4) and (5).

Therefore, without loss of generality we may search for $A_{n,s}$ -extremal polynomials (6) such that $G(t)$ has at most $\deg(G) - 1$ zeros in the interval $[-1, s]$. This allows us to use Karlin-Shapley's theorem [24], [25, p. 75]. It states that $G(t)$ can be represented as follows:

$$G(t) = \begin{cases} A_m^2(t) + (s-t)(1+t)B_{m-1}^2(t) & \text{if } \deg(G) = 2m, \\ (t+1)A_m^2(t) + (s-t)B_m^2(t) & \text{if } \deg(G) = 2m+1, \end{cases} \quad (7)$$

where $\deg(A_m) = \deg(B_m) = m$, $\deg(B_{m-1}) = m-1$, the polynomials $A_m(t)$, $B_m(t)$, $B_{m-1}(t)$ have exactly m , m , $m-1$ zeros in $[-1, s]$, respectively. Moreover, in (7) we have $G(t_0) = 0$ for some $t_0 \in (1, s)$ if and only if $A_m(t_0) = B_{m-1}(t_0) = 0$ (or $A_m(t_0) = B_m(t_0) = 0$). In other words, $G(t)$ has a double zero $t_0 \in (-1, s)$ if and only if $A_m(t)$ and $B_{m-1}(t)$ (or $B_m(t)$) have t_0 as a common zero.

In what follows in this section we assume $G(-1) > 0$ and $G(s) > 0$. Note that $G(-1) = 0$ or (and) $G(s) = 0$ implies that the number of the parameters in (6) decreases by one (or two). Of course, we consider such cases when we search for $A_{n,s}$ -extremal polynomials.

We prove that if $\deg(f)$ is large enough (with respect to $h(n, s)$, the degree of the corresponding Levenshtein polynomial (4) or (5)), then the polynomials $A_m(t)$ and $B_{m-1}(t)$ (or $B_m(t)$) from (7) have at least $r(n, s) + 1$ common zeros. In fact, we require $\deg(f) \geq h(n, s) + 3$.

The first step in our proof is the next lemma.

Lemma 2.1. Let $f(t) = G(t)(t - \xi) = \sum_{i=0}^k f_i P_i^{(n)}(t)$ ($k \geq 4$) be an $A_{n,s}$ -extremal polynomial such that $f_1 > 0$, $G(-1) > 0$, and $G(s) > 0$. Then $G(t)$ has at least one double zero in the interval $(-1, s)$.

Proof. Let us assume that $G(t)$ has no double zeros. Then it is strictly positive in $[-1, s]$. There are $\varepsilon > 0$ such that $G(t) \geq \varepsilon > 0$ for $-1 \leq t \leq s$. Now we consider, for $0 < \delta \leq \varepsilon$, polynomials

$$f_\delta(t) = f(t) - \delta(t - \xi) = (G(t) - \delta)(t - \xi) = \sum_{i=0}^k f_i^{(\delta)} P_i^{(n)}(t).$$

We have $f_\delta(t) \leq 0$ for $-1 \leq t \leq s$. So, $f_\delta(t)$ satisfies (A1). Moreover, since $f_0^{(\delta)} = f_0 + \xi\delta > f_0$, $f_1^{(\delta)} = f_1 - 2\delta/(n-1) \geq 0$ for small enough δ (for $\delta \leq (n-1)f_1/2$), and $f_i^{(\delta)} = f_i$ for $i \geq 2$, we have $f_\delta(t) \in A_{n,s}$ for some δ . However, we have

$$L(f_\delta) = \frac{f_\delta(1)}{f_0^{(\delta)}} = \frac{f(1) - \delta(1 - \xi)}{f_0 + \delta\xi} < \frac{f(1)}{f_0},$$

which contradicts the extremality of $f(t)$. □

Generalization of this lemma follows.

Theorem 2.2. Let $f(t) = C_r^2(t)G(t)(t - \xi) = \sum_{i=0}^k f_i P_i^{(n)}(t)$ be an $A_{n,s}$ -extremal polynomial such that:

- (C1) $C_r(t)$ has exactly $r = \deg(C_r)$ zeros in $(-1, s)$.
- (C2) $G(-1) > 0$, $G(s) > 0$, and $\deg(G) \geq 4$.
- (C3) The coefficients f_i , $i = 1, 2, \dots, k$, satisfy $f_1 > 0$, $f_2 > 0, \dots, f_{2r+1} > 0$, where $2r+1 \leq h(n, s)$, $h(n, s)$ being the degree of the corresponding Levenshtein polynomial (4) or (5).

Then $f(t)$ has at least $r+1$ double zeros in $[-1, s]$.

Proof. We have to prove that $G(t)$ has at least one double zero in $(-1, s)$. Let us suppose that $\varepsilon > 0$ exist such that $G(t) \geq \varepsilon > 0$ for $-1 \leq t \leq s$. We consider, for $0 < \delta \leq \varepsilon$, polynomials

$$f_\delta(t) = f(t) - \delta D(t) = (G(t) - \delta)D(t) = \sum_{i=0}^k f_i^{(\delta)} P_i^{(n)}(t),$$

where $D(t) = C_r^2(t)(t - \xi)$.

We have $f_\delta(t) \leq 0$ for $-1 \leq t \leq s$, i.e., (A1) is satisfied. Denoting the Gegenbauer coefficients of $D(t)$ by $f_i(D)$ we obtain, as in Lemma 2.1, $f_i^{(\delta)} = f_i - \delta f_i(D) \geq 0$ for small enough $\delta > 0$ and $i = 0, 1, \dots, 2r+1 = \deg(D)$. Further, $f_i^{(\delta)} = f_i$ for $i \geq 2r+2$. Therefore, we have $f_\delta(t) \in A_{n,s}$ for some $\delta > 0$. Now we consider two cases.

Case 1: $f_0(D) \leq 0$. We have

$$L(f_\delta) = \frac{f_\delta(1)}{f_0^{(\delta)}} = \frac{f(1) - \delta D(1)}{f_0 - \delta f_0(D)} < \frac{f(1)}{f_0} = L(f),$$

a contradiction (observe that $D(1) > 0$).

Case 2: $f_0(D) > 0$. Since $\deg(D) = 2r + 1 \leq h(n, s)$, by Corollaries 4.1 and 4.3 of Section 4 in [19] we obtain $D(1)/f_0(D) \geq$ (the Levenshtein bound (1)) $> L(f)$. However, it can be easily checked that this inequality is equivalent to $L(f_\delta) < L(f)$ — a contradiction that completes the proof. $\square$

We point out that a constant $c = c(r, s)$ exists such that, for any dimension $n \geq c$, $f_0(D) \leq 0$ (see [8]), i.e., Case 1 occurs. For example, we have $c(2, s) = 2 + 2/s^2$, $c(3, \frac{1}{2}) = 18$, and so on. However, in general this gives a lower bound $r(n, s)$ for the double zeros. So, Case 2 is more important in Theorem 2.2.

We now formulate an analog (for $f(-1) = 0$) of Theorem 2.2 having a similar proof.

Theorem 2.3. Let $f(t) = C_r(t)G(t)(t+1)(t-\xi) = \sum_{i=0}^k f_i P_i^{(n)}(t)$ be an $A_{n,s}$ -extremal polynomial such that:

- (D1) $C_r(t)$ has exactly $r = \deg(C_r)$ real zeros in $(-1, s)$.
- (D2) $G(-1) > 0$, $G(s) > 0$, and $\deg(G) \geq 4$.
- (D3) The coefficients f_i , $i = 1, 2, \dots, k$, satisfy $f_1 > 0$, $f_2 > 0, \dots, f_{2r+2} > 0$, where $2r + 2 \leq h(n, s)$, the degree of the corresponding Levenshtein polynomial (4) or (5).

Then $f(t)$ has at least $r + 1$ double zeros in $[-1, s]$.

It follows from Theorems 2.2 and 2.3 that (assuming $f_i > 0$ for all $i \leq h(n, s)$) we can count inductively $r(n, s) + 1$ double zeros for putative extremal polynomials having degrees at least $h(n, s) + 3$ or $h(n, s) + 4$ in some cases.

We next discuss the restrictions $f_i > 0$ in Theorems 2.2. and 2.3.

Definition 2.4. We denote $A_{n,s}(m) = \{f(t) = \sum_{i=0}^k f_i P_i^{(n)}(t) \in A_{n,s} : f_1 > 0, \dots, f_m > 0\}$, for $m = 1, 2, \dots$.

Definition 2.5. A polynomial $f(t) \in A_{n,s}(m)$ is called $A_{n,s}(m)$ -extremal if

$$L(f) = \min\{L(g) : g(t) \in A_{n,s}(m), \deg(g) \leq \deg(f)\}.$$

Theorem 2.6. Let the polynomials $f(t)$ and $g(t)$ be $A_{n,s}(h(n, s))$ - and $A_{n,s}$ -extremal, respectively, and suppose $\deg(f) = \deg(g) \geq h(n, s) + 3$. Then $L(f) = L(g)$. More-

over, $f(t)$ and $g(t)$ have as common zeros the $r(n, s) + 1$ double zeros of $f(t)$ that are given by Theorems 2.2 and 2.3.

Proof. By Definitions 1 and 2.5 we have $L(g) \leq L(f)$. Suppose that $L(g) < L(f)$. We consider the polynomials

$$H_{\alpha, \beta}(t) = \alpha f(t) + \beta g(t),$$

where α and β are positive. Then we have $H_{\alpha, \beta}(t) \in A_{n, s}(m)$ ($m = h(n, s)$) and

$$L(g) < L(H_{\alpha, \beta}) < L(f),$$

a contradiction with the extremality of $f(t)$ for $A_{n, s}(m)$. Therefore, $L(f) = L(g) = L(H_{\alpha, \beta})$.

Since $H_{\alpha, \beta}(t) \in A_{n, s}(m)$ is $A_{n, s}$ -extremal, it has at least $r(n, s) + 1$ double zeros. It is obvious that these zeros must coincide with $r(n, s) + 1$ double zeros of $f(t)$. Moreover, they must be double zeros of $g(t)$. $\square$

Theorem 2.6 shows that any $A_{n, s}(h(n, s))$ -extremal polynomial $f(t)$ of degree at least $h(n, s) + 3$ is $A_{n, s}$ -extremal as well. Moreover, any $A_{n, s}$ -extremal polynomial of the same degree must have the same $r(n, s) + 1$ double zeros as $f(t)$. So we can search for $A_{n, s}$ -extremal polynomials of degrees at least $h(n, s) + 3$ having $r(n, s) + 1$ double zeros in $[-1, s]$ without loss of generality.

2.2. Spherical Designs

Because of the weaker conditions of Theorem 2, the situation is simplified here. More precisely, we do not need the conditions $f_i > 0$. The corresponding assertion for the double zeros was proved by Nikova and the author in [9] and [16]. We only state it.

Theorem 2.7. *Let $f(t)$ be a $B_{n, \tau}$ -extremal ($n \geq 3$, $\tau \geq 4$) polynomial of degree $k \geq \tau + 3$. Then:*

- (a) *If τ is odd, then $f(t)$ has at least $\lfloor \tau/2 \rfloor + 1$ double zeros in $[-1, 1]$.*
- (b) *If τ is even and -1 is an even zero of $f(t)$ (in particular, if $f(-1) > 0$), then $f(t)$ has at least $\lfloor \tau/2 \rfloor + 1$ double zeros in $[-1, 1]$.*
- (c) *If τ is even and -1 is an odd zero of $f(t)$ (in particular $f(-1) = 0$), then $f(t)$ has at least $\tau/2$ double zeros in $[-1, 1]$.*

We remark that all the $B_{n, \tau}$ -extremal polynomials of degrees $\tau + 3$ we have found have $\lfloor \tau/2 \rfloor + 1$ double zeros.

3. Zero Gegenbauer Coefficients of Extremal Polynomials

3.1. Spherical Codes

The Levenshtein polynomials (4) and (5) have strictly positive Gegenbauer coefficients. We obtain $A_{n,s}$ -extremal polynomials that have several consecutive zero Gegenbauer coefficients beginning from $h(n, s)$ (or sometimes $h(n, s) \pm 1$). Then we use the conditions $f_i = 0$ in order to express some parameters of the $A_{n,s}$ -extremal polynomials in terms of the remaining unknown parameters.

We describe a method for finding the indices of zero Gegenbauer coefficients. In some cases we obtain $f_i = 0$ under some condition $L(f) \geq C = \text{const}$. When the constant C is less than the best-known upper bound for $\tau(n, s)$ it may be assumed, without loss of generality, that $f_i = 0$ for putative $A_{n,s}$ -extremal polynomials. Again $n \geq 3$ and $s \in (0, 1)$ are fixed.

The main theorem concerning zero Gegenbauer coefficients follows.

Theorem 3.1. *Let $E_q(t)$ ($q \geq h(n, s)$) be a real polynomial such that:*

- (E1) $E_q(t) \leq 0$ for $-1 \leq t \leq s$.
- (E2) *The Gegenbauer coefficients of $E_q(t)$ in $E_q(t) = \sum_{i=0}^k f_i(E_q) P_i^{(n)}(t)$ satisfy $f_0(E_q) > 0$, $f_q(E_q) < 0$, and $f_i(E_q) \geq 0$ for $i \neq q$.*
- (E3) $L(E_q) \leq C(n, s)$, $C(n, s)$ being the best-known lower bound for $\tau(n, s)$.

Then any $A_{n,s}$ -extremal polynomial $f(t)$ of degree $m \geq k$ has the Gegenbauer coefficient $f_q = 0$.

Proof. Let us suppose $f_q > 0$, contradicting the assertion. We construct polynomials

$$H_\alpha(t) = f(t) + \alpha E_q(t)$$

with $\alpha > 0$. We have $H_\alpha(t) \leq 0$ for $-1 \leq t \leq s$ (i.e., (A1) is satisfied). Moreover, $f_i(H_\alpha) = f_i + \alpha f_i(E_q) \geq 0$ for small enough $\alpha > 0$ (for $0 < \alpha \leq -f_q(E_q)/f_q$). Therefore, $H_\alpha(t) \in A_{n,s}$ for some $\alpha > 0$. Since

$$L(E_q) \leq C(n, s) < L(f)$$

(an upper bound exceeds the lower bound $C(n, s)$) we obtain

$$L(E_q) < L(H_\alpha) < L(f),$$

a contradiction. □

More often, we use the next theorem having a similar proof.

Theorem 3.2. Let $E_q(t)$ ($q \geq h(n, s)$) be a real polynomial such that:

(F1) $E_q(t) \leq 0$ for $-1 \leq t \leq s$.

(F2) The Gegenbauer coefficients of $E_q(t)$ in $E_q(t) = \sum_{i=0}^k f_i(E_q)P_i^{(n)}(t)$ satisfy $f_0(E_q) > 0$, $f_q(E_q) < 0$, and $f_i(E_q) \geq 0$ for $i \neq q$.

Then any $A_{n,s}$ -extremal polynomial $f(t)$ of degree $m \geq k$ such that $L(f) > L(E_q)$ has the Gegenbauer coefficient $f_q = 0$.

In our previous applications [8], [9] we found many suitable (for Theorems 3.1 or 3.2 with $s = \frac{1}{2}$) polynomials $E_q(t)$ for $h(n, s) \leq q \leq h(n, s) + 2$. In all cases we obtained several consecutive zero Gegenbauer coefficients.

Example 3.3. Let us consider the following polynomial of degree 9:

$$E_7(t) = t^2(t+1)^2(t+s)^2(t^2+at+b)(t-s) = \sum_{i=0}^k f_i(E_7)P_i^{(n)}(t),$$

where $a = -2 - s$ and $b = 2(1+s)^2/(2+s)$. It is easy to check that $E_7(t)$ satisfies (E1) (and (F1)) for $s \in (0, 1)$. Moreover, we have $f_i(E_7) > 0$ for $i = 0, 1, 2, \dots, 5, 9$, $f_6(E_7) = f_8(E_7) = 0$, and $f_7(E_7) < 0$ for $1/n \leq s^2 \leq 3/(n+2)$. Also,

$$\frac{E_7(1)}{f_0(E_7)} = \frac{2n(n+2)(1-s^2)}{3 - (n+2)s^2}.$$

Since, in the above interval for s , the last number is very small compared with the corresponding Levenshtein bound (1) we can apply Theorems 3.1 and 3.2. Therefore any $A_{n,s}$ -extremal polynomial ($1/n \leq s^2 \leq 3/(n+2)$) of degree at least 9 must have $f_7 = 0$.

3.2. Spherical Designs

In [9] and [15] Nikova and the author proved a general assertion for the zero Gegenbauer coefficient $f_{\tau+1} = 0$ as follows.

Theorem 3.4. Let $f(t)$ be a $B_{n,\tau}$ -extremal ($n \geq 3$, $\tau \geq 4$) polynomial of degree $k \geq \tau + 1$. If $L(f) < R(n, \tau + 1)$ ($R(n, \tau + 1)$ being the bound (3) for spherical $(\tau + 1)$ -designs on S^{n-1}), then $f_{\tau+1} = 0$ in the Gegenbauer expansion $f(t) = \sum_{i=0}^k f_i P_i^{(n)}(t)$.

In particular, Theorem 3.4 shows that Theorem 2.7 cannot be improved. Also, it shows that $B_{n,\tau}$ -extremal polynomials of degree $\tau + 1$ do not exist.

For the coefficient $f_{\tau+2}$ we use the analogs of Theorems 3.1 and 3.2. So, we search for extremal polynomials of degree $\tau + 3$ having $f_{\tau+1} = f_{\tau+2} = 0$.

4. Method for Finding Extremal Polynomials

4.1. Spherical Codes

Following Sections 2.1 and 3.1 we look for $A_{n,s}$ -extremal polynomials having degree $h(n, s) + 3$ or 4. These must have the form (see (6) and (7))

$$f(t) = A^2(t)G(t)(t - \xi), \quad (8)$$

where $\deg(A) = r(n, s) + 1$, $s \leq \xi < 1$, and

$$G(t) = \begin{cases} (t - p)^2 + q(s - t)(1 + t) & \text{if } \deg(f) \text{ is odd,} \\ (t + 1)(t - p)^2 + q(s - t)(t - r^2) & \text{if } \deg(f) \text{ is even,} \end{cases}$$

$0 < q < 1$, $p, r \in [-1, s]$. Since $L(af) = L(f)$ for any positive a , it may be assumed that the leading coefficient of $A(t)$ equals 1.

Finding suitable polynomials for Theorems 3.1 and 3.2. We obtain equations $f_i = 0$ for several consecutive indices i . Then we can express several coefficients of $A(t)$ by p, q, r, ξ , and the dimension n .

Next we consider the rational function

$$F = \frac{f(1)}{f_0}$$

depending on the remaining (after the expressing) parameters. The coefficient f_0 can be computed by the following standard formula [9], [15], [16]:

$$f_0 = a_0 + \frac{a_2}{n} + \frac{3a_4}{n(n+2)} + \cdots = a_0 + \sum_{i=1}^{[k/2]} \frac{(2i-1)!! a_{2i}}{n(n+2) \cdots (n+2i-2)}, \quad (9)$$

where $f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k$.

Writing the partial derivatives of the function F we obtain (after some trivial calculations) a system of equations that are linear with respect to some unknown parameters. We now express these parameters by the remaining. In fact, we can express all coefficients of $A(t)$.

Finally, we have to search for two, three or four still unknown parameters. It does not seem possible to apply further analytical methods. So we use a computer to determine the best values of the remaining parameters that minimize the function F under conditions (A1) and (A2). Since the unknowns are in a bounded domain, we can apply a Monte Carlo method very effectively, computing the values of F at suitable points.

4.2. Spherical Designs

As remarked, we search for $B_{n,\tau}$ -extremal polynomials of degree $\tau + 3$ having the following form:

$$f(t) = A^2(t)G(t), \quad (10)$$

where $\deg(A) = [\tau/2] + 1$, $0 < q < 1$, $p \in [-1, 1]$, and

$$G(t) = \begin{cases} q(1+t) + 1-t & \text{if } \tau \text{ is even,} \\ q(t-p)^2 + 1-t^2 & \text{if } \tau \text{ is odd.} \end{cases}$$

Following Section 3.2, we require $f_{\tau+1} = f_{\tau+2} = 0$ ($f_{\tau+3} < 0$ is equivalent to $q < 1$). We again take the leading coefficient of $A(t)$ to be equal to 1. The next two coefficients of $A(t)$ can be expressed by the equations $f_{\tau+1} = f_{\tau+2} = 0$.

As in Section 4.1 we examine the function $F = f(1)/f_0$. Again we can express all the remaining coefficients of $A(t)$ by setting the partial derivatives of the function F equal to 0.

Finally, it remains to find only one (q) or two (p and q) parameters maximizing F . These can be easily determined by a computer.

The $B_{n,\tau}$ -extremal polynomials we have found in this way are in the range $n_1(\tau) \leq n \leq n_2(\tau)$. In some cases we obtain the same bounds as (3) (for example, when $\tau = 5$). In fact, in this case we obtain infinitely many nonproportional extremal polynomials. Such bounds can be improved by investigation on the structure of the putative codes that would achieve them [4], [5], [10], [11].

5. On the Extremality

We begin this section with yet another definition for extremality.

Definition 5.1. A polynomial $f(t) = \sum_{i=0}^k a_i t^i \in A_{n,s}$ (or $B_{n,\tau}$) is called local $A_{n,s}$ -extremal (local $B_{n,\tau}$ -extremal) if a neighborhood B of the point $(a_0, a_1, \dots, a_k) \in \mathbf{R}^{k+1}$ exists such that one has $L(f) \leq L(g)$ (resp. $L(f) \geq L(g)$) for any polynomial $g(t) = \sum_{i=0}^k b_i t^i \in A_{n,s}$ ($B_{n,\tau}$) such that $(b_0, b_1, \dots, b_k) \in B$.

The next theorem shows that Definitions 2 and 5.1 are essentially equivalent.

Theorem 5.2. A polynomial $f(t) \in A_{n,s}$ is $A_{n,s}$ -extremal if and only if it is local $A_{n,s}$ -extremal.

Proof. Any extremal polynomial is local extremal by definition. Suppose that $f(t) = \sum_{i=0}^k a_i t^i \in A_{n,s}$ is local $A_{n,s}$ -extremal, but not $A_{n,s}$ -extremal. Then a polynomial $Q(t) \in A_{n,s}$ of degree at most k exists such that $L(Q) < L(f)$.

We now consider polynomials ($\alpha > 0$)

$$H(t; \alpha) = f(t) + \alpha Q(t) = \sum_{i=0}^k b_i(\alpha) t^i.$$

We have $H(t; \alpha) \in A_{n,s}$, and, moreover, $L(Q) < L(H) < L(f)$. It is clear that we can choose α to be small enough, such that $(b_0(\alpha), b_1(\alpha), \dots, b_k(\alpha)) \in B$. Thus, we obtain a contradiction with the local $A_{n,s}$ -extremality of $f(t)$. $\square$

The proof for spherical designs is parallel to the last one. Thus we have:

Theorem 5.3. *A polynomial $f(t) \in B_{n,\tau}$ is $B_{n,\tau}$ -extremal if and only if it is local $B_{n,\tau}$ -extremal.*

Therefore, we have to find local extremal polynomials. We point out that none, one, or infinitely many nonproportional extremal polynomials exist (for $(n, s), (n, \tau)$ fixed).

We consider the function $F = f(1)/f_0$ for polynomials from (8) and (10) (f_0 being expressed by (9)). Since $f_0 > 0$ in the bounded domain where $f(t) \in A_{n,s}$ (resp. $B_{n,\tau}$), this function is analytic in the domain mentioned. Therefore, it has its extremums (including infinity) at its critical points or on the boundary. Comparing the values at the critical points with those on the boundary, we can check if the corresponding polynomial is extremal. If an extremal polynomial exists and we have chosen the right form of our polynomial, then we find this extremal polynomial as a critical point.

In some cases we obtain extremums on the boundary $q = 1$ (see (8) and (10)). This shows that extremal polynomials of the degree and form considered do not exist.

We conclude this section by noting two interesting facts. We have obtained infinitely many nonproportional $B_{n,5}$ -extremal polynomials of degree 7. Moreover, we could not find any $B_{n,5}$ -extremal polynomial of degree 8. Next, all $A_{n,s}$ -extremal polynomials we have found vanish at s . However, we have not proved that this is true in general.

6. Some Examples

The bounds we have found for the kissing numbers $\tau_n = \tau(n, \frac{1}{2})$ (in dimensions $4 \leq n \leq 32$) were published in [9], [13], and [14]. We improve the Odlyzko–Sloane bounds [32] by one in dimensions 19, 21, and 23 [13] and obtain substantial improvements of (1) in dimensions $25 \leq n \leq 32$ [9], [14]. In [12] we give some $A_{n,1/2}$ -extremal polynomials of degrees 9, 10, and 11.

In this section we give some improvements on the bound (1) for $\tau(n, s)$ where $s = 1/\sqrt{5}$ and $s = 1/\sqrt{6}$, $3 \leq n \leq 10$. We show the degree of the extremal polynomials we have found, the corresponding number of double zeros, and the indices of the zero Gegenbauer coefficients. In some cases our improvements are not in the integral part, i.e., we have not obtained new bounds in these cases.

If an upper bound which is an integer number is found, then it is possible to find the distance distribution [7], [10], [17] of a code which attains this bound. In some cases (see [10]) this shows that such a code does not exist (see the last paragraph of Section 4.2).

The bound $\tau(3, 1/\sqrt{5}) \leq 12$ (from Table 1) is tight (the icosahedron attains it). However, the bound $\tau(4, 1/\sqrt{6}) \leq 20$ (from Table 2) can be improved by 1. Indeed, a code $W \subset S^3$ with 20 points and a minimum distance $d(W) = 1/\sqrt{5}$ would be a tight spherical 5-design [5], [20]. Since such a design does not exist [5], we obtain $\tau(4, 1/\sqrt{6}) \leq 19$. A similar argument shows that $\tau(n, 1/\sqrt{n+2}) \leq n(n+1) - 1$ instead of $n(n+1)$ unless $n+2$ is a square of an odd integer [11].

Table 1. Upper bounds for $\tau(n, 1/\sqrt{5})$, $3 \leq n \leq 9$ (R is the number of the double zeros of the corresponding extremal polynomial).

n	Bound (1)	New bound	$\deg(f)$	R	$f_i = 0$
3	12	12	4	1	—
4	22.158	22.133	8	2	$f_7 = 0$
5	38.09	37.698	9	3	$f_6 = f_7 = f_8 = 0$
6	62.803	61.213	9	3	$f_6 = f_7 = f_8 = 0$
7	101.305	97.72	9	3	$f_6 = f_7 = f_8 = 0$
8	160.68	156.28	9	3	$f_6 = f_7 = f_8 = 0$
9	245.172	244.085	10	3	$f_7 = f_8 = 0$

Table 2. Upper bounds for $\tau(n, 1/\sqrt{6})$, $3 \leq n \leq 10$ (R is the number of the double zeros of the corresponding extremal polynomial).

n	Bound (1)	New bound	$\deg(f)$	R	$f_i = 0$
3	10.982	10.943	7	2	$f_6 = 0$
4	20	20	4	1	—
5	33.013	33.136	8	2	$f_7 = 0$
6	52.34	52.246	8	2	$f_7 = 0$
7	80.141	79.229	9	3	$f_6 = f_7 = f_8 = 0$
8	120.29	118.005	9	3	$f_6 = f_7 = f_8 = 0$
9	178.7	175.02	9	3	$f_6 = f_7 = f_8 = 0$
10	265.22	261.23	9	3	$f_6 = f_7 = f_8 = 0$

As remarked, in [9], [15], and [16] we give new lower bounds for spherical designs. These are in the following dimensions:

- 6-designs—in dimensions $4 \leq n \leq 10$ [9], [16].
- 7-designs—in dimensions $5 \leq n \leq 7$ [9], [16].
- 8-designs—in dimensions $4 \leq n \leq 17$ [9], [16].
- 9-designs—in dimensions $4 \leq n \leq 14$ [9], [15].
- 10-designs—in dimensions $4 \leq n \leq 15$ [15].

7. Some Open Problems

In this section we give some open problems.

Open Problem 1. Find a polynomial (of degree depending on n and s) that improves the bound (1) in all dimensions.

Open Problem 2. Find exact conditions for $f_i = 0$. Prove (or disprove) that any $A_{n,s}$ -extremal polynomial must vanish at s .

Open Problem 3. Prove (or disprove) that our polynomials are extremal in general, i.e., no polynomials giving better bounds for spherical codes and designs exist.

Open Problem 4. Finding new upper bounds for $\tau(n+1, s)$, try to improve in some dimensions the upper bound for the maximum sphere packing density in n dimensions (see [17]) by the inequality

$$\Delta \leq (\sin \theta/2)^n \tau(n+1, s),$$

where $0 < \theta \leq \pi$, $\cos \theta = s$ (see [17, Chapter 9, Theorem 6], [23], and [29]).

Open Problem 5. Use suitable polynomials in (2) for obtaining restrictions on the distribution of the scalar products (x, y) , $x, y \in W$ for “very large” codes and “very small” designs. If $f(1)/f_0 = |W|$, then the distance distribution [7], [17, Chapter 14] of the code W can be computed [7], [9]–[11].

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