A rank number for a class of polygons.

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To Enrico Bompiani on his scientific Jubilee

Summary. - *Il the number of tangents of a polygon of real order n in real projective n-space which intersect an arbitrary $n-2$ — space is counted according to a certain convention this number is shown not to exceed $2n-2$.*

The r^{th} rank number of a differentiable curve of real order n in real projective n-space is defined to be the maximum number of osculating n-r-spaces of the curve which intersect an arbitrary linear r-1- subspace L_{r-1}. The first rank number was shown by SCHERK [4] to be n. For the case $n=3$, the second rank number was proved by JUEL [3] to be 4 at the International Mathematical Congress at Bologna. This is a special case of the n-^{1st}-rank number, i. e. the maximum number of tangents of a curve which intersect an arbitrary space L_{n-2}. The present paper shows that the maximum number of tangents, if these are counted with certain conventions, of a closed polygon of order n in n-space which intersect an arbitrary space L_{n-2} is $2n$-2. This is the generalization of a 3-dimensional result analogous to that of JUEL.

The proof depends on a structure theorem given by the author [1] for this class of polygons which is stated in § 1. The result is proved in § 3 for arbitrary spaces $L_{n\text{-}2}$ which are not within a certain nowhere dense set. In course of the proof the space L_{n-2} remains fixed but the polygon is replaced by a second polygon of order n for which the number of tangents which intersect L_{n-2} is not essentially less than the number of tangents of the original polygon which intersect L_{n-2}. This second polygon either intersects L_{n-2} in a point P or has one side less than the original polygon. In the first case the projection of the polygon from P has one side less than the original. The proof then follows by induction. The restriction to which the spaces L_{n-2} are subjected in § 3 is removed in the final section by displacing the polygon if it has no point in common with L_{n-2} and otherwise by a projection from such a point.

1. Definitions and Notation.

1.1. The symbol L_k, $0 \leqq k \leqq n$, is used to denote a projective k-space within the real projective n-space L_n. $A_1, A_2, \ldots, A_r$ will always designate the vertices of a closed polygon π, the sides of which are straight line segments A_1A_2, A_2A_3, ..., A_rA_1 within L_n, $n \geqq 1$. It is convenient to compute the subscripts of the vertices modulo r, e.g. A_1 and A_{r+1} represent the same point.

A point P of an intersection $L_r \frown \pi$ is *defined* to be an intersection point of the space L_r and the polygon π if P is a vertex of π or otherwise the only interior point of a side $A_i\, A_{i+1}$ of π within L_r.

If π is a closed polygon Hjelmslev [2] showed that all spaces L_{n-1} which do not contain any vertex of π either all intersect π in an even number of points or all intersect π in an odd number of points. In the first case the polygons are said to be *even*, in the second *odd*. Any open polygon A_1A_2, A_2A_3, ..., $A_{r-1}A_r$, $A_r \neq A_1$, can be closed by an appropriately chosen straight line segment A_rA_1 so that it becomes even. Frequent use will be made of even triangles constructed in this way.

A closed polygon π is *defined* to be a polygon of order n provided

π is not included in any subspace L_{n-1} of L_n

and

no space L_{n-1} intersects π in more than n points.

The symbol π_n is used exclusively to designate such a polygon.

If, for $n \geqq 2$, π_n is a polygon with vertices $B_1 = A_1$, B_2, ..., B_n, $B_{n+1} = A_{n+2}$, ..., A_r then a polygon with vertices A_1, A_2, ..., A_r is defined to be *inscribed* in π_n if A_i is an interior point of the side $B_{i-1}B_i$, $2 \leqq i \leq n+1$, while the side A_iA_{i+1} together with the segment $A_iB_iA_{i+1}$ of π_n forms an even triangle $2 \leqq i \leqq n$.

The following two results proved by the author in a previous paper [1] will be assumed.

1.2. *A polygon inscribed in polygon of order n is likewise a polygon of order n.*

1.3. *A polygon π_n either has exactly $n+1$ vertices or is inscribed in a polygon of order n.*

Moreover, the notation may be adjusted so that A_1 is an arbitrary vertex of π_n.

1.4. The symbol $[A, B, ...]$ is used to denote the linear subspace of a space L_n generated by points or point sets A, B, ...

If Q_1, Q_2, ..., Q_{k+1}, $0 \leqq k \leqq n$, are any $k+1$ distinct vertices of a polygon π_n then $[Q_1, Q_2, ..., Q_{k+1}]$ is a space L_k.

If this were false, a space generated by $n+1$ vertices of π_n could be constructed, contained within a space L_{n-1}, contrary to the definition of π_n.

2. The Tangents of the Polygons π_n.

2.1. If, for $n > 1$, A_{i-1}, A_i, A_{i+1} are consecutive vertices of a polygon π_n then by 1.4 $[A_{i-1}, A_i, A_{i+1}]$ is a space L_2. If $A_{i-1}A_{i+1}$ is the line segment which together with the segment $A_{i-1}A_iA_{i+1}$ of π_n forms an even triangle in L_2, then any line $L(A_i)$ with $A_i \in L(A_i) \subseteq L_2$ which does not contain an interior point of $A_{i-1}A_{i+1}$ is *defined* to be a tangent of π_n at A_i. The set of tangents $L(A_i)$ is a segment of the line pencil in L_2 through A_i bounded by $[A_{i-1}, A_i]$ and $[A_i, A_{i+1}]$.

2.2. *If a point P of a polygon π_2 is within a tangent $L(A_i)$ of π_2 then either $P \in A_{i-1}A_i$, or $P \in A_iA_{i+1}$.*

This follows from the order of π_2 by displacing $L(A_i)$.

2.3. *If, for $n \geqq 2$, P is an interior point of a side A_jA_{j+1} of a polygon π_n, then the projections of the sides $A_i A_{i+1}$, $i \neq j$ of π_n from P are the sides of a polygon π_{n-1}.*

If, for $n \geqq 3$, A'_i is the projection of a vertex A_i of π_n from P, $i \neq j$, $i \neq j+1$, then the projection of each tangent $L(A_i)$ of π_n from P is a tangent $L(A'_i)$ of π_{n-1} and each tangent $L(A'_i)$ is the projection of exactly one tangent $L(A_i)$.

PROOF. Each side A_iA_{i+1} of π_n, $i \neq j$, is projected into a line segment. To prove this it is sufficient to show $P \notin [A_i, A_{i+1}]$ for $i \neq j$. For $n=2$ this is clear because of the order of π_2. If, for $n > 2$, $P \in [A_i, A_{i+1}]$ then $[A_i, A_{i+1}, A_j, A_{j+1}]$ would have at most dimension 2 and so by 1.4 would contain at most three distinct vertices. Hence, as $i \neq j$ either $A_i = A_{j+1}$ or $A_{i+1} = A_j$. Consequently, A_iA_{i+1}, A_jA_{j+1} would be collinear which by 1.4 would imply $i = j$ which we assume not to be the case and so $P \notin [A_i, A_{i+1}]$.

The set of projections of the sides A_iA_{i+1}, $i \neq j$, is a closed polygon π as A_j and A_{j+1} are projected into the same point. Moreover π cannot be within a hyperplane of the projected space for then π_n would be within a hyperplane of the original space contrary to its definition.

Where L_{n-2} is any hyperplane of the projected space, to prove π has order $n-1$, we must show that L_{n-2} intersects it in at most $n-1$ points. Let L_{n-1} be the hyperplane of the original space which is projected into L_{n-2}. Every intersection point of L_{n-2} and π is the projection of an intersection point of L_{n-1} and π_n as, by the definition of π, its vertices are projections of the vertices of π_n. If $A_j \notin L_{n-1}$ then L_{n-1} intersects π_n in P

and so intersects π_n in at most $n-1$ points different from P. P not being projected into a point of π, it follows then that L_{n-2} intersects π in at most n—1 points. If $A_j \in L_{n-1}$ then $A_{j+1} \in L_{n-1}$ and both these vertices are projected into a single vertex of π within L_{n-2}. Again this implies that L_{n-2} intersects π in at most n—1 points. Therefore π is a polygon π_{n-1}.

If, for the vertex A_i of π_n, $i \neq j$, $i \neq j+1$, then $A_j A_{j+1}$ is not within the segment $A_{i-1} A_i A_{i+1}$ of π_n. We have just proved that the projection of this segment is a segment $A'_{i-1}A'_iA'_{i+1}$ of π_{n-1}. If $n > 2$, $[A'_{i-1}, A'_i, A'_{i+1}]$ is a space L_2 by 1.4. Hence $P \notin [A_{i-1}, A_i, A_{i+1}]$ and so the even triangle which contains the segment $A_{i-1}A_iA_{i+1}$ of π_n is projected into the even triangle which contains the segment $A'_{i-1}A'_iA'_{i+1}$ of π_{n-1}. It follows then from its definition that a tangent $L(A_i)$ of π_n is projected into a tangent $L(A'_i)$ of π_{n-1} and also that each tangent $L(A'_i)$ is the projection of a single tangent $L(A'_i)$ is the projection of a single tangent $L(A_i)$ of π_n. This completes the proof.

3. A Nowhere Dense Set of Spaces L_{n-2}.

3.1. For a given polygon π_n, $n > 1$, $S(\pi_n)$ is *defined* to be the set of all spaces L_{n-2} for which $L_{n-2} \frown L_1 \neq \emptyset$ for at least one line L_1 which contains a side of π_n. $S(\pi_n)$ is nowhere dense. If L_{n-2} is a space not within $S(\pi_n)$, then at most one tangent $L(A_i)$ at a vertex A_i intersects L_{n-2} for otherwise all the tangents at the vertex A_i including $[A_{i-1}, A_i]$, $[A_i, A_{i+1}]$ would intersect L_{n-2} contrary to the fact that $L_{n-2} \notin S(\pi_n)$.

3.2. *If, for a polygon π_n, $n \geqq 2$, L_{n-2} is a space for which $L_{n-2} \notin S(\pi_n)$ then at most $2n$—2 tangents of π_n intersect L_{n-2}.*

The proof is preceded by two lemmas.

Lemma 1. Where T is an arbitrary nowhere dense set of spaces L_{n-2} it is sufficient to prove 3.2 for all L_{n-2} for which $L_{n-2} \notin T \cup S(\pi_n)$.

Proof of lemma. Let L_{n-2} be a space for which $L_{n-2} \notin S(\pi_n)$. T, $S(\pi_n)$ both being nowhere dense $T \cup S(\pi_n)$ is also nowhere dense. Therefore, a sequence L^{μ}_{n-2} exists, $L^{\mu}_{n-2} \notin T \cup S(\pi_n)$, which approaches L_{n-2}. If A_i is a vertex of π_n for which a tangent $L(A_i)$ intersects L_{n-2} then we show that if L^{μ}_{n-2} is sufficiently close to L_{n-2} that a tangent $L'(A_i)$ intersects L^{u}_{n-2}. Because $[A_{i-1}, A_i] \frown L_{n-2} = \emptyset$, $[A_i, L_{n-2}] \frown [A_{i-1}, A_i, A_{i+1}]$ is a line. Moreover, as this is the only line within $[A_{i-1}, A_i, A_{i+1}]$ which contains A_i and intersects L_{n-2} it must be the tangent $L(A_i)$. If $A_{i-1} A_{i+1}$ be the third side of the even triangle which contains the segment $A_{i-1}A_iA_{i+1}$ of π_n then by the definition of a tangent $L(A_i)$ cannot intersect $[A_{i-1}, A_{i+1}]$ in a point interior to $A_{i-1}A_{i+1}$. A_{i-1}, $A_{i+1} \notin L(A_i)$ for otherwise $[A_{i-1}, A_i]$ or $[A_i, A_{i+1}]$ would intersect L_{n-2} contrary to the fact that $L_{n-2} \notin S(\pi_n)$. It follows then, as $L^{\mu}_{n-2} \notin S(\pi_n)$, that provided L^{μ}_{n-2} is sufficiently close to L_{n-2} $[A_i, L^{\mu}_{n-2}] \cap [A_{i-1}, A_i, A_{i+1}]$ does not intersect $[A_{i-1}, A_{i+1}]$ in an inner point of $A_{i-1} A_{i+1}]$ Therefore this intersection which is a line which intersects L^{μ}_{n-2} must be a tangent $L'(A_i)$. Consequently if L^{μ}_{n-2} is sufficiently close to L_{n-2} the number of tangents of π_n which intersect L^{μ}_{n-2} is at least as great as the number of tangents which intersect L_{n-2}. If, then, 3.2 is true for all spaces L^{μ}_{n-2}, it is also true for L_{n-2}, and the lemma is established.

LEMMA 2. The number of tangents of π_n which intersect a space L_{n-2}, $L_{n-2} \notin S(\pi_n)$, is even.

PROOF OF LEMMA. As no point s of π_n is within L_{n-2}, $[s, L_{n-2}]$ is a space $L_{n-1}(s)$. As s moves continuously around π_n in a fixed direction $L_{n-1}(s)$ moves continuously within the pencil through L_{n-2}. As no line $[A_i, A_{i+1}]$ intersects L_{n-2}, $L_{n-1}(s)$ cannot experience a reversal of direction within its pencil except when s is a vertex A_i and then only if $L_{n-1}(A_i)$ supports the even triangle containing the segment $A_{i-1}A_iA_{i+1}$ of π_n. It follows, then, from the definition of a tangent that $L_{n-1}(s)$ reverses its direction if and only if s is a vertex at which a tangent intersects L_{n-2}. As π_n is closed the total number of reversals of direction of $L_{n-1}(s)$ is even. This proves the number of tangents which intersect L_{n-2} is even.

The proof of 3.2 is by induction according to the number of vertices of π_n. π_n has at least $n+1$ vertices as it is not contained in a hyperplane. By 3.1 at most one tangent at each vertex intersects L_{n-2}. Therefore, if $r=n+1$, there are at most $n+1$ tangents which intersect L_{n-2}. In this case at most $2n-2$ tangents intersect L_{n-2} as $n+1 \leqq 2n-2$ for $n \geqq 3$ while for $n=2$ at most 2 tangents contain L_0 as this number cannot exceed 3 and must be even by Lemma 2. Thus 3.2 is proved for $r=n+1$.

We assume it true for all polygons π_n with $r-1$ vertices, $r>n+1$. If, then, a polygon π_n has r vertices $A_1, A_2, \ldots, A_r$ it can, by 1.3, be inscribed in a polygon π'_n with $r-1$ vertices $A_1=B_1, B_2, \ldots, B_n, B_{n+1}=A_{n+2}, A_{n+3}, \ldots, A_r$ where A_i is an interior point of the side $B_{i-1}B_i$, $2 \leqq i \leqq n+1$, of π'_n while each triangle T_i consisting of the side A_iA_{i+1} of π_n and the segment $A_iB_iA_{i+1}$ of π'_n, $2 \leqq i \leqq n$, is even. We now assume that L_{n-2} contains no point of any line through a side of π'_n nor any point of any line through two distinct vertices of π_n. As this only excludes L_{n-2} from a nowhere dense set it does not, by Lemma 1, affect the generality of the proof. With this restriction no point of any side of a triangle T_i, $2 \leqq i \leqq n$, can be within L_{n-2}.

We first prove the result with the assumption that a point P of L_{n-2} exists within the interior of a triangle T_j. We displace the side A_jA_{j+1} of T_j continuously so that A_{j+1} remains on the side B_jA_{j+1} of T_j to a position $A_jA'_{j+1}$ so that $P \in A_jA'_{j+1}$ and displace $A_{j+1}A_{j+2}$ continuously to $A'_{j+1}A_{j+2}$.

The displaced position $A_1A_2 \ldots A_jA'_{j+1}A_{j+2} \ldots A_r$ of π_n is inscribed in π'_n and so by 1.2 is a polygon of order n which we designate by $\bar{\pi}_n$. If A_i is a vertex of π_n $i \neq j$, $i \neq j+1$, $i \neq j+2$, the segment $A_{i-1}A_iA_{i+1}$ of π_n is also a segment of $\bar{\pi}$ and so every tangent $L(A_i)$ of π_n is also a tangent of $\bar{\pi}_n$. If $n = 2$, L_{n-2} becomes a single point which must be P. In this case, by 2.2, no tangent $L(A_i)$ can contain P as P is an interior point of $A_jA'_{j+1}$. Hence at most trhee tangents of π_n contain P all of which must be at one of the vertices A_j, A_{j+1}, A_{j+2}. By the second lemma the total number of tangents which contain P is even and so can be at most 2 which establishes the result in this case. We may assume therefore that $n > 2$. Because P is an interior point of the side $A_jA'_{j+1}$ of $\bar{\pi}_n$, the projection of $\bar{\pi}_n$ from P is, by 2.3, a polygon π_{n-1}. As $P \in L_{n-2}$, the projection of L_{n-2} is a space L_{n-3}. As A_j, A'_{j+1} are projected into a single vertex of π_{n-1}, π_{n-1} has $r-1$ vertices. No line containing a side of π_{n-1} intersects L_{n-3}. This is clear if such a line is the projection of a line which contains a side of π_n for we assume that none of these lines intersect L_{n-2}. The only other line which contains a, side of π_{n-1} is the projection of $[A'_{j+1}, A_{j+2}]$. As P is an interior point of A_jA_{j+1} this is the same as the projection of $[A_j, A_{j+2}]$ from P. We are assuming that no line containing two distinct vertices of π_n intersects L_{n-2}. Therefore $[A_j, A_{j+2}]$ does not intersect L_{n-2} and its projection does not intersect L_{n-3}. We may now apply the induction assumption to L_{n-3} and π_{n-1}. Hence at most $2n-4$ tangents of π_{n-1} intersect L_{n-3}. It follows from 2.3 that every tangent $L(A_i)$ is projected into a tangent of π_{n-1} and that different tangents are projected into different tangents. Hence at most $2n-4$ such tangents intersect L_{n-2}. Consequently at most $2n-1$ tangents of π_n including those of the type $L(A_j)$, $L(A_{j+1})$, $L(A_{j+2})$ intersect L_{n-2}. It follows then fron the second lemma that at most $2n-2$ tangents of π_n intersect L_{n-2}.

This completes the proof if L_{n-2} contains an inner point of at least one triangle T_i. Accordingly, we now assume that no inner point of any triangle T_i, $2 \leqq i \leqq n$, is a point of L_{n-2}. L_{n-2} intersects the plane $[A_i, B_i, A_{i+1}]$ of a triangle T_i in at least one point Q. L_{n-2} cannot intersect this plane in more than one point. This is trivial if $n = 2$ and clear for $n > 2$ for otherwise the lines containing a side of T_i would all intersect L_{n-2} contrary to the assumption that no line containing either a side of π_n or of π'_n intersects L_{n-2}. If, for a vertex A of T_i, $[Q, A]$ supports T_i, A is defined to be a support vertex otherwise an intersection vertex. As Q is not within the interior of T_i, T_i has two support vertices and one intersection vertex. Thus of the $3n-3$ vertices of the triangles T_i, $n-1$ are intersection vertices. (A point A_i common to T_{i-1}, T_i is counted twice in this classification once as a vertex of T_{i-1} and again as a vertex of T_i).

We now show that if, for a vertex A_j, $2 \leqq j \leqq n+1$, a tangent $L(A_j)$ intersects L_{n-2} then A_j is an intersection vertex of one of the triangles T

Because no line which contains a side of either π_n or π'_n intersects L_{n-2}, $[L_{n-2}, A_j]$ is a hyperplane which intersects the line segment $B_{j-1}A_jB_j$ in the single point A_j and the plane $[A_{j-1}, A_j, A_{j+1}]$ in a line. If a tangent $L(A_j)$ intersects L_{n-2} this line must be $L(A_j)$ and $[L_{n-2}, A_j]$ can only contain the one point namely A_j of the even triangle which contains the segment $A_{j-1}A_jA_{j+1}$ of π_n. In other words, as the space is locally affine, $[L_{n-2}, A_j]$ supports $A_{j-1}A_jA_{j+1}$ at A_j and separates it from one of the segments $B_{j-1}A_j$, A_jB_j. If $j=2$, $B_1A_2 = A_1A_2$ and so $[L_{n-2}, A_2]$ separates A_2B_2 from A_2A_3. This means A_2 is an intersection vertex of the triangle T_2. Likewise if $j = n+1$ A_{1+n} is an intersection vertex of T_{n+1}. For $2 < j < n+1$, the segments $B_{j-1}A_j$, A_jB_j are sides of T_{j-1}, T_j respectively an so A_j is and intersection vertex of T_{j-1} or T_j according as $[L_{n-2}, A_j]$ separates $B_{j-1}A_j$ or A_jB_j from $A_{j-1}A_jA_{j+1}$. Thus in every case A_j is an intersection vertex of a triangle T_i.

The set of vertices of the triangles T_i consists of $A_2, A_3, \ldots, A_{n+1}$ together with $B_2, B_3, \ldots, B_n$. It follows then from the last paragraph that if p be the number of tangents $L(A_j)$, $2 \leqq j \leqq n+1$, which intersect L_{n-2}, at most $n-1-p$ of the $n-1$ vertices $B_2, B_3, \ldots, B_n$ are intersection vertices of the triangles T_i. Hence at least $n-1-(n-1-p)=p$ of these are support vertices. It follows that at least p tangents $L(B_i)$, $2 \leqq i \leqq n$, of π'_n intersect L_{n-2}. Except for the p tangents of π_n at the vertices $A_2, A_3, \ldots, A_{n+1}$ which intersect L_{n-2} every tangent of π_n which intersects L_{n-2} is also a tangent of π'_n which intersects L_{n-2}. Hence the number of tangents of π'_n which intersect L_{n-2} is at least as great as the number of those tangents of π_n which intersect L_{n-2}. However, π'_n has $r-1$ vertices and no line containig one of its sides intersects L_{n-2}. Hence, by the induction assumption, at nost $2n-2$ tangents of π'_n intersect L_{n-2}. This implies that at most $2n-2$ tangents of π_n intersect L_{n-2} and completes the proof.

4. The Rank Number.

4.1. In this section the restriction that $L_{n-2} \notin S(\pi_n)$, imposed in the previous section, is removed. The result we shall prove uses in fact a definition analogous to the definition of an intersection point of a polygon and a space L_r. Let $E(\pi_n, L_{n-2})$ be the set of all tangents $[A_i, A_{i+1}]$ of a polygon π_n which intersect L_{n-2} together with all tangents $L(A_j)$ of π_n which intersect L_{n-2} but for which $[A_{j-1}, A_j]$, $[A_j, A_{j+1}]$ do not intersect L_{n-2}. Our final rank number result which includes 3.2 is:

4.2. *No set $E(\pi_n, L_{n-2})$ contains more than $2n-2$ tangents.*

The proof depends on the following lemma.

If, for a polygon π_n, $n \geqq 2$, with vertices $A_1, A_2, \ldots, A_i, \ldots, A_r$ the sides $A_{i-1}A_i$, A_iA_{i+1} are displaced continuously to $A_{i-1}A'_i$, A'_iA_{i+1} so that the segment $A_{i-1}A'_i$ contains $A_{i-1}A_i$ as a subsegment, then the polygon $A_1A_2 \dots A_{i-1}A'_iA_{i+1} \dots A_r$ has order n provided A'_i is sufficiently close to A_i.

PROOF OF LEMMA. If π_n has exactly $n+1$ vertices the result follows from the fact that if π_n is even (odd) a sufficiently small displacement of π_n is even (odd) while the vertex points of π_n remain linearly independent. If $r > n+1$ the notation may be adjusted so that A_i is the vertex A_2. By 1.3 π_n may be inscribed in a polygon π'_n with vertices $B_1 = A_1$, B_2, ..., $B_{n+1} = A_{n+2}$, ..., A_r. If π'_n is kept fixed while A_2 is displaced to an interior point A'_2 of the side B_1B_2 of π'_n and the sides A_1A_2, A_2A_3 displaced to $A_1A'_2$, A'_2A_3 the resultant polygon will be inscribed in π'_n provided A'_2 is sufficiently close to A_2 and so by 1.2 be a polygon of order n. Thus the lemma is proved.

We assume first that $L_{n-2} \cap \pi_n = \emptyset$. If no line containing a side of π_n intersects L_{n-2} the result becomes 3.2. We suppose then the result is true if at most $p-1$, $p > 1$, of the sides of π_n are within lines which intersect L_{n-2} and proceed by induction. Let π_n now be a polygon for which exactly p lines containing a side of π_n intersect L_{n-2}. At least one line $[A_j, A_{j+1}]$ intersects L_{n-2}. As $A_j \notin L_{n-2}$, $[A_j, L_{n-2}]$ is a space L_{n-1}. $A_{j+1} \in L_{n-1}$ but as π_n is not contained in any hyperplane a segment $A_i \dots A_jA_{j+1} \dots A_{i+k-1}$ of π_n exists in L_{n-1} for which A_{i-1}, $A_{i+k} \notin L_{n-1}$. Let $A_1A_2 \dots A_{i-1}A'_iA_{i+1} \dots A_r$ be a polygon π'_n constructed as in the lemma, The line $[A'_i, A_{i+1}]$ does not intersect L_{n-2} for otherwise $[A'_i, A_{i+1}] \subseteq L_{n-1}$ and $[A_{i-1}, A_i] = [A_i, A'_i] \subseteq L_{n-1}$ contrary to the assumption $A_{i-1} \notin L_{n-1}$. Only $p-1$ lines which contain a side of π'_n intersect L_{n-2} for other than $[A'_i, A_{i+1}]$ every line which contains a side of π'_n also contains a side of π_n. Hence by the induction assumption $E(\pi'_n\ L_{n-2})$ contains at most $2n-2$ tangents. Every tangent of π_n except those at the vertices A_i, A_{i+1} is also a tangent of π'_n. Besides $[A_i, A_{i+1}]$ the only other tangent of π_n at the vertices A_i, A_{i+1} which could be within $E(\pi_n, L_{n-2})$ is $[A_{i+1}, A_{i+2}]$ which is also a tangent of π'_n. Thus apart from $[A_i, A_{i+1}]$ every tangent contained in $E(\pi_n, L_{n-2})$ is also contained in $E(\pi'_n, L_{n-2})$. Because $A_i \notin L_{n-2}$ A'_i can be chosen so close to A_i that $[A'_i, L_{n-2}]$ is a space L'_{n-1}. As $A_{i-1} \notin L_{n-1}$ and $A'_i \neq A_i$, $L'_{n-1} \neq L_{n-1}$. Therefore L'_{n-1} contains no point of the side A_iA_{i+1} of π_n. Because of the construction of π'_n A_iA_{i+1} together with the segment $A_iA'_iA_{i+1}$ of π'_n is an even triangle. Hence L'_{n-1} supports this triangle at A'_i. Therefore a tangent $L(A'_i)$ of π'_n different from $[A_i, A'_i]$, $[A'_i, A_{i+1}]$ intersects L_{n-2} and so is within $E(\pi'_n, L_{n-2})$. It follows that $E(\pi_n, L_{n-2})$ contains at most as many tangents as $E(\pi'_n, L_{n-2})$ and consequently contains at most $2n-2$ tangents. This proves the result if $L_{n-2} \cap \pi_n = \emptyset$.

To complete the proof for the remaining case in which $L_{n-2} \cap \pi_n \neq \emptyset$ it is sufficient to show, where $s(\pi_n)$ is the number of points of intersection of L_{n-2} and π_n which are not vertices and $t(\pi_n)$ the number of tangents in $E(\pi_n, L_{n-2})$, that $s(\pi_n) + t(\pi_n) \leqq 2n-2$. For $n = 2$ this is an immediate consequence of 2.2.

We assume the result proved for polygons π_{n-1}, $n > 2$, and proceed by induction. If a point Q of L_{n-2} exists in the interior of a side A_jA_{j+1} of π_n then by 2.3 the projection of π_n from Q is a polygon π_{n-1}. Let L_{n-3} be the projection of L_{n-2} from Q. If (π_{n-1}), $t(\pi_{n-1})$ be defined for π_{n-1} and L_{n-3} analogously to $s(\pi_n)$, $t(\pi_n)$ respectively then by the induction assumption $s(\pi_{n-1}) + t(\pi_{n-1}) \leqq 2n - 4$. Every one of the $s(\pi_{n-1})$ intersections of L_{n-3} and π_{n-1} which are not vertices is, by 2.3, the projection of exactly one similar intersection of L_{n-2} and π_n different from Q. Hence $s(\pi_n) - 1 \leqq s(\pi_{n-1})$. Except for $[A_j, A_{j+1}]$ every tangent of $E(\pi_n, L_{n-2})$ can be written as $L(A_i)$, $i \neq j$, $i \neq j+1$, as, besides, $[A_j, A_{j+1}]$, the only tangents $L(A_j)$, $L(A_{j+1})$ which could be within $E(\pi_n, L_{n-2})$ are $[A_{j-1}, A_j]$, $[A_{j+1}, A_{j+2}]$ which can also be written as $L(A_{j-1})$, $L(A_{j+2})$ respectively. Where A'_i is the projection of a vertex A_i, $i \neq j$, $i \neq j+1$ from Q every tangent $L(A'_i)$ of π_{n-1} is the projection of exactly one tangent $L(A_i)$ of π_n. Consequently $t(\pi_n) - 1 \leqq t(\pi_{n-1})$. By combining the above three results we obtain $s(\pi_n) - 1 + t(\pi_n) - 1 \leqq s(\pi_{n-1}) + t(\pi_{n-1}) \leqq 2n - 4$. Hence $s(\pi_n) + t(\pi_n) \leqq 2n - 2$ which establishes the result if $L_{n-2} \cap \pi_n$ contains at least one point which is not a vertex of π_n. If $L_{n-2} \cap \pi_n$ consists only of vertices let A_i be such a vertex. As no interior point of $A_{i-1}A_i$ is within L_{n-2}, $A_{i-1} \notin L_{n-2}$. Hence if $A_1 \dots A_{i-1}A'_iA_{i+1} \dots A_r$ be a polygon π'_n constructed as in the lemma, L_{n-2} contains the point A_i of the side $A_{i-1}A'_i$ of π'_n and no other point of this side. Hence $s(\pi'_n) \geqq 1$ and by the above result $s(\pi'_n) + t(\pi'_n) \leqq 2n - 2$. As $[A_i, A_{i+1}] \in E(\pi_n, L_{n-2})$ the only other tangent $L(A_{i+1})$ which could be within $E(\pi_n, L_{n-2})$ is $[A_{i+1}, A_{i+2}]$ which is also a tangent of π'_n. Thus apart from $[A_i, A_{i+1}]$ every tangent in $E(\pi_n, L_{n-2})$ is also in $E(\pi'_n, L_{n-2})$ and so $t(\pi_n) - 1 \leqq t(\pi'_n)$. As $s(\pi_n) = 0$ it now follows that $s(\pi_n) + t(\pi_n) = 1 + (t(\pi'_n) - 1) \leqq s(\pi'_n) + t(\pi'_n) \leqq 2n-2$. The result is now established in all cases and the proof is complete.

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