Some theorems on spinor-tensors.

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Summary. - *Some of the operators employed by A. Lichnerowicz in his study of spinor fields and propagators in general relativity are considered for arbitrary (r, s)-spinor (p, q)-tensor fields. The general properties of those operators are developed and applied to a Laplacian operator proposed by the author.*

INTRODUCTION

Recently LICHNEROWICZ [3], [4], [5], in his study of quantization in a curved space-time, has introduced and established a number of interesting results concerning operators defined on certain simple familiar quantum field theory spinor and spinor-tensor fields [1]. For instance, in the theory of DIRAC (spin 1/2) one encounters (1, 0) and (0, 1)-spinor fields; for the RARITA-SCHWINGER theory (spin (3/2), (1, 0)-spinor (0, 1)-tensor fields occur; and in the theory of PETIAU-DUFFIN-KEMMER (maximum spin 1), (1, 1)-spinor fields aee utilized. A comprehensive treatment of these theories within the framework of general relativity is given in [4]. In this paper, without reference to any particular physical theory, the general properties of these operators are developed for arbitrary (r, s)-spinor (p, q)-tensor fields in a four dimensional RIEMANNIAN manifold of hyperbolic normal signature. Conceivably, such spinor-tensor fields might be employed in physical theories considering particles of arbitrary spin. In §1 the basic notions about spinor-tensors are reviewed. The DIRAC adjoint, charge conjugation and related operators are studied in §2, and in the final section these operators are applied to a LAPLACIAN for (r, s)-spinor (p, q)-tensors recently proposed by the author [8].

§ 1. - Basic notions on spinor-tensors.

Let V_4 be a four dimensional differentiable manifold with a RIEMANNIAN metric $g_{\alpha\beta}(x^\lambda)$ of hyperbolic normal signature, and a space-time orientation ρ. The (1, 0)-spinors (*) at a point $x \in V_4$ are elements of a complex vector space $\mathbb{S}_x$ and the (0, 1)-spinors at this point are elements of the complex vector space $\mathbb{S}_x^*$ which is the dual of $\mathbb{S}_x$. By forming repeated tensor products of $\mathbb{S}_x$ and $\mathbb{S}_x^*$ one obtains the notion of an (r, s)-spinor, *i.e.*, an element of the

(*) A spinor of type (1, 0), *i. e.*, a contravariant spinor, will be referred to as a (1,0)-spinor. The same convention will be used for spinor-tensors of arbitrary type.

product space $\overset{r}{\otimes} \mathfrak{S}_x \overset{s}{\otimes} \mathfrak{S}_x^*$. If, in addition, the tangent and dual tangent spaces at $x \in V_4$, denoted by T_x and T_x^* respectively, are introduced into the tensor product one is led to the notion of an (r, s)-spinor (p, q)-tensor, *i.e.*, an element of the product space $\overset{r}{\otimes} \mathfrak{S}_x \overset{s}{\otimes} \mathfrak{S}_x^* \overset{p}{\otimes} T_x \overset{q}{\otimes} T_x^*$. In terms of its components an (r, s)-spinor (p, q)-tensor may be written as $\zeta^{a_1 \dots a_r, \omega_1 \dots \omega_p}_{b_1 \dots b_s, \lambda_1 \dots \lambda_q}$, or more briefly as $\zeta^{A, \Omega}_{B, \Lambda}$ where $A = (a_1 \dots a_r), \dots, \Lambda = (\lambda_1 \dots \lambda_q)$ are used as collective indices.

The four (4×4) DIRAC matrices are defined on V_4 by the equations

$$\gamma^a_{\alpha r}\gamma^r_{\beta b} + \gamma^a_{\beta r}\gamma^r_{\alpha b} = -2g_{\alpha\beta}e^a_b \tag{1}$$

where e^a_b is the (4×4) identity matrix. By adopting an orthonormal frame the metric tensor reduces to the MINKOWSKI metric $\eta_{\alpha\beta}$ and (1) merely summarizes the well-known properties of the DIRAC matrices [1]. In this paper we will choose, as in [4], the following representation for the $\gamma^a_{\alpha b}$:

$$\gamma^a_{0b} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad \gamma^a_{1b} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^a_{2b} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^a_{3b} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \tag{2}$$

In addition to these matrices we introduce a pair of real (4×4) matrices β^a_b and α^a_b such that

$$\beta^a_r\beta^r_b = e^a_b \tag{3}$$

and

$$\alpha^a_r\alpha^r_b = e^a_b\,. \tag{4}$$

By virtue of our choice of representation of the DIRAC matrices these conditions may be satisfied by choosing $\alpha^a_b = \gamma^a_{1b}$ and $\beta^a_b = -i\gamma^a_{0b}$. The anticommutation relations (1) for the DIRAC matrices immediately yield

$$\alpha^a_r\beta^r_b = -\beta^a_r\alpha^r_b \tag{5}$$

Let A be an arbitrary matrix. We denote by A^* its complex conjugate, by A^T its transpose, and by $\tilde{A} \overset{\text{def}}{=} A^{T*}$ its adjoint. We state now two results which are proven in [4]:

$$\tilde{\gamma}^{a}_{ar}\beta^{r}_{b} = -\beta^{a}_{r}\gamma^{r}_{ab} \tag{6}$$

$$\gamma^{a}_{ar}\alpha^{r}_{b} = \overset{*}{\alpha}{}^{a}_{r}\gamma^{r}_{ab}. \tag{7}$$

All the concepts and details indicated in the next two sections are presented only for V_4 with signature $(+ - - -)$. They could immediately be extended to higher dimensional V_n, for instance to the unified field theories of JORDAN–THIRY $n = 5$ $(+ - - - -)$ [7] and RENAUDIE $n=6$ $(- - + - - -)$ [6], provided one could find a representation of the DIRAC matrices such that equations (1)-(7) would be satisfied. The existence of such matrices for each n and signature is not considered in this paper.

§ 2. - The Dirac Adjoint and Charge conjugation.

DEFINITION. - The DIRAC adjoint $\mathfrak{A}$ is the anti-linear mapping of an (r, s)-spinor (p, q)-tensor onto an (s, r)-spinor-(p, q)-tensor defined by

$$\mathfrak{A}\zeta^{a_1 \dots a_r, \Omega}_{b_1 \dots b_s, \Lambda} \overset{\text{def}}{=} \rho\beta^{d_1}_{b_1} \dots \beta^{d_s}_{b_s}\tilde{\zeta}^{c_1 \dots c_r, \Omega}_{d_1 \dots d_s, \Lambda}\beta^{a_1}_{c_1} \dots \beta^{a_r}_{c_r} \tag{8}$$

where ρ is the space-time orientation of V_4.

The following theorem is a generalization of the results of LICHNEROWICZ [4] and summarizes the most important properties of the DIRAC adjoint.

THEOREM 1. - Let $\zeta^{A, \Omega}_{B, \Lambda}$ be an (r, s)-spinor (p, q)-tensor. Then we have

$$\mathfrak{A}^2(\zeta^{A, \Omega}_{B, \Lambda}) = \zeta^{A, \Omega}_{B, \Lambda} \tag{9}$$

$$\mathfrak{A}\left(\sum_{i=1}^{r} \gamma^{\alpha a_i}_{c}\zeta^{a_1 \dots c \dots a_r, \Omega}_{B, \Lambda}\right) = -\sum_{i=1}^{r} \mathfrak{A}(\zeta^{a_1 \dots c \dots a_r, \Omega}_{B, \Lambda})\gamma^{\alpha a_i}_{c} \tag{10}$$

$$\mathfrak{A}\left(\sum_{j=1}^{s} \zeta^{A, \Omega}_{b_1 \dots d \dots b_s, \Lambda}\gamma^{\alpha d}_{b_j}\right) = -\sum_{j=1}^{s} \gamma^{\alpha d}_{b_j}\mathfrak{A}(\zeta^{A, \Omega}_{b_1 \dots d \dots b_s, \Lambda}) \tag{11}$$

$$\mathfrak{A}\left(\sum_{i=1}^{r}\sum_{j=1}^{s} \gamma^{\alpha a_i}_{c}\zeta^{a_1 \dots c \dots a_r, \Omega}_{b_1 \dots d \dots b_s, \Lambda}\gamma^{\beta d}_{b_j}\right) = \sum_{i=1}^{r}\sum_{j=1}^{s} \gamma^{\beta d}_{b_j}\mathfrak{A}(\zeta^{a_1 \dots c \dots a_r, \Omega}_{b_1 \dots d \dots b_s, \Lambda})\gamma^{\alpha a_i}_{c} \tag{12}$$

$$\mathfrak{A}(\nabla_\alpha\zeta^{A, \Omega}_{B, \Lambda}) = \nabla_\alpha(\mathfrak{A}\zeta^{A, \Omega}_{B, \Lambda}). \tag{13}$$

PROOF. - Equation (9) is a direct consequence of (8) and (3).

To establish (10) we note that $\mathcal{A}$ and the summation sign are permutable. Then using (8) and (6) the result follows. For instance for a (2, s)-spinor (p, q)-tensor one has

$$\mathcal{A}(\gamma_c^{\alpha a_1}\zeta_{B}^{c a_2,}{}_{;\Lambda}^{\Omega} + \gamma_c^{\alpha a_2}\zeta_{B}^{a_1 c,}{}_{;\Lambda}^{\Omega}) = \mathcal{A}(\gamma_c^{\alpha a_1}\zeta_{B}^{c a_2;}{}_{;\Lambda}^{\Omega}) + \mathcal{A}(\gamma_c^{\alpha a_2}\zeta_{B}^{a_1 c,}{}_{;\Lambda}^{\Omega})$$

$$= \rho\beta_{b_1}^{d_1} \dots \beta_{b_s}^{d_s}(\tilde{\zeta}_{d_1 \dots d_s}^{c_1 c}{}_{;\Lambda}^{,\Omega}\gamma_c^{\alpha c_2}\beta_{c_2}^{a_2}\beta_{c_1}^{a_1} + \zeta_{d_1 \dots d_s}^{c c_2}{}_{;\Lambda}^{,\Omega}\tilde{\gamma}_c^{\alpha c_1}\beta_{c_1}^{a_1}\beta_{c_2}^{a_2})$$

$$= -\mathcal{A}(\zeta_{B}^{a_1 c,}{}_{;\Lambda}^{\Omega})\gamma_c^{\alpha a_2} - \mathcal{A}(\zeta_{B}^{c a_2;}{}_{;\Lambda}^{\Omega})\gamma_c^{\alpha a_1}$$

$$= -\sum_{i=1}^{2} \mathcal{A}(\zeta_{B}^{a_1 \dots c \dots a_r;}{}_{;\Lambda}^{\Omega})\gamma_c^{\alpha a_i}.$$

Equation (11) is proven in the same way.

Equation (12) is a generalization of (10) and (11) which is obtained in the same manner. We note only that whereas (10) and (11) required the use of (6) once in each term, (12) by virtue of the double summation employs this relation twice in each term. Hence the negative sign appearing on the right-hand side of (10) and (11) does not occur in (12).

Equation (13) is proven for (1, 0)-spinor (0, q)-tensor in [4] and the same technique immediately extends to (r, s)-spinor (p, q)-tensors.

DEFINITION. - The charge conjugation $\mathcal{C}$ is the anti-linear mapping of an (r, s)-spinor (p, q)-tensor onto an (r, s)-spinor (p, q)-tensor defined by

$$(14) \qquad \mathcal{C}(\zeta_{b_1 \dots b_s,}^{a_1 \dots a_r,}{}_{\Lambda}^{\Omega}) \overset{\text{def}}{=} \alpha_{c_1}^{a_1} \dots \alpha_{c_r}^{a_r}\overset{*}{\zeta}{}_{d_1 \dots d_s,}^{c_1 \dots c_r,}{}_{\Lambda}^{\Omega}\alpha_{b_1}^{d_1} \dots \alpha_{b_s}^{d_s}.$$

The following theorem is also a generalization of the results of LICHNEROWICZ [4], and contains the basic properties of $\mathcal{C}$.

THEOREM 2. - Let $\zeta_{B,}^{A,}{}_{\Lambda}^{\Omega}$ be an (r, s)-spinor (p, q)-tensor. Then we have

$$(15) \qquad \mathcal{C}^2(\zeta_{B,}^{A,}{}_{\Lambda}^{\Omega}) = \zeta_{B,}^{A,}{}_{\Lambda}^{\Omega}$$

$$(16) \qquad \mathcal{C}\Big(\sum_{i=1}^{r} \gamma_c^{\alpha a_i}\zeta_{B}^{a_1 \dots c \dots a_r;}{}_{;\Lambda}^{\Omega}\Big) = \sum_{i=1}^{r} \gamma_c^{\alpha a_i}\mathcal{C}(\zeta_{B}^{a_1 \dots c \dots a_r;}{}_{;\Lambda}^{\Omega})$$

$$(17) \qquad \mathcal{C}\Big(\sum_{j=1}^{s} \zeta_{b_1 \dots d \dots b_s;}^{A}{}_{;\Lambda}^{\Omega}\gamma_{b_j}^{\alpha d}\Big) = \sum_{j=1}^{s} \mathcal{C}(\zeta_{b_1 \dots d \dots b_s;}^{A}{}_{;\Lambda}^{\Omega})\gamma_{b_j}^{\alpha d}$$

(18)
$$\mathcal{C}\Big(\sum_{i=1}^{r}\sum_{j=1}^{s}\gamma_c^{\alpha a_i}\zeta^{a_1\dots c\dots a_r,\,\Omega}_{b_1\dots d\dots b_s,\,\Lambda}\gamma^{\beta d}_{b_j}\Big) =$$
$$= \sum_{i=1}^{r}\sum_{j=1}^{s}\gamma_c^{\alpha a_i}\mathcal{C}(\zeta^{a_1\dots c\dots a_r,\,\Omega}_{b_1\dots d\dots b_s,\,\Lambda})\gamma^{\beta d}_{b_j}.$$

(19)
$$\mathcal{C}(\nabla_\alpha\zeta^{A,\,\Omega}_{B,\,\Lambda}) = \nabla_\alpha\mathcal{C}(\zeta^{A,\,\Omega}_{B,\,\Lambda}).$$

PROOF. - Equation (15) follows immediately from (14) and (4).

The proof of (16) is indicated by considering a (2, s)-spinor (p, q)-tensor

$$\mathcal{C}(\gamma_c^{\alpha a_1}\zeta_B^{ca_2,\,\Omega}{}_{,\,\Lambda} + \gamma_c^{\alpha a_2}\zeta_B^{a_1c,\,\Omega}{}_{,\,\Lambda}) = \mathcal{C}(\gamma_c^{\alpha a_1}\zeta_B^{ca_2,\,\Omega}{}_{,\,\Lambda}) + \mathcal{C}(\gamma_c^{\alpha a_2}\zeta_B^{a_1c,\,\Omega}{}_{,\,\Lambda})$$
$$= (\overset{*}{\alpha}{}^{a_1}_{c_1}\gamma_c^{\alpha c_1}\overset{*}{\alpha}{}^{a_2}_{c_2}\zeta^{cc_2,\,\Omega}_{d_1\dots d_s,\,\Lambda} + \overset{*}{\alpha}{}^{a_2}_{c_2}\gamma_c^{\alpha c_2}\overset{*}{\alpha}{}^{a_1}_{c_1}\zeta^{c_1c,\,\Omega}_{d_1\dots d_s,\,\Lambda})\alpha^{d_1}_{b_1}\dots\alpha^{d_s}_{b_s}$$

which by (7) is

$$= (\gamma_c^{\alpha a_1}\alpha^{c}_{c_1}\overset{*}{\alpha}{}^{a_2}_{c_2}\zeta^{c_1c_2,\,\Omega}_{d_1\dots d_s,\,\Lambda} + \gamma_c^{\alpha a_2}\alpha^{a_1}_{c_1}\overset{*}{\alpha}{}^{c}_{c_2}\zeta^{c_1c_2,\,\Omega}_{d_1\dots d_s,\,\Lambda})\alpha^{d_1}_{b_1}\dots\alpha^{d_s}_{b_s}$$
$$= \gamma_c^{\alpha a_1}\mathcal{C}(\zeta_B^{ca_2,\,\Omega}{}_{,\,\Lambda}) + \gamma_c^{\alpha a_2}\mathcal{C}(\zeta_B^{a_1c,\,\Omega}{}_{,\,\Lambda})$$

which completes the proof.

Equation (17) is established in the same way and Equation (18) is an immediate generalization of (16) and (17).

Equation (19) is proven for (1, 0)-spinor (0, q)-tensors in [4] and the technique obviously applies to (r, s)-spinor (p, q)-tensors.

THEOREM 3. - If $\zeta^{A,\,\Omega}_{B,\,\Lambda}$ is an (r, s)-spinor (p, q)-tensor then

(20)
$$\mathcal{A}\mathcal{C}(\zeta^{A,\,\Omega}_{B,\,\Lambda}) = (-1)^{r+s}\mathcal{C}\mathcal{A}(\zeta^{A,\,\Omega}_{B,\,\Lambda}).$$

PROOF. - It suffices to consider the case of a (2, 1)-spinor (p, q)-tensor for which we find that

$$\mathcal{C}\mathcal{A}(\zeta^{a_1a_2,\,\Omega}_{b_1,\,\Lambda}) = \rho\alpha^{f_1}_{b_1}\beta^{c_1}_{f_1}\overset{T}{\zeta}{}^{d_1d_2,\,\Omega}_{c_1,\,\Lambda}\beta^{e_1}_{d_1}\alpha^{a_1}_{e_1}\beta^{e_2}_{d_2}\alpha^{a_2}_{e_2}$$
$$\mathcal{A}\mathcal{C}(\zeta^{a_1a_2,\,\Omega}_{b_1,\,\Lambda}) = \rho\beta^{f_1}_{b_1}\alpha^{c_1}_{f_1}\overset{T}{\zeta}{}^{d_1d_2,\,\Omega}_{c_1,\,\Lambda}\alpha^{e_1}_{d_1}\beta^{a_1}_{e_1}\alpha^{e_2}_{d_2}\beta^{a_2}_{e_2}.$$

By using the anti-commutation relation (5) for α^a_b and β^a_b in each factor of either of these equations we see that they are identical up to a factor of (-1). Hence the general formula with $(-1)^{r+s}$ occurs because of the appearance of $r+s$ negative signs when (5) is used.

We note that in the theories of DIRAC and RARITA-SCHWINGER, $\mathcal{C}$ and $\mathcal{A}$ anti-commute, while in the PETIAU-DUFFIN-KEMMER theory they commute. For later purposes it will be convenient to denote

$$\Gamma \overset{\text{def}}{=} \mathcal{A}\mathcal{C}. \tag{21}$$

DEFINITION. - The linear differential operator $\mathfrak{F}$ which maps $(r, 0)$-spinor (p, q)-tensor fields onto $(r, 0)$-spinor (p, q)-tensor fields defined by

$$\mathfrak{F}\zeta^{A,}{}_{,}{}^{\Omega}_{\Lambda} \overset{\text{def}}{=} \sum_{i=1}^{r} \gamma_c^{aa_i} \nabla_a \zeta^{a_1 \ldots c \ldots a_r,}{}_{,}{}^{\Omega}_{\Lambda} \tag{22}$$

is called the DIRAC operator.

This definition is suggested by the familiar operator appearing in the DIRAC equations for $(1, 0)$-spinors. It is easy to verify that the adjoint of $\mathfrak{F}$, denoted by $\tilde{\mathfrak{F}}$, is the linear differential operator which maps $(0, s)$-spinor (p, q)-tensors onto $(0, s)$-spinor (p, q)-tensors

$$\tilde{\mathfrak{F}}\zeta_{B,}{}^{\Omega}_{\Lambda} = -\sum_{j=1}^{s} \nabla_a \zeta_{b_1 \ldots d \ldots b,}{}^{\Omega}_{\Lambda} \gamma^{ad}_{b_j}. \tag{23}$$

It is important to note that by definition $\mathfrak{F}$ acts only on contravariant spinor indices and $\tilde{\mathfrak{F}}$ acts only on covariant spinor indices.

THEOREM 3. - If $\zeta^{A,}_{B,}{}^{\Omega}_{\Lambda}$ is an (r, s)-spinor (p, q)-tensor then

$$\mathcal{A}\mathfrak{F}\zeta^{A,}_{B,}{}^{\Omega}_{\Lambda} = \tilde{\mathfrak{F}}\mathcal{A}\zeta^{A,}_{B,}{}^{\Omega}_{\Lambda} \tag{24}$$

$$\mathcal{C}\mathfrak{F}\zeta^{A,}_{B,}{}^{\Omega}_{\Lambda} = \mathfrak{F}\mathcal{C}\zeta^{A,}_{B,}{}^{\Omega}_{\Lambda} \tag{25}$$

$$\mathcal{C}\tilde{\mathfrak{F}}\zeta^{A,}_{B,}{}^{\Omega}_{\Lambda} = \tilde{\mathfrak{F}}\mathcal{C}\zeta^{A,}_{B,}{}^{\Omega}_{\Lambda}. \tag{26}$$

PROOF. - Equation (23) is a consequence of (10) and (13) since

$$\mathcal{A}\left(\sum_{i=1}^{r} \gamma_c^{aa_i} \nabla_a \zeta_B^{a_1 \ldots c \ldots a_r,}{}^{\Omega}_{\Lambda}\right) = -\sum_{i=1}^{r} \mathcal{A}(\nabla_a \zeta_B^{a_1 \ldots c \ldots a_r,}{}^{\Omega}_{\Lambda}) \gamma_c^{aa_i}$$

$$= -\sum_{i=1}^{r} \nabla_a (\mathcal{A}\zeta^{a_1 \ldots c \ldots a_r,}{}^{\Omega}_{\Lambda}) \gamma_c^{aa_i}.$$

By noting that $\mathcal{A}$ maps (r, s)-spinors onto (s, r)-spinors this is precisely (23).

Equations (25) and (26) follow similarly by virtue of (16) and (19).

The theory of PETIAU-DUFEIN-KEMMER, together with the principle of fusion of L. DE BROGLIE, suggests the introduction of the following differential operators:

DEFINITION. - Let $\mathcal{M}$ and $\mathcal{N}$ be the linear differential operators, called the PETIAU operators, which map (r, s)-spinor (p, q)-tensors onto (r, s)-spinor (p, q)-tensors

$$\mathcal{M} \overset{\text{def}}{=} \frac{1}{2}(\mathcal{S} + \tilde{\mathcal{S}}) \tag{27}$$

$$\mathcal{N} \overset{\text{def}}{=} \frac{1}{2}(\mathcal{S} - \tilde{\mathcal{S}}). \tag{28}$$

Using the PETIAU operators one has the following as a direct consequence of Theorem 3:

THEOREM 4. - If $\zeta^{A;\,\Omega}_{B;\,\Lambda}$ is an (r, s)-spinor (p, q)-tensor then

$$\mathcal{A}\mathcal{M}(\zeta^{A;\,\Omega}_{B;\,\Lambda}) = \mathcal{M}\mathcal{A}(\zeta^{A;\,\Omega}_{B;\,\Lambda}) \tag{29}$$

$$\mathcal{C}\mathcal{M}(\zeta^{A;\,\Omega}_{B;\,\Lambda}) = \mathcal{M}\mathcal{C}(\zeta^{A;\,\Omega}_{B;\,\Lambda}) \tag{30}$$

$$\mathcal{A}\mathcal{N}(\zeta^{A;\,\Omega}_{B;\,\Lambda}) = -\mathcal{N}\mathcal{A}(\zeta^{A;\,\Omega}_{B;\,\Lambda}) \tag{31}$$

$$\mathcal{C}\mathcal{N}(\zeta^{A;\,\Omega}_{\ ;\,\Lambda}) = \mathcal{N}\mathcal{C}(\zeta^{A;\,\Omega}_{B;\,\Lambda}). \tag{32}$$

3. - Laplacians for spinor-tensors.

In a recent note, [8], the author proved the following:

THEOREM 5. - Let $\zeta^{A;\,\Omega}_{B;\,\Lambda}$ and $\zeta'^{B;\,\Lambda}_{A;\,\Omega}$ be (r, s)-spinor (p, q)-tensors and (s, r)-spinor (q, p)-tensors respectively, which are defined on a compact orientable n-dimensional RIEMANNIAN manifold V_n with hyperbolic normal signature. If the intersection of the supports of $\zeta^{A;\,\Omega}_{B;\,\Lambda}$ and $\zeta'^{B;\,\Lambda}_{A;\,\Omega}$ is compact then

$$< \Delta\zeta^{A;\,\Omega}_{B;\,\Lambda},\ \zeta'^{B;\,\Lambda}_{A;\,\Omega} > = < \zeta^{A;\,\Omega}_{B;\,\Lambda},\ \Delta\zeta'^{B;\,\Lambda}_{A;\,\Omega} > \tag{33}$$

where $<, >$ is the global scalar product V_n and

$$\Delta\zeta^{A;\,\Omega}_{B;\,\Lambda} \overset{\text{def}}{=} \frac{1}{r+s}\Big(\sum_{i=1}^{r} \gamma^{\alpha a_i}_{m}\gamma^{\beta m}_{c} \nabla_\alpha \nabla_\beta \zeta^{a_1 \dots c \dots a_r;\,\Omega}_{B;\,\Lambda} + \sum_{j=1}^{s} \nabla_\alpha \nabla_\beta \zeta^{A;\,\Omega}_{b_1 \dots d \dots b_s;\,\Lambda} \gamma^{\beta d}_{m} \gamma^{\alpha m}_{b_j} \Big). \tag{34}$$

The DIRAC matrices are defined as in (1). It was suggested that since Δ is linear and self-adjoint it might be an appropriate LAPLACIAN operator for spinor-tensors. In fact

THEOREM 6. - Let $\zeta_{B,\ \Lambda}^{A,\ \Omega}$ be an (r, s)-spinor (p, q)-tensor then

$$\mathfrak{A}(\Delta\zeta_{B,\ \Lambda}^{A,\ \Omega}) = \Delta(\mathfrak{A}\zeta_{B,\ \Lambda}^{A,\ \Omega}) \tag{35}$$

$$\mathfrak{C}(\Delta\zeta_{B,\ \Lambda}^{A,\ \Omega}) = \Delta(\mathfrak{C}\zeta_{B,\ \Lambda}^{A,\ \Omega}) \tag{36}$$

$$\Gamma(\Delta\zeta_{B,\ \Lambda}^{A,\ \Omega}) = \Delta(\Gamma\zeta_{B,\ \Lambda}^{A,\ \Omega}). \tag{37}$$

PROOF. - Equation (24) is established immediately by noting that

$$\mathfrak{A}(\Delta\zeta_{B,\ \Lambda}^{A,\ \Omega}) = \frac{1}{r+s}\{\,\mathfrak{A}(\sum_{i=1}^{r}\gamma_{m}^{\alpha a_i}\gamma_{c}^{\beta m}\nabla_\alpha\nabla_\beta\zeta_{B}^{a_1\ldots c\ldots a_r,\ \Omega}{}_{,\ \Lambda})$$

$$+\,\mathfrak{A}(\sum_{j=1}^{s}\nabla_\alpha\nabla_\beta\zeta^{A}_{b_1\ldots d\ldots b_s,\ \Lambda}{}^{,\ \Omega}\cdot\gamma_{m}^{\beta d}\gamma_{b_j}^{\alpha m}).$$

and applying (10) and (11) twice in the first and second terms respectively.

Equation (36) follows in the same way by applying (16) and (17) twice in the respective terms.

Equation (37) is clearly a consequence of the previous two equations.

The properties indicated in Theorems 5 and 6 are precisely those possessed by the LAPLACIANS defined by LICHNEROWICZ, [4], for the DIRAC, RARITA-SCHWINGER and PETIAU-DUFFIN-KEMMER theories. The LAPLACIAN defined in (34) reduced to those defined by LICHNEROWICZ for the first two theories. Furthermore, his LAPLACIANS, which we denote by Δ_l, have the interesting properties that

$$\Delta_l = \mathfrak{F}^2 = \tilde{\mathfrak{F}}^2$$

$$\Delta_l = \mathfrak{M}^2 + \mathfrak{N}^2$$

and since $\mathfrak{M}\mathfrak{N} = \mathfrak{N}\mathfrak{M} = 0$

$$\mathfrak{M}^3 = \mathfrak{M}\Delta_l = \Delta_l\mathfrak{M}$$

$$\mathfrak{N}^3 = \mathfrak{N}\Delta_l = \Delta_l\mathfrak{N},$$

which are not shared by the generalization (34). This is due primarily to the fact that for (r, s)-spinors squaring $\mathfrak{F}$, $\tilde{\mathfrak{F}}$, $\mathfrak{M}$ or $\mathfrak{N}$ introduces cross terms which do not occur in the simpler cases studied by LICHNEROWICZ. Thus it seems unlikely to the author that the DIRAC and PETIAU operators are really intrinsic operators to employ in the general study of (r, s)-spinor (p, q)-

tensors. In particular the author has been unable to construct a LAPLACIAN for (r, s)-spinor (p, q)-tensors which is self-adjoint, commutes with $\mathcal{A}$, $\mathcal{C}$, Γ, $\mathfrak{M}$ and $\mathfrak{N}$ and is equal to $\mathfrak{F}^2$, $\tilde{\mathfrak{F}}^2$ and $\mathfrak{M}^2 + \mathfrak{N}^2$. An interesting discussion of some of the proposed LAPLACIANS (not including (34)) is given by COLLEAU [2]. He comments that none of the proposed LAPLACIANS he considers possesses all the properties suitable for the purposes of mathematical physics. However, he does not indicate what these properties might be.

BIBLIOGRAPHY

[1] N. N. BOGOLIUBOV - D. V. SHIROKOV, *Introduction to the Theory of Quantized Fields* « Interscience », (1960).

[2] J. COLLEAU, *Étude des propagateurs tensoriels et spinoriels,* « Ann. Inst. H. Poincaré » IIIA(3), (1965), pp. 195-337.

[3] A. LICHNEROWICZ, *Propagateurs, Commutateurs et Anticommutateurs en Relativité générale Les Houches Lectures 1963 in Relativity,* Groups and Topology by C. M. & B. S. DeWitt p. 823-861, Gordon & Breach.

[4] A. LICHNEROWICZ, *Champs spinoriels et propagateurs en Relativité générale,* Bull. Soc. Math. de France, 92(1964), pp. 11-100.

[5] A. LICHNEROWICZ, *Champ de Dirac, champ du neutrino et transformations C, P, T sur un espace-temps courbe,* « Ann. Inst. H. Poincaré », IA(3), (1964), pp. 233-290.

[6] J. RENAUDIE, *Théorie hexadimensionelle du champ de certaines particules,* « Sém. Janet (Paris) », 1958-1959, No 2, pp. 1-8.

[7] Y. THIRY, *Théorie pentadimensionelle de la gravitation et de l'électromagnétisme,* « Sém. Janet (Paris), » 1958 (1959), No. 1· pp, 1-15.

[8] J. D. ZUND, *A theorem on spinor-tensors,* « Jour. Math. Mechanics, 15 » (1966), pp. 221-226.