Subdivision Algorithms and the Kernel of a Polyhedron*

R. J. Gardner¹ and M. Kallay²

¹ Department of Mathematics, Western Washington University,
 Bellingham, WA 98225-9063, USA

² Electronic Data Systems Corporation, C4 Technology West,
 Suite 300, 13555 SE 36th Street, Bellevue, WA 98006, USA

Abstract. The kernel K of a convex polyhedron P_0 , as defined by L. Fejes Tóth, is the limit of the sequence (P_n) , where P_n is the convex hull of the midpoints of the edges of P_{n-1} . The boundary ∂K of the convex body K is investigated. It is shown that ∂K contains no two-dimensional faces and that ∂K need not belong to C^2 . The connection with similar algorithms from CAD (computer aided design) is explained and utilized.

1. Introduction

Suppose P_0 is a planar convex polygon. A sequence (P_n) of polygons can be defined by letting P_n be the *midpoint polygon* of P_{n-1} ; that is, the convex hull of the midpoints of all the edges of P_{n-1} .

It can be shown $\lim_n P_n = \{c_0\}$, where c_0 is the centroid of P_0 (and of all the polygons P_n). Much more interesting is the beautiful result that the sequence $\{\alpha_{2n} P_{2n}\}$, where α_{2n} are suitable scaling factors, converges to an affinely regular polygon. (This theorem, attributed in [9] to Neumann, was actually discovered earlier by Darboux [5]; it has been rediscovered many times (see p. 322 of [9]).) Moreover, apart from its aesthetic appeal, there is an important application, since the result is a central ingredient in the solution of Hammer's X-ray problem [10].

Several generalizations of the midpoint process in $\mathbf{R}^2$ have been studied (e.g., [2] and [12]), but here we are concerned with the three-dimensional analogue proposed by Fejes Tóth in [13]. If P_0 is a convex polyhedron in $\mathbf{R}^3$, P_n is defined exactly as above. In this case, however, $\lim_n P_n = K(P_0)$, where K is a convex body

* R. J. Gardner was supported in part by a von Humboldt fellowship.

in $\mathbf{R}^3$ called the *kernel* of P_0 . To see this, observe that since the midpoint algorithm in $\mathbf{R}^2$ preserves centroids, the set of all centroids of facets of the polyhedra P_n lies in the boundary ∂K of K . As a preliminary example, the reader may check that if P_0 is a regular tetrahedron, then P_1 is a regular octahedron, and P_2 is a cuboctahedron; and if P_0 is a cube, P_1 is again a cuboctahedron, yielding the interesting fact that all these polyhedra have the same kernel (up to homothety).

Fejes Tóth asked what can be said about the structure of ∂K . As far as we know, nothing is yet known. From examples the impression that, as n increases, the size of the facets of P_n should decrease (and so their number should be unbounded), is obtained. We confirm this in Theorem 4, and deduce (Corollary 6) that ∂K contains no facets (two-dimensional faces). We also show that ∂K need not belong to C^2 (Theorem 8). The obvious questions, whether ∂K is strictly convex or smooth, remain unanswered (the latter was also posed by Fejes Tóth in [13]). However, our results are evidence for the intriguing conjecture, suggested by Zamfirescu, that the kernel is in some sense a “typical” convex body. Indeed, Klee showed that most convex bodies, in the sense of Baire category, are strictly convex and have a smooth boundary, and Gruber proved that most do not have a boundary belonging to C^2 (see [14] and the references given there).

We have a second motivation in this note, namely, to draw the attention of mathematicians to the connection between Fejes Tóth’s problem and some work in CAD (computer aided design). In CAD, processes called subdivision algorithms are sometimes used to model curves or surfaces. It has been found efficient to store a finite set of points, edges, and facets, and then generate new ones by some algorithm, formulated so that successive iterations converge to a limiting curve or surface, preferably smooth. Several such algorithms have been suggested, and in some of these the midpoint operation or a similar one plays a role.

We discuss two of these algorithms in Section 4, and apply one of them to Fejes Tóth’s problem. This is a planar algorithm which was, in fact, considered by de Rham long before CAD existed. Its analysis is quite complicated, and de Rham devoted several papers to it. The proof of Theorem 8 of Section 4 indicates that the midpoint operation in three dimensions is actually analogous to de Rham’s algorithm, rather than to the planar midpoint algorithm. Furthermore, the convex hull operation in three dimensions can create new edges in a way that is not easy to foresee. All this suggests that a full solution of Zamfirescu’s conjecture may be difficult to obtain.

We thank David Levin for a careful reading of the manuscript which revealed a flaw in our previous version of Theorem 8.

2. Definitions

If E is a set, we denote by $d(E)$, ∂E , $\text{int } E$, and $\text{relint } E$, the *diameter*, *boundary*, *interior*, and *relative interior* of E , respectively.

Suppose P is an m -dimensional convex polytope in $\mathbf{R}^n$. Let $M(P)$ be the *midpoint polytope* of P , i.e., the m -dimensional polytope in $\mathbf{R}^n$ formed by taking the convex

hull of the midpoints of the edges of P . We write $M^n(P)$ for $M(M^{n-1}(P))$, $n = 2, 3, \dots$, where $M^1(P) = M(P)$.

If P_0 is a convex polyhedron in $\mathbf{R}^3$, let $P_n = M(P_{n-1})$ for $n = 1, 2, \dots$. Write $K = \lim_n P_n$. Then $K = K(P_0)$ is called the *kernel* of P_0 (see [13]).

If K is the kernel of P_0 , we define the *centroid set* $C(K)$ of K by $C(K) = \{x: x \text{ is a centroid of a facet of some } P_n, n \in \mathbf{N}\}$.

Suppose F is a facet, i.e., a two-dimensional face, of the polyhedron P_n . If there is a facet G of P_{n-1} such that $F = M(G)$, we say that F is of *type 1*. Now suppose F is a facet of P_n which has the following property: there is a vertex v of P_{n-1} such that each vertex w of F is the midpoint of some edge of P_{n-1} with v as one endpoint. Then we call F a *type 2* facet of P_n . For each n , we let

$$\delta_n = \max\{d(F): F \text{ is a facet of } P_n\}$$

and

$$\delta_n^{(2)} = \max\{d(F): F \text{ is a type 2 facet of } P_n\}.$$

As is usual, we denote by C^k the class of curves (or surfaces) whose every point has a relative neighborhood which is the range of an open set in $\mathbf{R}$ (or $\mathbf{R}^2$, respectively) under a k -times continuously differentiable mapping. *Smooth* means belonging to C^1 .

3. The Kernel Has No Facets

Lemma 1. *For each n , every facet of P_n is either of type 1 or of type 2.*

Proof. Let F be a facet of P_n . Let proj_π denote the orthogonal projection onto a plane Π orthogonal to F , and let h be the line in Π containing $\text{proj}_\pi F$.

If $\text{proj}_\pi P_{n-1}$ lies on one side of h , it is easy to see that F must be a type 1 facet of P_n . Suppose, then, that $h \cap \text{int}(\text{proj}_\pi P_{n-1}) \neq \emptyset$. By our choice of Π , $\text{proj}_\pi P_n$ lies on one side of h . Let E be the open half-plane in Π on the other side of h . If two or more vertices of P_{n-1} project into E , there is an edge of P_{n-1} whose midpoint projects into E , contradicting $E \cap \text{proj}_\pi P_n = \emptyset$. Therefore a unique vertex v of P_{n-1} projects into E . It follows that each vertex of F must be the midpoint of an edge of P_{n-1} which has v as one endpoint, and so F is of type 2. $\square$

Lemma 2. *For each n , $\delta_n < \delta_{n-1}$.*

Proof. If F is a facet of P_n of type 1, then $F = M(G)$ for some facet G of P_{n-1} . The diameter $d(G)$ is the distance between some pair of vertices of G , and F contains none of these vertices. Therefore $d(F) < d(G) \leq \delta_{n-1}$.

Suppose F is a type 2 facet of P_n . Let v be the vertex of P_{n-1} from the definition. Let $\{w_i\}$ be the vertices of F and, for each i , let e_i be the edge of P_{n-1} containing v and w_i . Finally, let G_i be any facet of P_{n-1} containing e_i . Then

$$\begin{aligned} d(F) &< \frac{1}{2} \max_{i,j} \{d(e_i) + d(e_j)\} \\ &\leq \frac{1}{2} \max_{i,j} \{d(G_i) + d(G_j)\} \\ &\leq \frac{1}{2} \cdot 2\delta_{n-1} = \delta_{n-1}, \end{aligned}$$

and the lemma follows. $\square$

Remark 3. We do not know if there is a $c < 1$, depending on P_0 , such that $\delta_n < c\delta_{n-1}$ for each n . On the other hand, given $c < 1$, there is a P_0 with $\delta_1 > c\delta_0$. To see this, let H be a regular hexagon in the xy -plane centered at the origin, and for $\varepsilon > 0$ let $P_0(\varepsilon) = \text{conv}\{(0, 0, \pm\varepsilon), H\}$, i.e., a regular double pyramid over H . Then, given $c < 1$, we can choose ε small enough so that for $P_0(\varepsilon)$ we have $\delta_1^{(2)} > c\delta_0$. More elaborate examples show that, given $c < 1$ and $n \geq 1$, there is a P_0 with $\delta_n^{(2)} > c\delta_0$.

Theorem 4. $\delta_n \rightarrow 0$.

Proof. Suppose the sequence (δ_n) does not converge to zero. Since $\delta_n < \delta_{n-1}$ by Lemma 2, we have $\delta_n \rightarrow a > 0$ and $\delta_n > a$ for all n . Therefore there are sequences $(u_n^{(1)})$ and $(u_n^{(2)})$, where $u_n^{(1)}$ and $u_n^{(2)}$ are vertices of the same facet F_n of P_n , such that $\delta_n = d(F_n) = \|u_n^{(1)} - u_n^{(2)}\| \rightarrow a$.

Suppose there is an n_0 such that F_n is of type 1 for $n \geq n_0$. For each $n \geq n_0$, there is a least integer k_n such that there are facets $G_i^{(n)}$ of P_i with $F_n = G_{k_n}^{(n)}$ and $G_{i+1}^{(n)} = M(G_i^{(n)})$, $k_n \leq i \leq n-1$. Suppose there is an m such that $k_n \leq m$ for all $n \geq n_0$. Then each F_n , $n \geq n_0$, is descended from some facet of some P_i , $0 \leq i \leq m$, via the midpoint operation. However, there are only finitely many such facets, and for each one successive midpoint polygons converge to its centroid. Therefore, for sufficiently large n , we must have $d(F_n) < a$, which contradicts $\delta_n > a$. Thus $k_n \rightarrow \infty$ as $n \rightarrow \infty$. Let $F'_n = G_{k_n}^{(n)}$. Then F'_n is not of type 1, so by Lemma 1 it must be of type 2. Also, $d(F_n) < d(F'_n) \leq \delta_{k_n}$, so $d(F'_n) \rightarrow a$.

From the above it follows that we can assume there is a subsequence S of $\mathbb{N}$, such that F_n is of type 2 for $n \in S$ and $d(F_n) \rightarrow a$ as $n \rightarrow \infty$ through S . Choose a subsequence T of S such that the line segments $s_n = [u_n^{(1)}, u_n^{(2)}]$ converge in the Hausdorff metric to a line segment $s = [u^{(1)}, u^{(2)}]$ of length a in ∂K , where K is the kernel of P_0 , as $n \rightarrow \infty$ through T . Let t be the line containing s .

For each $n \in T$, there are vertices $v_n^{(i)}$, $1 \leq i \leq 3$, of P_{n-1} such that $u_n^{(i)} = \frac{1}{2}(v_n^{(i)} + v_n^{(i+1)})$, and $e_n^{(i)} = [v_n^{(i)}, v_n^{(i+1)}]$ is an edge of P_{n-1} , $i = 1, 2$. By taking a further subsequence if necessary, we may suppose that $e_n^{(1)}$ is the longer of these edges, i.e., $d(e_n^{(2)}) = c_n d(e_n^{(1)})$ for some $c_n \leq 1$. Let ϑ_n be the angle between $e_n^{(1)}$ and $e_n^{(2)}$. Then

$$[d(F_n)]^2 = \frac{1}{4}([d(e_n^{(1)})]^2 + [d(e_n^{(2)})]^2 - 2d(e_n^{(1)})d(e_n^{(2)}) \cos \vartheta_n),$$

giving $d(e_n^{(1)}) = \alpha_n d(F_n)$, where

$$\alpha_n = \frac{2}{(1 + c_n^2 - 2c_n \cos \vartheta_n)^{1/2}}.$$

Now $\alpha_n \geq 1$, so $d(F_n) \leq d(e_n^{(1)}) \leq \delta_{n-1}$ for $n \in T$. Thus $\alpha_n \rightarrow 1$, as $n \rightarrow \infty$ through T . This implies that $c_n \rightarrow 1$ and $\vartheta_n \rightarrow \pi$, as $n \rightarrow \infty$ through T . It follows that $v_n^{(i)} \rightarrow v^{(i)}$ as $n \rightarrow \infty$ through T , where $v^{(i)}$ lies in $t \cap \partial K$, $1 \leq i \leq 3$, $\|v^{(1)} - v^{(2)}\| = \|v^{(2)} - v^{(3)}\| = a$, and $v^{(2)}$ is the midpoint of s .

The edges $e_n^{(1)}$ of P_{n-1} converge to the segment $[v^{(1)}, v^{(2)}]$ of length a . Each $e_n^{(1)}$ belongs to a facet of type 1 or type 2. In either case there are vertices $w_n^{(i)}$ of P_{n-2} , $1 \leq i \leq 3$, such that $v_n^{(i)} = \frac{1}{2}(w_n^{(i)} + w_n^{(i+1)})$, and $f_n^{(i)} = [w_n^{(i)}, w_n^{(i+1)}]$ is an edge of P_{n-2} , $i = 1, 2$. Now the argument above shows that $w_n^{(i)} \rightarrow w^{(i)}$, where $w^{(i)}$ lies in $t \cap \partial K$, $1 \leq i \leq 3$, $\|w^{(1)} - w^{(2)}\| = \|w^{(2)} - w^{(3)}\| = a$, and $w^{(2)} = u^{(1)}$ is the midpoint of $[v^{(1)}, v^{(2)}]$. Hence $\|w^{(1)} - v^{(3)}\| = 5a/2$.

Replacing $e_n^{(1)}$ by $f_n^{(1)}$ and repeating the argument yields a point $x^{(3)}$ in $t \cap \partial K$ with $\|w^{(1)} - x^{(3)}\| = 3a$, and so on. We deduce that ∂K contains line segments of arbitrary length, which is impossible. $\square$

Corollary 5. *The centroid set $C(K)$ is dense in ∂K .*

Proof. Let $\varepsilon > 0$ and $x \in \partial K$ be given. There is an n_0 such that, for each $n \geq n_0$, there is a point $x_n \in \partial P_n$ with $\|x - x_n\| < \varepsilon/2$.

By Theorem 4 there is an $n_1 > n_0$ such that $\delta_n < \varepsilon/2$ for each $n \geq n_1$. If $x_n \in F_n$, where F_n is a facet of P_n , and c_n is the centroid of F_n , we have

$$\begin{aligned} \|x - c_n\| &\leq \|x - x_n\| + \|x_n - c_n\| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for $n \geq n_1$, as required. $\square$

Corollary 6. *∂K contains no facets.*

Proof. Suppose D is a facet in ∂K . Then there is a $c \in C(K)$ with $c \in \text{relint } D$, by Corollary 5. For some n , c is the centroid of a facet F_n of P_n . Then $D \subset F_n$. However, if H is the plane containing F_n , then $P_n \cap H \rightarrow \{c\}$, a contradiction. $\square$

4. Subdivision Algorithms

We begin by describing a planar algorithm studied by de Rham, which we then apply to obtain information about the kernel of a polyhedron.

Let Q_0 be a planar convex polygon with vertices $q_0, \dots, q_{n-1}$. (We always take the indices of q_i modulo n .) On the edge $[q_i, q_{i+1}]$, consider the points q'_{2i} and

q'_{2i+1} which divide the edge into three segments, $[q_i, q'_{2i}]$, $[q'_{2i}, q'_{2i+1}]$, and $[q'_{2i+1}, q_{i+1}]$, in the proportions β_1 , α , and β_2 , respectively, where $\beta_1 + \alpha + \beta_2 = 1$. The points q'_i , $i = 0, \dots, 2n - 1$, form the vertices of a polygon Q_1 . The process may be repeated, with the same proportions, to obtain a sequence (Q_n) of polygons. Then $Q = \lim_n Q_n$ is a convex body in $\mathbb{R}^2$.

Let $\gamma_i = \beta_i/\alpha$, $i = 1, 2$. De Rham's interest in the limiting curve ∂Q stems from the surprising fact that, for all but a single pair of values of the parameters γ_i , ∂Q has singular properties. In a series of papers (see, for example, [6]) he showed that if $\gamma_i \leq 1$, $i = 1, 2$ (which includes the case where each side of the polygon Q_n is trisected), then ∂Q is smooth, but if $\gamma_i > 1$, $i = 1, 2$, then ∂Q has no tangent at a countable dense subset of ∂Q ; further, unless $\gamma_1 = \gamma_2 = \frac{1}{2}$, no subarc of ∂Q belongs to C^2 .

In contrast to this is the special case $\gamma_1 = \gamma_2 = \frac{1}{2}$. Here it can be shown (e.g., Section 2 of [3]) that ∂Q is a continuously differentiable curve consisting of a finite union of parabolic arcs, each joining adjacent midpoints of the sides of the original polygon Q_0 . This was also known to de Rham, but in the CAD literature it is called Chaikin's algorithm (see [4], [11], or [3]), since until recently it was not realized that de Rham has priority. We need the following result concerning this algorithm.

Lemma 7. *Let Q_0 be a planar convex polygon with vertices q_i , $i = 0, \dots, n - 1$, and let Q be the limiting convex body obtained from Q_0 by Chaikin's algorithm. Then ∂Q belongs to C^2 if and only if $[q_{i-1}, q_{i+2}]$ is parallel to $[q_i, q_{i+1}]$ for $i = 0, \dots, n - 1$.*

Proof. It is known (see p. 120 of [3]) that ∂Q is a finite union of parabolic arcs given by

$$p_i(t) = (\tfrac{1}{2}q_{i-1} - q_i + \tfrac{1}{2}q_{i+1})t^2 + (q_i - q_{i-1})t + \tfrac{1}{2}(q_{i-1} + q_i)$$

for $0 \leq t \leq 1$, $i = 0, \dots, n - 1$. The arc $p_i(t)$, $0 \leq t \leq 1$, joins the midpoints $p_i(0) = \frac{1}{2}(q_{i-1} + q_i)$ and $p_i(1) = \frac{1}{2}(q_i + q_{i+1})$ of two adjacent sides of Q_0 . Note that

$$p'_i(1) = q_{i+1} - q_i = p'_{i+1}(0),$$

$i = 0, \dots, n - 1$, which shows that ∂Q belongs to C^1 . Also, $p''_i(t)$ is constant, and the curvature at $p_i(t)$ is the length of the vector

$$\frac{p'_i(t) \times p''_i}{|p'_i(t)|^3},$$

$i = 0, \dots, n - 1$. Using $p'_i(1) = p'_{i+1}(0)$, we see that ∂Q belongs to C^2 if and only if

$$p'_i(1) \times p''_i = p'_i(1) \times p''_{i+1}$$

or

$$p'_i(1) \times (p''_{i+1} - p''_i) = 0,$$

$i = 0, \dots, n-1$. This is equivalent to the condition that $p''_{i+1} - p''_i$ is parallel to $p'_i(1)$, $i = 0, \dots, n-1$. Since

$$\begin{aligned} p''_{i+1} - p''_i &= (q_i - 2q_{i+1} + q_{i+2}) - (q_{i-1} - 2q_i + q_{i+1}) \\ &= (q_{i+2} - q_{i-1}) + 3(q_i - q_{i+1}) \\ &= (q_{i+2} - q_{i-1}) - 3p'_i(1), \end{aligned}$$

this reduces to the condition that $q_{i+2} - q_{i-1}$ is parallel to $q_{i+1} - q_i$, $i = 0, \dots, n-1$, as required. $\square$

Theorem 8. *There exists a polyhedron P_0 such that the boundary of the kernel of P_0 does not belong to C^2 .*

Proof. Let Q_0 be an quadrilateral in the xy -plane in $\mathbf{R}^3$ which is not a parallelogram. To ease our description, we also assume that Q_0 is not a trapezium, though this is not actually necessary. Suppose q_i , $0 \leq i \leq 3$, are the vertices of Q_0 . The polyhedron P_0 is the right cylinder with vertices $(q_i, \pm 1)$, $0 \leq i \leq 3$. We need to follow Fejes Tóth's algorithm through several steps.

Let R_i be the rectangular facet of P_0 with vertices $(q_i, \pm 1)$ and $(q_{i+1}, \pm 1)$. Note that each P_n will be symmetric about the xy -plane. For our purposes, it will suffice to fix i and consider at the n th stage of the algorithm the facet $M^n(R_i)$ of P_n and the set $\mathcal{F}_i^+(P_n)$ of facets of P_n which meet both $M^n(R_i)$ and the open half-space $\{(x, y, z): z > 0\}$.

The facet $M(R_i)$ of P_1 is a parallelogram, and the set $\mathcal{F}_i^+(P_1)$ consists of the n -gon $M(Q_0 \times \{1\})$ and two triangles, S_i and S_{i+1} , with one vertex at q_i, q_{i+1} , respectively (see Fig. 1). The facet $M^2(R_i)$ of P_2 is rectangular. Two rectangles, $R_i^{(1)}, R_i^{(2)}$, each of which share a vertical edge with $M^2(R_i)$, are contained in $\mathcal{F}_i^+(P_2)$. The latter also contains two triangles, $S_i^{(1)}$ and $S_i^{(2)}$, which have an edge in common

Fig. 1

Fig. 2

with $M^2(R_i)$ and $M^2(Q_0 \times \{1\})$, respectively. Finally, $\mathcal{F}_i^+(P_2)$ contains the two triangles $M(S_i)$ and $M(S_{i+1})$. We also assume without loss of generality that $S_i^{(1)}$ and $M(S_i)$ meet in an edge (see Fig. 2). The key observation here is that one edge of $M(S_i)$ is parallel to one diagonal of $M^2(R_i)$, and one edge of $M(S_{i+1})$ is parallel to the other diagonal. This parallelism produces two trapezoidal facets at the next stage.

The following facets comprise $\mathcal{F}_i^+(P_3)$ (see Fig. 3). Firstly, the three parallelograms $M^3(R_i)$, $M(R_i^{(1)})$, and $M(R_i^{(2)})$; two trapeziums, T_i and T_{i+1} , which share an edge with $M^3(R_i)$ and with $M^2(S_i)$, $M^2(S_{i+1})$, respectively; a triangle S'_i , which has

Fig. 3

Fig. 4

common edges with $M(S_i^{(1)})$, $M(S_i^{(2)})$, and T_{i+1} ; and the triangle $M(S_i^{(1)})$. Note that the latter has a horizontal edge, which is parallel to the horizontal diagonal of $M^3(R_i)$; this results in a trapezium at the next stage.

The next iteration produces the facets in $\mathcal{F}_i^+(P_4)$, which form a configuration (see Fig. 4) that is preserved by alternate steps of the algorithm. We have the rectangles $M^4(R_i)$, $M^2(R_i^{(1)})$, and $M^2(R_i^{(2)})$; two parallelograms, $M(T_i)$ and $M(T_{i+1})$; a trapezium T'_i , which has common edges with $M^4(R_i)$ and $M(T_i)$; and a triangle S''_i , which meets both T'_i and $M(T_{i+1})$ in an edge.

The parallelograms $M(T_i)$ and $M(T_{i+1})$ each have a diagonal which is parallel to a diagonal of $M^4(R_i)$. Using this fact it is easy to see that $\mathcal{F}_i^+(P_6)$, and by induction, $\mathcal{F}_i^+(P_{2n})$, for $n \geq 3$, also contains three rectangles, two parallelograms, a trapezium, and a triangle, in the same configuration. In particular, the rectangles $M^{2n}(R_i)$, $M^{2n-2}(R_i^{(1)})$, and $M^{2n-2}(R_i^{(2)})$ are facets of P_{2n} , and their centroids lie in the boundary of the kernel of P_0 . The intersections of these rectangles with the xy -plane form three consecutive edges of the polygon Q_n obtained from Q_0 by Chaikin's algorithm; namely, the edge whose midpoint is $\frac{1}{2}(q_i + q_{i+1})$, together with the adjacent edge on either side. The midpoints of these edges, which are precisely the centroids of the just-mentioned rectangular facets of P_{2n} , lie in the limiting curve ∂Q obtained from Q_0 by Chaikin's algorithm.

It follows that the kernel $K(P_0)$ contains two sequences of points in the xy -plane, each converging to $\frac{1}{2}(q_i + q_{i+1})$. One of these sequences belongs to the parabolic arc of ∂Q with endpoints $\frac{1}{2}(q_{i-1} + q_i)$ and $\frac{1}{2}(q_i + q_{i+1})$, and the other to the parabolic arc of ∂Q with endpoints $\frac{1}{2}(q_i + q_{i+1})$ and $\frac{1}{2}(q_{i+1} + q_{i+2})$. Our assumptions on Q_0 imply, by Lemma 7 and its proof, that the intersection of the boundary of $K(P_0)$ with the xy -plane does not belong to C^2 . Therefore the boundary of $K(P_0)$ itself is not in C^2 . $\square$

Chaikin's algorithm—de Rham's with $\gamma_1 = \gamma_2 = \frac{1}{2}$ —motivated several algorithms in $\mathbf{R}^3$. For example, the Doo algorithm in [7], following earlier work of

Catmull and Clark, “smoothes” a polyhedron by an iteration very similar in spirit to Fejes Tóth’s algorithm. For a convex polyhedron P_0 , the Doo algorithm proceeds as follows. If F is a facet of P_0 , a new facet is formed by joining, by line segments, the midpoints of the line segments whose endpoints are the centroid of F and a vertex of F . If F is an m -gon, the new facet is also an m -gon, called a *type F* facet. Suppose F_1 and F_2 are facets of P_0 with an edge e in common. If G_1 and G_2 are the corresponding new type F facets, then G_1 and G_2 each have an edge parallel to e and of length half that of e . Joining the corresponding endpoints of these edges by line segments, we obtain a parallelogram. Such parallelograms will also be new facets, called *type E* (for edge) facets. Finally, suppose v is a vertex of P_0 . For each facet F of P_0 with v as a vertex, there is a new type F facet G contained in F . Facet G has a vertex v_F at the midpoint of the line segment joining the centroid of F with v . We consider the set of points v_F for such F to be the vertices of a new “facet,” which however will not in general be planar. We call these new “facets” *type V*.

In this way a new “polyhedron” P_1 is defined. We now iterate this construction—using all facets, planar or not—to obtain a sequence (P_n) of “polyhedra.” The limiting body H we call the *Doo kernel* of P_0 .

Although this algorithm is more complicated to describe than Fejes Tóth’s, the analysis of ∂H is in some ways easier, since its construction does not employ the rather unpredictable convex hull operation. For example, the type F and type V facets at each stage are surrounded by a carpet of type E facets (parallelograms). In the CAD literature some efforts have been made to determine the smoothness properties of ∂H and the boundaries of kernels of other algorithms (see [1], [7], [8], and Section 4 of [3]). Our final result, however, shows that the Doo kernel suffers from the disadvantage that it does not preserve convexity.

Theorem 9. *There is a convex polyhedron whose Doo kernel is not convex.*

Proof. Let P_0 be a convex polyhedron with the following properties:

- (i) $P_0 \cap \{(x, y, z): z \geq 0\}$ is the pyramid with vertices $(0, 0, 1)$, $(\pm 1, 0, 0)$, and $(0, \pm 1, 0)$.
- (ii) The segments $[(0, 1, 0), (1, 0, 0)]$ and $[(0, -1, 0), (-1, 0, 0)]$ are edges, so that there are two triangular facets, T_1 and T_2 , above the first and third quadrants in the xy -plane.
- (iii) The two facets containing $(0, 0, 1)$ above the second and fourth quadrants are hexagons, H_1 and H_2 , which extend below the xy -plane and are such that H_2 is the reflection of H_1 in the z -axis.
- (iv) The lowest facet of P_0 is horizontal and has its centroid, b say, on the z -axis.

Suppose that t_i is the centroid of T_i and h_i is the centroid of H_i for $i = 1, 2$. The type V “facet” of P_1 corresponding to the vertex $v = (0, 0, 1)$ of P_0 has four vertices, at $\frac{1}{2}(t_i + v)$, $\frac{1}{2}(h_i + v)$, $i = 1, 2$. The centroid of this “facet” is $c = (v/2) + (t_1 + t_2 + h_1 + h_2)/8$. Our assumptions (i)–(iii) above imply that c lies on the z -axis, and it is clear that $c \neq b$.

Now, for $i = 1, 2$, t_i lies in the boundary ∂H of the Doo kernel H of P_0 . If H

is convex, the point p at the intersection of $[t_1, t_2]$ and the z -axis belongs to H and has positive z -coordinate. However, both b and c belong to ∂H , and by choosing H_i sufficiently long we may ensure that c has a negative z -coordinate, as does b . This is impossible, so H cannot be convex. $\square$

References

- [1] A. A. Ball and D. J. T. Storry. Conditions for tangent plane continuity over recursively generated B-spline surfaces. *ACM Trans. Graphics*, 7:83–102, 1988.
- [2] M. Bourdeau and S. Dubuc. L'itération de Fejes Tóth sur un polygone. *J. Geom.*, 6:65–75, 1975.
- [3] A. S. Caveretta and C. A. Micchelli. The design of curves and surfaces by subdivision algorithms. In *Mathematical Methods in Computer Aided Design*, T. Lyche and L. L. Schumaker, eds., pp. 115–153. Academic Press, New York, 1989.
- [4] G. M. Chaikin. An algorithm for high speed curve generation. *Comput. Graphics Image Process.*, 3:346–349, 1974.
- [5] M. G. Darboux. Sur un problème de géométrie élémentaire. *Bull. Sci. Math.*, 2:298–304, 1878.
- [6] G. de Rham. Sur les courbes limites de polygones obtenus par trisection. *L'Enseign. Math.*, 5:729–747, 1959.
- [7] D. W. H. Doo. A subdivision algorithm for smoothing down irregular shaped polyhedrons. In *Proceedings: Interactive Techniques in Computer Aided Design*, Bologna, 1978, pp. 157–165.
- [8] D. Doo and M. Sabin. Behaviour of recursive division surfaces near extraordinary points. *Comput. Aided Design*, 10:356–360, 1978.
- [9] J. C. Fischer, D. Ruoff, and J. Shilleto. Polygons and polynomials. In *The Geometric Vein*, C. Davis, B. Grünbaum, and F. A. Sherk, eds., pp. 321–333. Springer-Verlag, New York, 1981.
- [10] R. J. Gardner and P. McMullen. On Hammer's x-ray problem. *J. London Math. Soc.*, 21:171–175, 1980.
- [11] R. F. Reisenfeld. On Chaikin's algorithm. *Comput. Graphics-Image Process.*, 4:304–310, 1975.
- [12] G. Fejes Tóth. Iteration processes leading to a regular polygon (in Hungarian). *Mat. Lapok*, 23:135–141, 1972. (*Math. Rev.*, 49, #3683.)
- [13] L. Fejes Tóth. Sequences of polyhedra. *Amer. Math. Monthly*, 88:145–146, 1981.
- [14] T. Zamfirescu. Nearly all convex surfaces are smooth and strictly convex. *Monatsh. Math.*, 103:57–62, 1987.

Received December 14, 1990, and in revised form October 21, 1991.