A New Step in the Solution of the Szökefalvi-Nagy Problem

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Abstract. A description of vector systems $H \subset R^d$ that satisfy the condition $\dim H = 2$ is given. With the help of this description the main result is obtained. It consists in a listing of all compact, 2-Helly-dimensional, convex bodies $M \subset R^d$. The listing is made in terms of polar bodies M^* .

1. Introduction

Let $M \subset R^d$ be a compact, convex body, and let $T(M)$ be the family of all its translates. By $\dim T(M)$ we denote the *Helly dimension* of the family, that is, the least of integers $m \geq 0$ that have the following property: For every collection of sets $M_1, \dots, M_s \in T(M)$, $s > m + 1$, if each $m + 1$ of them has a nonempty intersection, then the intersection $M_1 \cap \dots \cap M_s$ of all sets taken is also nonempty.

It is easily shown that if $M \subset R^d$ is a d -dimensional parallelotope, then $\dim T(M) = 1$. This means that if $M_1, \dots, M_s$, $s > 2$, are translates of a parallelotope, such that every two of them have a nonempty intersection, then $M_1 \cap \dots \cap M_s \neq \emptyset$. The famous Hungarian mathematician Szökefalvi-Nagy has proved [1] that the converse assertion is also true:

Theorem 1 (Szökefalvi-Nagy). *If $M \subset R^d$ is a compact convex body and $\dim T(M) = 1$, then M is a parallelotope.*

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This interesting result leads us to the following problem of combinatorial geometry:

The Szökefalvi-Nagy Problem. To find all compact, convex bodies $M \subset R^d$ for which $\text{him } T(M) = s$ where s is a given integer ($2 \leq s \leq d$).

We indicate partial results that were obtained in this direction. In order to state these results we recall some notions.

Definition 1. Let $M_1, \dots, M_k$ be convex sets in R^d . The set of all points $x_1 + \dots + x_k$, where $x_1 \in M_1, \dots, x_k \in M_k$, is said to be the *vector sum* (or Minkowski's sum) of the convex sets $M_1, \dots, M_k$. If there exist subspaces $L_1, \dots, L_k$ of R^d , such that $R^d = L_1 \oplus \dots \oplus L_k$ and, for every $i = 1, \dots, k$, a translate of M_i is contained in L_i , then the vector sum $M_1 + \dots + M_k$ is said to be the *direct vector sum* and is denoted by $M_1 \oplus \dots \oplus M_k$.

The following theorem was proved in [2].

Theorem 2 (Boltyanski). *Let $M \subset R^d$ be a compact, centrally symmetric, convex body. If $\text{him } T(M) = 2$, then M is a direct vector sum of convex sets each of which has dimension ≤ 2 .*

We are now going to formulate the results that were established in [3] and [4], but first we describe two classes of the convex bodies (see [5]–[7]).

Definition 2. A convex polytope $M \subset R^d$ is said to be a *zonotope* if it may be represented as a vector sum (in the general case, nondirect) of a finite number of closed segments. A compact, convex body $M \subset R^d$ is said to be a *zonoid* if it is a limit (in the Hausdorff metric) of a convergent sequence of zonotopes,

Theorem 3 (Baladze). *Let $M \subset R^d$ be a zonoid (in particular, a zonotope). The inequality $\text{him } T(M) \leq s$ is true if and only if M is a direct vector sum of convex sets, each of which has dimension $\leq s$.*

The aforesaid theorems are in accordance with the following:

Direct Sum Conjecture. Let $M \subset R^d$ be a compact, convex body. If $\text{him } T(M) \leq s$, then M is a direct vector sum of convex sets, each of which has dimension $\leq s$.

More exactly, in Theorems 1–3 a class K of convex bodies in R^d and a subset $S = \{1, 2, \dots, d\}$ are indicated, such that for every convex body $M \subset K$ and every $s \in S$ the direct sum conjecture holds. Indeed, in Theorem 1 K is the class of *all* convex bodies and $S = \{1\}$. Furthermore, in Theorem 2 K is the class of all *centrally symmetric* convex bodies and $S = \{1, 2\}$. Finally, in Theorem 3 K is the class of all *zonoids* in R^d (in particular, zonotopes) and $S = \{1, 2, \dots, d\}$. In each of these cases the direct sum conjecture is confirmed.

Besides that, there are some results on the Szökefalvi-Nagy problem which fall outside the framework of the direct sum conjecture. In [8] Boltyanski and Chabukiani have listed all convex bodies in R^3 that satisfy the condition $\text{him } T(M) = 2$ (without the assumption of central symmetry). In [9] Kincses has listed all *centrally symmetric* convex bodies in R^d that satisfy the conditions $\text{him } T(M) = 3$ or $\text{him } T(M) = 4$. Furthermore, we mention one more result [2], [10] that gives a formal algebraic solution of the Szökefalvi-Nagy problem in the general case. For this purpose we need several definitions.

Definition 3. Vectors $a_0, \dots, a_m \in R^d$ are said to be *minimally dependent* if they are the vertices of an m -dimensional simplex T , such that the origin 0 belongs to the relative interior $\text{relint } T$ of T . Equivalently, the vectors $a_0, \dots, a_m$ are minimally dependent if and only if (i) they are positively dependent:

$$\lambda_0 a_0 + \dots + \lambda_m a_m = 0, \quad \lambda_0 > 0, \dots, \lambda_m > 0,$$

and (ii) any m of them (consequently, each m of them) are linearly independent.

Definition 4. Let H be a nonempty subset of the unit sphere $S^{d-1} \subset R^d$. By $\text{md } H$ we denote the largest of integers m , such that there is in H a system of $m + 1$ minimally dependent vectors. (If there is no system of minimally dependent vectors in H , then $\text{md } H = 0$.) Furthermore, let $M \subset R^d$ be a convex body that does not coincide with R^d . The set of all unit vectors $x \in S^{d-1}$ that are outward normals of the body M at its regular boundary points will be denoted by $H(M)$, and the number $\text{md } H(M)$ will be denoted by $\text{md } M$. We recall that a boundary point a of a body M is said to be *regular* if there exists only one hyperplane L through a that supports the body M .

Let us note that if a convex body $M \subset R^d$ is compact, then the set $H(M)$ is *not one-sided*, that is, there is no closed hemisphere $P \subset S^{d-1}$ that contains $H(M)$ (or, what is the same, there exists no vector $p \in S^{d-1}$, such that $\langle p, x \rangle \leq 0$ for every $x \in H(M)$).

Theorem 4 (Boltyanski). *Let $M \subset R^d$ be a compact, convex body. The Helly dimension of the family $T(M)$ is equal to $\text{md } M$, that is, $\text{him } T(M) = \text{md } M$.*

This theorem is an essential step for a complete *geometrical* solution of the Szökefalvi-Nagy problem, that is, for a description of all compact, convex bodies $M \subset R^d$ with $\text{him } T(M) = r$ in geometrical terms (for example, to be a direct vector sum, to be a parallelotope, to have dimension $\leq r$, and so on).

Besides that, we indicate two theorems (that are, at least for us, auxiliary ones). The first of them was established by Givalevich [11], the second is a trivial consequence of a classical theorem of the theory of convex sets.

Theorem 5 (Givalevich). *If a vector system $H \subset R^d$ is not one-sided, then $\text{md } H = \text{md cl } H$.*

Definition 5. Let $M \subset R^d$ be a compact, convex body. A boundary point $x \in \text{bd } M$ is said to be an *extremal point* of M if there exists no segment $I \subset M$, such that $x \in \text{relint } I$. The set of all extremal points of the convex body M is denoted by $\text{ext } M$.

Definition 6. Let $M \subset R^d$ be a set of vectors. By norm E we denote the set of all *normed* nonzero vectors from E , that is, $y \in \text{norm } E$, if and only if there exists a vector $x \in E$, such that $x \neq 0$ and $y = x/|x|$.

Definition 7. Let $X \subset R^d$ be a convex set and let $a \in R^d$. The *polar set* X_a^* of the set X with respect to the point a is defined by

$$X_a^* = \{y \in R^d: \langle x - a, y \rangle \leq 1 \text{ for all } x \in X\}.$$

Theorem 6. Let $M \subset R^d$ be a compact, convex body and let $a \in \text{int } M$. Then the inclusions $H(M) \subset \text{norm ext } M_a^* \subset \text{cl } H(M)$ are true.

The main purpose of the paper is to state and to prove a theorem that contains a listing of all compact, convex bodies $M \subset R^d$ with $\text{him } T(M) = 2$ (or, what is the same, $\text{md } M = 2$). This result will be obtained in the general case, that is, *without* the assumption of central symmetry. However, we use a new approach to a solution of the Szökefalvi-Nagy problem. This approach (applied in [3], [4], [8], and, in particular, [9]) involves describing not the bodies M with $\text{md } M = r$, but their polar bodies M^* instead.

2. Statements of the Results

First we give a description of non-one-sided vector systems $H \subset R^d$ that satisfy the condition $\text{md } H = 2$. With the help of this description we obtain a listing of compact, convex bodies in R^d with $\text{md } M = 2$ (in terms of their polars).

Definition 8. A vector system $H \subset S^{d-1}$ is said to be *decomposed* if there exists a direct decomposition $R^d = L_1 \oplus \cdots \oplus L_s$ (where $\dim L_i < d$ for all $i = 1, \dots, s$), such that $H \subset L_1 \cup \cdots \cup L_s$.

Definition 9. A vector system $H \subset R^d$ is said to be *2-system* if it is non-one-sided and satisfies the condition $\text{md } H = 2$.

By virtue of this definition, for every 2-system $H \subset S^{d-1}$ there exists a direct decomposition $R^d = L_1 \oplus \cdots \oplus L_s$ ($s \geq 1$), such that $H \subset L_1 \cup \cdots \cup L_s$ and each vector system $H \cap L_i$ is an indecomposable 2-system. So it is sufficient to describe all indecomposable 2-systems.

Furthermore, according to the following theorem, it is sufficient to find all *maximal* 2-systems (since if a 2-system $H \subset L_i$ is indecomposable, then a maximal 2-system in L_i that contains H is indecomposable, too.)

Theorem 7. *Each indecomposable 2-system $H \subset R^d$ is contained in a maximal indecomposable 2-system.*

Proof. By virtue of Theorem 5, each 2-system is contained in a closed 2-system (because each 2-system $H \subset S^{d-1}$ is not one-sided). So it is sufficient to consider only closed 2-systems. Furthermore, for every 2-system $H \subset S^{d-1}$ its convex hull $\text{conv } H$ is a convex body in R^d that is uniquely defined by H . This means that we must find a maximal element in the family (ordered by inclusion) consisting of all convex bodies $\text{conv } H'$ for closed 2-systems H' , each of which contains the given 2-system H . For closed, convex bodies in R^d an application of the Zorn lemma does not require use of the Zermelo axiom. So it is sufficient to prove that each chain $F' \subset F$ is bounded. Indeed, the set

$$H_1 = \text{cl} \bigcup_{H' \in F'} H'$$

is, obviously, a closed 2-system that satisfies the inequality $H' \leq H_1$ for all $H' \in F'$. This completes the proof. $\square$

Now we introduce notions that help us describe all maximal 2-systems.

Definition 10. Let $H \subset S^{d-1}$ be a non-one-sided vector system. A vector $e \in R^d$ is said to be *particular* with respect to H , if it cannot be represented as a positive linear combination of vectors from $H \sim \{e\}$.

It is evident that if $e \in R^d$ is a particular vector with respect to a 2-system H , then $e \in H$ and $H \sim \{e\} \subset P$, where P is a closed half-space with boundary hyperplane through the origin.

In the remaining part of this section theorems will be only stated. The proofs are contained in the following sections.

Theorem 8. Let $H \subset S^{d-1}$ be an indecomposable 2-system, $d \geq 3$. If H contains no particular vector, then $d = 4$ and the system has (in a basis e_1, e_2, e_3, e_4 of the space R^4) the form $H = H^{(0)}$, where

$$H^{(0)} = \text{norm}\{e_1, e_2, e_3, e_4, -e_1 - e_2, -e_1 - e_3, -e_2 - e_4, -e_3 - e_4, e_1 + e_2 + e_3 + e_4\}.$$

Theorem 9. Let $H \subset S^{d-1}$ be a maximal, indecomposable 2-system that contains, at least, one particular vector, $d \geq 3$. We denote by H^Δ the set of all particular vectors of the system H . Then $H^\Delta = \{e_1, \dots, e_k\}$, where $e_1, \dots, e_k$ are linearly independent. If $k > 1$ we denote by F_{ij} the angle $F_{ij} = \{v: v = x_i e_i + x_j e_j; x_i \leq 0, x_j \leq 0\}$. Then

$$H = \text{norm}\left(\{e_1, \dots, e_d\} \cup \left(\bigcup_{i < j \leq d} F_{ij}\right)\right) \quad \text{for } k = d. \quad (1)$$

Further, let $k < d$. Then there exist vectors $e_{k+1}, \dots, e_d$ and a map

$$\varphi: \{k+1, \dots, d\} \rightarrow \{1, \dots, k\},$$

such that $e_1, \dots, e_d$ is a basis in R^d (not necessarily orthonormal) and, denoting by $\square_{l,i}$ the closed half-plane $\{v: v = x_i e_1 + x_i e_i; x_i \leq 0\}$, the following relations are true:

$$H = \text{norm}\left(\{e_1\} \cup \left(\bigcup_{l>1} \square_{l,1}\right)\right) \quad \text{for } k=1, \quad (2)$$

$$H = \text{norm}\left(\{e_1, e_2\} \cup F_{12} \cup \left(\bigcup_{l>2} \square_{l,\varphi(l)}\right)\right) \quad \text{and } \varphi \text{ is onto} \quad \text{for } k=2, \quad (3)$$

$$H = \text{norm}\left(\{e_1, \dots, e_k\} \cup \left(\bigcup_{i<j\leq k} F_{ij}\right) \cup \left(\bigcup_{i>k} \square_{i,\varphi(i)}\right)\right) \quad \text{for } 2 < k < d. \quad (4)$$

Remark. We may suppose (using a suitable permutation of the indices $1, \dots, d$) that the function φ is nonstrictly increasing and the inverse images $\varphi^{-1}(l)$ satisfy the condition $|\varphi^{-1}(1)| \geq \dots \geq |\varphi^{-1}(k)|$. Under these conditions the function φ is defined uniquely by the maximal indecomposable 2-system H .

Theorems 8 and 9 give a complete listing of all maximal indecomposable 2-systems. So, in R^3 we have only two types of maximal indecomposable 2-systems:

$$(A) \quad H = \text{norm}(\{e_1\} \cup \square_{21} \cup \square_{31}) \quad (k=1),$$

$$(B) \quad H = \text{norm}(\{e_1, e_2, e_3\} \cup F_{12} \cup F_{13} \cup D_{23}) \quad (k=3)$$

(we note that the case $k=2$ is nonrealizable because there is no mapping of the set $\{3\}$ onto $\{1, 2\}$). This result was announced in [8].

Further, in R^4 we have five types of maximal indecomposable 2-systems:

$$(A) \quad H = \text{norm}(\{e_1\} \cup \square_{21} \cup \square_{31} \cup \square_{41}) \quad (k=1),$$

$$(B) \quad H = \text{norm}(\{e_1, e_2\} \cup F_{12} \cup \square_{31} \cup \square_{42}) \quad (k=2),$$

$$(C) \quad H = \text{norm}(\{e_1, e_2, e_3\} \cup F_{12} \cup F_{13} \cup F_{23} \cup \square_{41}) \quad (k=3),$$

$$(D) \quad H = \text{norm}\left(\{e_1, e_2, e_3, e_4\} \cup \left(\bigcup_{i<j\leq 4} F_{ij}\right)\right) \quad (k=4),$$

$$(E) \quad H = H^{(0)} \quad (\text{see Theorem 8}).$$

This result is announced in a new paper by Boltyanski and Chabukiani (to appear in *Soobtsheniya AN GSSR*, Tbilisi, 1992).

Finally, we formulate the following theorem that contains a listing of all convex, compact, indecomposable bodies $M \subset R^d$ with $\text{md } M = 2$. In the statement of the theorem we preserve the notations of Theorem 9 (see also the Remark after Theorem 9).

Theorem 10. *Let $M \subset R^d$ be a compact, convex body; $d \geq 3$. Let us suppose that $0 \in \text{int } M$ and M is not decomposable into a direct vector sum of convex sets, each of which has dimension $< d$. The inequality $\text{md } M \leq 2$ holds in and only in the following cases:*

- (A) *There exist a basis $e_1, \dots, e_d$ in R^d and closed, two-dimensional convex sets B_{ij} ($i < j \leq d$), such that*
 - (i) $B_{ij} \subset \text{lin}(e_i, e_j)$ and $\text{ext } B_{ij} \subset \{e_i, e_j\} \cup F_{ij}$,
 - (ii) $M^* = \text{conv}(\bigcup_{i < j \leq d} B_{ij})$.
- (B) *There exist a basis $e_1, \dots, e_d$ in R^d and closed, two-dimensional convex sets $C_2, \dots, C_d$, such that*
 - (i) $C_l \subset \text{lin}(e_1, e_l)$ and $\text{ext } C_l \subset \{e_1\} \cup \bigcap_{l,1} \text{ for } l = 2, \dots, d$,
 - (ii) $M^* = \text{conv}(C_2 \cup \dots \cup C_d)$.
- (C) *There exist a basis $e_1, \dots, e_d$ in R^d , an integer k ($2 \leq k < d$), a map $\varphi: \{k+1, \dots, d\} \rightarrow \{1, \dots, k\}$ (that is onto in the case $k=2$), and closed, two-dimensional convex sets B_{ij}, C_l ($i < j \leq k; l > k$), such that*
 - (i) $B_{ij} \subset \text{lin}(e_i, e_j)$ and $\text{ext } B_{ij} \subset \{e_i, e_j\} \cup F_{ij}$,
 - (ii) $C_l \subset \text{lin}(e_l, e_{\varphi(l)})$ and $\text{ext } C_l \subset \{e_{\varphi(l)}\} \cup \bigcap_{l, \varphi(l)}$.
- (D) $d = 4$ and there exists a basis e_1, e_2, e_3, e_4 in R^4 , such that $H(M) = H^{(0)}$ (see Theorem 8).

3. Auxiliary Propositions

In this section we prove lemmas that are necessary for the proofs of Theorems 8–10.

Lemma 1. *Let $a_0, a_1, \dots, a_m$ be positively dependent vectors. If they are not minimally dependent, then it is possible to choose among $a_1, \dots, a_m$ fewer than m vectors, such that a_0 and the chosen vectors are minimally dependent.*

Proof. Let $\alpha_0 a_0 + \dots + \alpha_m a_m = 0$ be a positive dependence. Since $a_0, \dots, a_m$ are not minimally dependent, then the vectors $a_1, \dots, a_m$ are linearly dependent. Let

$$\lambda_1 a_1 + \dots + \lambda_m a_m = 0$$

be a nontrivial dependence (among the coefficients there may be positives, as well as negatives or zeros). Without loss of generality we may suppose that $\lambda_1 > 0$. If $\varepsilon > 0$ is small enough, then the dependence

$$\alpha_0 a_0 + \alpha_1 a_1 + \dots + \alpha_m a_m - \varepsilon(\lambda_1 a_1 + \dots + \lambda_m a_m) = 0 \quad (5)$$

is positive, that is, $\alpha_1 - \varepsilon\lambda_1 > 0, \dots, \alpha_m - \varepsilon\lambda_m > 0$. Let ε_0 be the exact upper bound of the numbers $\varepsilon > 0$ for which the dependence (5) is positive (the number ε_0 exists, since, for $\varepsilon > \alpha_1/\lambda_1$, the dependence (5) is not positive). Thus for $\varepsilon = \varepsilon_0$ at least one of the coefficients in (5) turns to zero. So, substituting $\varepsilon = \varepsilon_0$ we obtain from (5) a positive dependence between a_0 and some of the vectors $a_1, \dots, a_m$, and moreover the number of summands in the dependence obtained is less than earlier. Continuing this process, we obtain, finally, a minimal dependence between a_0 and some of the vectors $a_1, \dots, a_m$. This completes the proof. $\square$

Lemma 2. *Let $M \subset S^{d-1}$ be a 2-system. Let, furthermore, L be a subspace, such that $0 < \dim L < d$ and each vector $v \in L$ is nonparticular with respect to the vector system $H \cap L$. Then the system H is decomposable.*

Proof. Let K be the set of all nonnegative linear combinations of the vectors from $H \sim L$. Then K is a convex cone with apex 0. First we prove that $L \cap K = \{0\}$. Let us suppose, counterwise, that there exists a nonzero vector $v_0 \in L \cap K$. Each nonzero vector $v \in L \cap K$ is representable as a positive linear combination of several vectors from $H \sim L$. Let m be the least of the integers, such that a nonzero vector $v_0 \in L \cap K$ is positively representable through m vectors from $H \sim L$, and $v_0 = \lambda_1 a_1 + \dots + \lambda_m a_m$ be the positive dependence. Then $m \geq 2$ and no nontrivial positive linear combination of fewer than m of the vectors $a_1, \dots, a_m$ belongs to L . Besides that, the vectors $-v_0, a_1, \dots, a_m$ are minimally dependent (by virtue of Lemma 1). Since, furthermore, $-v_0 \in L$ is a nonparticular vector with respect to the vector system $H \cap L$, we have $-v_0 = \mu_1 b_1 + \dots + \mu_k b_k$, where $b_1, \dots, b_k \in (H \cap L) \sim \{-v_0\}$ and all the coefficients are positive ($k \geq 2$). We may suppose that the vectors $b_1, \dots, b_k$ are linearly independent (see Lemma 1). So we obtain the positive dependence

$$\lambda_1 a_1 + \dots + \lambda_m a_m + \mu_1 b_1 + \dots + \mu_k b_k = 0$$

with not less than four summands. Moreover, the vectors $a_1, \dots, a_{m-1}, b_1, \dots, b_k$ are linearly independent (indeed, if $\gamma_1 a_1 + \dots + \gamma_{m-1} a_{m-1} + v_1 b_1 + \dots + v_k b_k = 0$, then $\gamma_1 a_1 + \dots + \gamma_{m-1} a_{m-1} \in L$; consequently, $\gamma_1 = \dots = \gamma_{m-1} = 0$, and then $v_1 = \dots = v_k = 0$). This means that the vectors $a_1, \dots, a_m, b_1, \dots, b_k$ are minimally dependent, and, hence, $\text{md } H \geq 3$, contradicting the condition of the lemma.

The contradiction obtained shows that $L \cap K = \{0\}$. We now prove that $K = \text{aff } K$, that is, K is a subspace. Let us suppose that $K \neq \text{aff } K$. Then $0 \notin \text{relint } K$ and, consequently, the convex cones L, K with common apex 0 are separated. This means that there exists a hyperplane Γ through the origin, such that $L \subset \Gamma$ and $K \subset P$ where P is a closed half-space with boundary Γ . Consequently, $H = (H \cap L) \cup (H \sim L) \subset L \cup K \subset P$, contradicting the assumption that H is a 2-system (and, hence, H is non-one-sided).

So, $K = \text{aff } K$, that is, K is a subspace of R^d . By virtue of the relations $L \cap K = \{0\}$, $H \subset L \cup K$, we conclude that $R^d = L \oplus K$ (since H is non-one-sided), and, consequently, the vector system H is decomposable (we recall that $0 < \dim L < d$). This completes the proof. $\square$

Lemma 3. *Let H be a 2-system, and let $a_0, a_1, a_2 \in H$ be minimally dependent vectors. Then each unit vector v of the subspace $L = \text{lin}(a_0, a_1, a_2)$, except maybe a_0, a_1, a_2 , is a nonparticular vector with respect to the vector system $H \cap L$.*

This assertion is evident.

Lemma 4. *Let $H \subset R^d$ be an indecomposable 2-system and let $a_0, a_1, a_2 \in H$ be three minimally dependent vectors. If $d > 2$, then at least one of the vectors a_0, a_1, a_2 is particular with respect to the vector system $H \cap \text{lin}(a_0, a_1, a_2)$. This means that there exists a closed half-plane $P \subset \text{lin}(a_0, a_1, a_2)$ with boundary line through the origin, such that $H \cap \text{lin}(a_0, a_1, a_2) \subset \{e\} \cup P$, where e is one of the vectors a_0, a_1, a_2 .*

This is an evident consequence of Lemmas 2 and 3.

Lemma 5. *Let $H \subset R^d$ be a 2-system and let $a_0, a_1, a_2 \in H$ be three minimally dependent vectors, none of which is particular with respect to H . If $-a_0 \in H$ and $d \geq 3$, then the system H is decomposable.*

Proof. We assume that H is indecomposable and show that this assumption leads to a contradiction. We may suppose that $\mu_0 a_0 + \mu_1 a_1 + \mu_2 a_2 = 0$ with positive coefficients μ_0, μ_1, μ_2 . Since the vector a_0 is nonparticular, then

$$a_0 = \lambda_1 b_1 + \dots + \lambda_s b_s \quad (b_i \neq a_0, b_i \in H, \lambda_i > 0, s \geq 0).$$

By virtue of Lemma 1, we may suppose that the vectors $-a_0, b_1, \dots, b_s$ are minimally dependent. Consequently, $s = 2$ (since $\text{md } H = 2$), that is, $a_0 = \lambda_1 b_1 + \lambda_2 b_2$. If now $b_1, b_2 \notin \text{lin}(a_0, a_1, a_2)$, then the vectors a_0, a_1, b_1 are linearly dependent, and, moreover, the vectors a_1, a_2, b_1, b_2 are positively dependent (since $a_0 = -(\mu_1/\mu_0)a_1 - (\mu_2/\mu_0)a_2 = \lambda_1 b_1 + \lambda_2 b_2$). This means that $\text{md } H \geq 3$ contradicting the condition of the lemma. Consequently, b_1, b_2 belong to the subspace $L = \text{lin}(a_0, a_1, a_2)$, that is, a_0 is a nonparticular vector with respect to the system $H \cap L$. Therefore, by virtue of Lemma 4 (and according to our assumption about the indecomposability of H), one of the vectors a_1, a_2 (say a_1) is particular with respect to the vector system $H \cap L$. This means that $H \cap L \subset \{a_1\} \cup P$, where $P \subset L$ is a closed half-plane with boundary line through the origin, such that $a_0, -a_0, a_2 \in P$. Now from the equality $a_0 = \lambda_1 b_1 + \lambda_2 b_2$ (where $\lambda_1 > 0, \lambda_2 > 0, b_1, b_2 \in H \cap L$) we deduce that $b_1 = a_1$ and $b_2 \in \text{relint } P, b_2 \neq a_2$. Moreover, the vectors a_1, a_2, b_2 are minimally dependent.

Since a_1 is a nonparticular vector with respect to H , then $a_1 = v_1 c_1 + \dots + v_t c_t$ ($c_i \neq a_1, c_i \in H, v_i > 0; t \geq 2$). Applying Lemma 1 to the vectors $-a_1, c_1, \dots, c_t$, we may suppose that the vectors $c_1, \dots, c_t$ are linearly independent. Let us denote by l_1 the line through the origin that contains a_1 . The equality $a_1 = v_1 c_1 + \dots + v_t c_t$ implies $l_1 \subset \text{lin}(c_1, \dots, c_t)$. So the following two cases are possible: (a) $L \cap \text{lin}(c_1, \dots, c_t) = l_1$ and (b) $L \subset \text{lin}(c_1, \dots, c_t)$. We consider these two cases separately.

(a) Since the vectors a_1, a_2, b_2 are minimally dependent, then $a_2 \notin l_1$. This means that $a_2 \in L \sim \text{lin}(c_1, \dots, c_t)$, and, therefore, $a_2 \notin \text{lin}(c_1, \dots, c_t)$. Consequently, the vectors $a_2, c_1, \dots, c_t$ are linearly independent. On the other hand, there exist positive numbers α, β , such that $a_1 + \alpha a_2 + \beta b_2 = 0$ (since a_1, a_2, b_2 are minimally dependent), and, hence, $-\alpha a_2 - \beta b_2 = v_1 c_1 + \dots + v_t c_t$. We have obtained $t + 2 \geq 4$ minimally dependent vectors $a_2, b_2, c_1, \dots, c_t \in H$, contradicting the condition $\text{md } H = 2$. So, case (a) leads to a contradiction.

(b) Let us denote by K the set of all vectors $x = \gamma_1 c_1 + \dots + \gamma_t c_t$ with nonnegative coefficients $\gamma_1, \dots, \gamma_t$. The set K is a closed convex cone with apex at the origin. By virtue of the relation $a_1 = v_1 c_1 + \dots + v_t c_t$, with positive $v_1, \dots, v_t$, we have $a_1 \in \text{relint } K$. Furthermore, since the vectors $c_1, \dots, c_t$ are linearly independent, then the cone K is pointed (that is, it does not contain any line). Consequently, $K \cap L$ is a convex angle distinct from a half-plane, and $a_1 \in \text{relint}(K \cap L)$. Since $K \cap L$ is not a half-plane, then at least one of the vectors $a_0, -a_0$ does not belong to $K \cap L$; let, for example, $a_0 \notin K \cap L$. This means that one side of the convex angle $K \cap L$ has with the closed half-plane P the only common point 0 and is situated between the vectors a_0, a_1 . So, denoting by q the unit vector directed along this side, we conclude that $-q = \sigma b_2 + \tau(-a_0)$ with positive σ, τ . Furthermore, since $q \in \text{relbd } K$ then we may choose from $c_1, \dots, c_t$ several vectors, say $c_1, \dots, c_p$ ($p < t$), such that $q = \rho_1 c_1 + \dots + \rho_p c_p$ with positive $\rho_1, \dots, \rho_p$ and L is not contained in $\text{lin}(c_1, \dots, c_p)$. Consequently, $\sigma b_2 + \tau(-a_0) + \rho_1 c_1 + \dots + \rho_p c_p = 0$. Moreover, the vectors $c_1, \dots, c_p$ are linearly independent and $a_0 \notin \text{lin}(c_1, \dots, c_p)$, and, consequently, $a_0, c_1, \dots, c_p$ are linearly independent. Thus, the vectors $b_2, -a_0, c_1, \dots, c_p$ are minimally dependent, and, hence, $\text{md } H \geq p + 1 \geq 3$, which is false. So, case (b) leads to a contradiction, too.

We see that the assumption about the indecomposability of H leads to a contradiction. This completes the proof. $\square$

Lemma 6. *Let $H \subset \mathbb{R}^d$ be an indecomposable 2-system, and let $a_0, a_1, a_2 \in H$ be minimally dependent vectors, none of which is particular with respect to H . Then there exist vectors b_1, b_2 , each of which is distinct from a_2 , such that the vectors a_0, b_1, b_2 are minimally dependent.*

Proof. We may suppose that $\mu_0 a_0 + \mu_1 a_1 + \mu_2 a_2 = 0$ with positive coefficients. Since the vector a_2 is not particular, then $a_2 = \lambda_1 c_1 + \dots + \lambda_k c_k$, where $c_1, \dots, c_k \in H$, $c_i \neq a_2$, $\lambda_i > 0$. Consequently,

$$\frac{\mu_0}{\mu_2} a_0 + \frac{\mu_1}{\mu_2} a_1 + \lambda_1 c_1 + \dots + \lambda_k c_k = 0.$$

We may suppose that each of the vectors $c_1, \dots, c_k$ is distinct from a_0 and a_1 . Let us choose from $a_1, c_1, \dots, c_k$ vectors, such that a_0 and the chosen vectors are minimally dependent (see Lemma 1). It is clear that this minimally dependent system is a triplet a_0, b_1, b_2 (because $-a_0 \notin H$, see Lemma 5), and each of the vectors b_1, b_2 is distinct from a_2 (but it is possible that one of the vectors b_1, b_2 coincides with a_1). $\square$

Lemma 7. *Let $H \subset R^d$ be an indecomposable 2-system and let $a_0, a_1, a_2 \in H$ be minimally dependent vectors, none of which is particular with respect to H . If $d > 2$, then the intersection $H \cap \text{lin}(a_0, a_1, a_2)$ consists only of the vectors a_0, a_1, a_2 .*

Proof. According to Lemma 4, there exists a vector among a_0, a_1, a_2 , say a_0 , that is particular with respect to the vector system $H \cap \text{lin}(a_0, a_1, a_2)$. This means that $H \cap \text{lin}(a_0, a_1, a_2) \subset \{a_0\} \cup P_0$, where P_0 is a closed half-plane with boundary line through the origin. Let us suppose that there exists a vector $e \in H \cap \text{lin}(a_0, a_1, a_2)$ that does not belong to $\{a_0, a_1, a_2\}$. Then, according to Lemma 5, $-e$ does not coincide with any vector a_0, a_1, a_2 . Consequently, $-e$ is a positive linear combination of two of the vectors a_0, a_1, a_2 , say, $-e = \lambda_0 a_0 + \lambda_1 a_1$, where $\lambda_0 > 0, \lambda_1 > 0$. This means that the vectors e, a_0, a_1 are minimally dependent. By virtue of Lemma 6 there exist vectors b_1, b_2 , each of which is distinct from a_0 , such that e, b_1, b_2 are minimally dependent, that is, $-e = \mu_1 b_1 + \mu_2 b_2$ with positive coefficients μ_1, μ_2 . It is evident that the vectors b_1, b_2 did not belong to $\text{lin}(a_0, a_1, a_2)$, since otherwise we should have $e, b_1, b_2 \in P_0$, contradicting the minimal dependence of these vectors. On the other hand, e is a positive linear combination of some two vectors from a_0, a_1, a_2 (since e coincides with no vector a_0, a_1, a_2), say, $e = \nu_0 a_0 + \nu_2 a_2$. So we have $\nu_0 a_0 + \nu_2 a_2 + \mu_1 b_1 + \mu_2 b_2 = 0$. Moreover, the vectors a_0, a_2, b_1 are linearly independent, since $b_1 \notin \text{lin}(a_0, a_1, a_2)$. Therefore, the vectors $a_0, a_2, b_1, b_2 \in H$ are minimally dependent, contradicting the condition $\text{md } H = 2$. This completes the proof. $\square$

Lemma 8. *Let $H \subset S^{d-1}$ be a 2-system and let a_0, a_1, a_2 be three minimally dependent vectors, none of which is particular with respect to H . If $d > 2$, then there exist vectors b_1, b_2 not belonging to $\text{lin}(a_0, a_1, a_2)$, such that a_0, b_1, b_2 are minimally dependent.*

This is an evident consequence of Lemmas 6 and 7.

Lemma 9. *Let $H \subset R^d$ be a maximal 2-system and let H^Δ be the set of all its particular vectors. If $H^\Delta \neq \emptyset$, then for every vector $e \in H^\Delta$ we have $-e \in H$.*

Proof. We must prove that $\text{md}(H \cup \{-e\}) = 2$ (it would follow from this that $-e \in H$, since H is maximal). Indeed, let $\alpha(-e) + \gamma_1 c_1 + \dots + \gamma_m c_m = 0$ be a minimal dependence, $c_1, \dots, c_m \in H$. Since e is a particular vector, then there exists a hyperplane N through the origin, such that $H \sim \{e\} \subset P$, where P is one of the closed half-spaces determined by N . Besides that, we have $e \notin P$, and, consequently, $-e \in \text{int } P$. So, if each of the vectors $c_1, \dots, c_m$ is distinct from e , then all the vectors $c_1, \dots, c_m$ belong to H , and $-e \in \text{int } P$, contradicting the minimal dependence of these vectors. This contradiction shows that one of the vectors $c_1, \dots, c_m$ coincides with e , say $c_1 = e$. However, then we have $m = 1$, and the minimal dependence has the form $e + (-e) = 0$. Thus if we have a minimal dependence $\beta_0 b_0 + \dots + \beta_s b_s = 0$ (where $b_0, \dots, b_s \in H \cup \{-e\}$), then either $-e \notin \{b_0, \dots, b_s\}$, that is, $b_0, \dots, b_s \in H$, and then $s \leq 2$, or $-e \in \{b_0, \dots, b_s\}$ and then $s = 1$. In any case $s \leq 2$, that is, $\text{md}(H \cup \{-e\}) = 2$. $\square$

Lemma 10. *Let $H \subset R^d$ be a maximal 2-system and let H^Δ be the set of all its particular vectors. If $H^\Delta \neq \emptyset$, then the vector system $H \cap \text{lin } H^\Delta$ is non-one-sided in $\text{lin } H^\Delta$.*

This follows immediately from Lemma 9.

Lemma 11. *Let $H \subset R^d$ be an indecomposable maximal 2-system and let $H^\Delta = \{e_1, \dots, e_k\}$ be the set of all its particular vectors. Then the vectors $e_1, \dots, e_k$ are linearly independent.*

Proof. Let us suppose that the vectors $e_1, \dots, e_k$ are linearly dependent. Without loss of generality we may suppose that a linear dependence has a form $\lambda_1 e_1 + \dots + \lambda_q e_q = 0$, where $e_1, \dots, e_{q-1}$ are linearly independent, $q \leq k$, $\lambda_i > 0$ for $i \leq s$ and $\lambda_i < 0$ for $s < i \leq q$. Then we have

$$\lambda_1 e_1 + \dots + \lambda_s e_s + |\lambda_{s+1}|(-e_{s+1}) + \dots + |\lambda_q|(-e_q) = 0,$$

that is, the vectors $e_1, \dots, e_s, -e_{s+1}, \dots, -e_q \in H$ (see Lemma 7) are minimally dependent. Consequently, $q \leq 3$. So we have two cases: (i) $q = 2$ and (ii) $q = 3$. In case (i) the linear dependence has the form $e_1 + e_2 = 0$, that is, $e_2 = -e_1$ where $e_1, e_2 \in H^\Delta$. This means that $H \sim \{e_1 \cup e_2\} \subset P_1 \cap P_2$, $e_1 \notin P_1$, $e_2 \notin P_2$ where P_1, P_2 are closed half-spaces with boundary hyperplanes through the origin. If now $\text{bd } P_1 = \text{bd } P_2$, then $H \subset \text{lin}\{e_1\} \oplus \text{bd } P_1$, contradicting the indecomposability of H . If $\text{bd } P_1 \neq \text{bd } P_2$, then it is easily shown that $P_1 \cap P_2 \cup \{e_1 \cup e_2\}$ is contained in a half-space with boundary hyperplane through the origin, contradicting the non-one-sidedness of H . In case (ii) the linear dependence has the form $\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 = 0$. This means, according to Lemma 9, that the vectors $e_1, e_2, e_3, -e_1, -e_2, -e_3$ belong to the two-dimensional plane $\text{lin}(e_1, e_2, e_3)$. Then each vector $v \in \text{lin}(e_1, e_2, e_3)$ is nonparticular with respect to the vector system $H \cap \text{lin}(e_1, e_2, e_3)$, contradicting the indecomposability of the system H (see Lemma 2). This completes the proof. $\square$

4. Proof of Theorem 8

Let $H \subset S^{d-1}$ be a maximal, indecomposable 2-system without particular vectors; $d \geq 3$. Let, furthermore, a_0, a_1, a_2 be three minimally dependent vectors. According to Lemma 8 there exist vectors $b_0, b_1, b_2, c_0, c_1, c_2$ not belonging to $\text{lin}(a_0, a_1, a_2)$, such that the triplets $(a_0, b_0, c_0), (a_1, b_1, c_1), (a_2, b_2, c_2)$ are minimally dependent. Then the spaces

$$L_0 = \text{lin}(a_0, a_1, a_2, b_0, c_0), \quad L_1 = \text{lin}(a_0, a_1, a_2, b_1, c_1), \quad L_2 = \text{lin}(a_0, a_1, a_2, b_2, c_2)$$

are three-dimensional. First we prove that these three subspaces are pairwise distinct (and, consequently, $d \geq 4$).

Let us suppose, counterwise, that two of these subspaces coincide, say $L_1 = L_2$. Then the vectors a_1, a_2, b_1 form a basis in the three-dimensional space $L_1 = L_2$. Without loss of generality we may suppose that b_1, b_2 lie in one open half-space with boundary $\text{lin}(a_0, a_1, a_2)$, and c_1, c_2 lie in the other open half-space. Since a_1, b_1, c_1 are minimally dependent, then

$$b_1 = -\alpha_1 a_1 - \gamma_1 c_1 \quad (\alpha_1 > 0, \gamma_1 > 0). \quad (6)$$

Similarly,

$$b_2 = -\alpha_2 a_2 - \gamma_2 c_2 \quad (\alpha_2 > 0, \gamma_2 > 0). \quad (7)$$

Furthermore, since b_1, b_2 lie in one open half-space with boundary $\text{lin}(a_0, a_1, a_2)$, we have

$$\lambda b_2 = b_1 + \mu a_1 + \nu a_2 \quad (\lambda > 0). \quad (8)$$

Without loss of generality we may suppose (changing the roles of the indices 1, 2, if necessary) that $\mu \geq 0$. Now we obtain, from the relations (7) and (8),

$$\lambda \gamma_2 c_2 + b_1 + \mu a_1 + (\nu + \lambda \alpha_2) a_2 = 0 \quad (\lambda > 0, \mu \geq 0, \gamma_2 > 0). \quad (9)$$

Besides that, we have a positive dependence

$$p_0 a_0 + p_1 a_1 + a_2 = 0 \quad (10)$$

(since a_0, a_1, a_2 are minimally dependent).

If the coefficient $\nu + \lambda \alpha_2$ is negative, that is, $\nu + \lambda \alpha_2 = -\sigma$ ($\sigma > 0$), then we obtain, from (9) and (10),

$$\lambda \gamma_2 c_2 + b_1 + \sigma p_0 a_0 + (\sigma p_1 + \mu) a_1 = 0.$$

This is a minimal dependence (since all the coefficients are positive, and a_0, a_1, b_1 are linearly independent), contradicting the equality $\text{md } H = 2$. Consequently, $\nu + \lambda \alpha_2 \geq 0$.

Furthermore, if both the coefficients $\mu, \nu + \lambda \alpha_2$ in (9) are positive, then (9) is a minimal dependence, contradicting the equality $\text{md } H = 2$. So both the coefficients $\mu, \nu + \lambda \alpha_2$ are nonnegative, and at least one of them is equal to 0. However, the case $\mu = \nu + \lambda \alpha_2 = 0$ is impossible. Indeed, in this case we would have $\lambda \gamma_2 c_2 + b_1 = 0$ and, consequently, $c_2 + b_1 = 0$ (since $|c_2| = |b_1| = 1$), that is, $-b_1 = c_2 \in H$, contradicting Lemma 5.

Thus, the following two cases are possible:

$$(a) \quad \lambda \gamma_2 c_2 + b_1 + \mu a_1 = 0 \quad (\lambda \gamma_2 > 0, \mu > 0),$$

$$(b) \quad \lambda \gamma_2 c_2 + b_1 + (\nu + \lambda \alpha_2) a_2 = 0 \quad (\lambda \gamma_2 > 0, \nu + \lambda \alpha_2 > 0).$$

In the case of (a) the vectors c_2, b_1, a_1 are minimally dependent, and by virtue of Lemma 7 we have $c_2 = c_1$. In case (b) the vectors c_2, b_1, a_2 are minimally dependent, and by virtue of Lemma 7 we have $b_1 = b_2$. Uniting the two cases (a) and (b), we obtain that in any case there are among b_1, b_2, c_1, c_2 only three distinct vectors. Without loss of generality we can suppose that $b_1 = b_2$, that is, we have three distinct vectors $b_1 = b_2, c_1, c_2$.

Now let us consider the third three-dimensional space L_0 . Since a_0, b_0, c_0 are minimally dependent, then we have

$$b_0 = -\alpha_0 a_0 - \gamma_0 c_0 \quad (\alpha_0 > 0, \gamma_0 > 0). \quad (11)$$

Consequently (by virtue of relation (10)),

$$\begin{aligned} p_0 \alpha_1 \alpha_2 (b_0 + \gamma_0 c_0) + p_1 \alpha_0 \alpha_2 (b_1 + \gamma_1 c_1) + p_2 \alpha_0 \alpha_1 (b_2 + \gamma_2 c_2) \\ = -\alpha_0 \alpha_1 \alpha_2 (p_0 a_0 + p_1 a_1 + p_2 a_2) = 0. \end{aligned} \quad (12)$$

This is a positive dependence among the vectors b_0, b_1, c_0, c_1, c_2 (since $b_2 = b_1$). If now $b_0 \notin L_1$, then the vectors b_1, c_1, c_2, b_0 are linearly independent, and, according to (12), the vectors b_0, c_0, b_1, c_1, c_2 are minimally dependent, contradicting the equality $\text{md } H = 2$. Hence, $b_0 \in L_1$, that is, $L_0 = L_1 = L_2$.

Thus, from the equality $L_1 = L_2$ we have deduced that there are among b_1, b_2, c_1, c_2 only three distinct vectors; say, $b_1 = b_2$. Furthermore, from the equality $L_0 = L_1$ we obtain, similarly, that there are among b_0, b_1, c_0, c_1 only three distinct vectors. So we have two possibilities: (i) $b_0 = b_1$ and (ii) $c_0 = c_1$. In case (ii) we have $b_1 = b_2, c_1 \neq c_2$ and $c_0 = c_1, b_0 \neq b_1$. This means that $b_0 \neq b_2, c_0 \neq c_2$, contradicting the equality $L_0 = L_2$. Consequently, case (i) must hold, that is, $b_0 = b_1 = b_2$, and c_0, c_1, c_2 are pairwise different. In this case we have (by virtue of (12))

$$p_0 \alpha_1 \alpha_2 (b_0 + \gamma_0 c_0) + p_1 \alpha_0 \alpha_2 (b_0 + \gamma_1 c_1) + p_2 \alpha_0 \alpha_1 (b_0 + \gamma_2 c_2) = 0,$$

that is, the vectors b_0, c_0, c_1, c_2 are positively dependent. Besides that, the vectors b_0, c_0, c_1 are linearly independent. Consequently, the vectors b_0, c_0, c_1, c_2 are minimally dependent, contradicting the equality $\text{md } H = 2$. The contradiction obtained shows that all the three-dimensional spaces L_0, L_1, L_2 are pairwise different.

Let us denote by Q the four-dimensional space $\text{lin}(a_0, a_1, a_2, b_1, b_2, c_1, c_2)$. If $b_0 \notin Q$, then the vectors b_1, b_2, c_1, c_2, b_0 are linearly independent. Consequently, the vectors $b_0, b_1, b_2, c_0, c_1, c_2$ are minimally dependent, contradicting the equality $\text{md } H = 2$. This shows that all the vectors $a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2$ lie in the four-dimensional space Q .

Let us now consider two triplets (a_1, b_1, c_1) and (a_2, b_2, c_2) . We have $b_2 \notin \text{lin}(a_1, a_2, b_1) = L_1$ (since $L_1 \neq L_2$). Consequently, a_1, a_2, b_1, b_2 is a basis in Q . Now we have $-b_0 = p_1 + p_2$, where $p_1 \in \text{lin}(a_1, b_1), p_2 \in \text{lin}(a_2, b_2)$. It is clear that $p_1 \neq 0, p_2 \neq 0$ (since $b_0 \notin \text{lin}(a_1, b_1, c_1), b_0 \notin \text{lin}(a_2, b_2, c_2)$). Furthermore, if p_1 coincides with no vector a_1, b_1, c_1 (up to a positive coefficient), then p_1 can be

written as a positive linear combination of two of the vectors a_1, b_1, c_1 , say, $p_1 = \beta b_1 + \gamma c_1$. Hence, $b_0 + \beta b_1 + \gamma c_1 = -p_2 \in \text{lin}(a_2, b_2, c_2)$. Now, $-p_2 \neq 0$ is a positive linear combination of one or two vectors a_2, b_2, c_2 , that is, we obtain four or five minimally dependent vectors $b_0, b_1, c_1, \dots$, contradicting the equality $\text{md } H = 2$. Thus, p_1 must coincide (up to a positive coefficient) with one of the vectors a_1, b_1, c_1 . Similarly, p_2 must coincide with one of the vectors a_2, b_2, c_2 . The same reasoning shows that $-c_0 = q_1 + q_2$, where q_1 coincides with one of the vectors a_1, b_1, c_1 , and q_2 coincides with one of the vectors a_2, b_2, c_2 .

Let us show that $p_1 \neq a_1$. Indeed, in the case $p_1 = a_1, p_2 = a_2$ we obtain $b_0 = -p_1 - p_2 \in \text{lin}(a_1, a_2) = \text{lin}(a_0, a_1, a_2)$, which is impossible by virtue of Lemma 7 (we recall that the nine vectors $a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2$ are pairwise different). Furthermore, in the case $p_1 = a_1, p_2 = b_2$ we conclude that b_0, a_1, b_2 are linearly dependent, and, consequently, $a_1 \in \text{lin}(b_0, b_2) = \text{lin}(b_0, b_1, b_2)$, which is impossible. Finally, in the case $p_1 = a_1, p_2 = c_2$ we find that the vectors b_0, a_1, c_2 are linearly dependent, and, consequently, the vectors b_0, a_1, c_2, a_2 lie in one three-dimensional space. This means that $\text{lin}(a_1, a_2, b_0) = \text{lin}(a_1, a_2, c_2)$, that is, $L_0 = L_2$, contradicting what was proved above. Thus, $p_1 \neq a_1$ (up to a positive coefficient). Similarly, $p_2 \neq a_2, q_1 \neq a_1, q_2 \neq a_2$.

We see that p_1 is equal to either b_1 or c_1 , and p_2 is equal to either b_2 or c_2 (up to a positive coefficient). Let us now remark that (by virtue of pairwise distinctness of the spaces L_0, L_1, L_2) we may interchange the symbols b_1 and c_1 ; similarly, we may interchange the symbols b_2 and c_2 . Therefore, we may suppose that $p_1 = \lambda_1 b_1, p_2 = \lambda_2 b_2$ ($\lambda_1 > 0, \lambda_2 > 0$), that is, the three vectors $b_0 = -p_1 - p_2, b_1, b_2$ are minimally dependent.

Now we conclude that q_1 is equal to either b_1 or c_1 and q_2 is equal to either b_2 or c_2 , but neither $q_1 = b_1$ nor $q_2 = b_2$ is valid. For, if q_1 was equal to b_1 , then the vectors b_0, c_0 (and consequently a_0, a_2) would lie in one three-dimensional space $\text{lin}(b_1, b_2, c_2)$, that is, $L_0 = L_2$, contradicting what was previously proved. Consequently, $q_1 = \mu_1 c_1$. Similarly, $q_2 = \mu_2 c_2$. This means that the three vectors $c_0 = -q_1 - q_2, c_1, c_2$ are minimally dependent.

We may sum up. The vectors a_1, b_1, a_2, b_2 form a basis in Q . Moreover, each triplet $(a_0, a_1, a_2), (b_0, b_1, b_2), (c_0, c_1, c_2), (a_0, b_0, c_0), (a_1, b_1, c_1), (a_2, b_2, c_2)$ is minimally dependent, that is, we have positive dependences:

$$a_0 + \lambda_1 a_1 + \lambda_2 a_2 = 0, \quad b_0 + \mu_1 b_1 + \mu_2 c_2 = 0, \quad p a_0 + p' b_0 + c_0 = 0, \quad (13)$$

$$c_0 + v_1 c_1 + v_2 c_2 = 0, \quad q a_1 + q' b_1 + c_1 = 0, \quad \tau a_2 + \tau' b_2 + c_2 = 0. \quad (14)$$

We now fix the basis e_1, e_2, e_3, e_4 in Q by the relations

$$e_1 = p \lambda_1 a_1, \quad e_2 = p \lambda_2 a_2, \quad e_3 = p' \mu_1 b_1, \quad e_4 = p' \mu_2 b_2. \quad (15)$$

Then, by virtue of (13),

$$a_0 = -\frac{1}{p} (e_1 + e_2), \quad b_0 = -\frac{1}{p'} (e_3 + e_4), \quad c_0 = e_1 + e_2 + e_3 + e_4. \quad (16)$$

Furthermore, according to (14), we have

$$c_0 = v_1(qa_1 + q'b_1) + v_2(\tau a_2 + \tau'b_2) = \frac{v_1 q}{p\lambda_1} e_1 + \frac{v_1 q'}{p'\mu_1} e_3 + \frac{v_2 \tau}{p\lambda_2} e_2 + \frac{v_2 \tau'}{p'\mu_2} e_4.$$

Consequently,

$$\frac{v_1 q}{p\lambda_1} = 1, \quad \frac{v_1 q'}{p'\mu_1} = 1, \quad \frac{v_2 \tau}{p\lambda_2} = 1, \quad \frac{v_2 \tau'}{p'\mu_2} = 1.$$

According to these relations, the two last equalities in (14) take the form

$$\frac{p\lambda_1}{v_1} a_1 + \frac{p'\mu_1}{v_1} b_1 + c_1 = 0, \quad \frac{p\lambda_2}{v_2} a_2 + \frac{p'\mu_2}{v_2} b_2 + c_2 = 0,$$

that is,

$$c_1 = -\frac{1}{v_1} (e_1 + e_3), \quad c_2 = -\frac{1}{v_2} (e_2 + e_4). \quad (17)$$

By virtue of (15)–(17) (and taking into account that $|a_i| = |b_i| = |c_i| = 1$, since $H \subset S^{d-1}$) we obtain $\{a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2\} = \text{norm}\{-e_1 - e_2, e_1, e_2, -e_3 - e_4, e_3, e_4, e_1 + e_2 + e_3 + e_4, -e_1 - e_3, -e_2 - e_4\} = H^{(0)}$ (see the statement of Theorem 8). Consequently, $H \supset H^{(0)}$ (because $a_i, b_i, c_i \in H$).

It remains to prove that $d = 4$, and H does not contain vectors except the vectors belonging to $H^{(0)}$. For this purpose we remark first that each vector $v \in Q$ is nonparticular with respect to vector system $H^{(0)}$. Indeed, $H^{(0)}$ is non-one-sided in the space Q (even the vector system $e_1, e_2, -e_1 - e_2; e_3, e_4, -e_3 - e_4$ is not one-sided). Furthermore, if $v \notin H^{(0)}$, then v is nonparticular (since $H^{(0)}$ is non-one-sided). Finally, if $v \in H^{(0)}$, then v is nonparticular too, since

$$e_1 = (e_1 + e_2 + e_3 + e_4) + e_4 + (-e_2 - e_4) + (-e_3 - e_4)$$

(similarly for e_2, e_3, e_4).

$$-e_1 - e_2 = (-e_1 - e_3) + (-e_2 - e_4) + e_3 + e_4$$

(similarly for $-e_1 - e_3, -e_2 - e_3, -e_2 - e_4$), and e_1, e_2, e_3 is, evidently, nonparticular, too. Now Lemma 2 shows that $d = 4$ (i.e., $R^d = Q$), since the vector system H is indecomposable.

It remains to prove that the 2-system coincides with $H^{(0)}$. Indeed, let $v \in H$. If $v \in \text{lin}(e_1, e_3)$, then either $v = e_1$, or $v = e_3$, or $v = -e_1 - e_3$ (see Lemma 7), that is, $v \in H^{(0)}$. Similarly, if $v \in \text{lin}(e_2, e_4)$, then $v \in H^{(0)}$. Let, finally, $v \in H \sim \text{lin}(e_1, e_3) \sim \text{lin}(e_2, e_4)$. Then $v = v^* + v^{**}$ where $v^* \in \text{lin}(e_1, e_3)$, $v^{**} \in \text{lin}(e_2, e_4)$ and $v^* \neq 0$, $v^{**} \neq 0$. The vector $-v^*$ is representable as a positive linear combination of one or two vectors $e_1, e_3, -e_1 - e_3$. If $-v^*$ is expressed by two vectors, say

$-v^* = \lambda e_1 + \mu(-e_1 - e_3)$ ($\lambda > 0, \mu > 0$), then we have $v + \lambda e_1 + \mu(-e_1 - e_3) - v^{**} = 0$, and (representing $-v^{**}$ as a positive linear combination of one or two vectors $e_2, e_4, -e_2 - e_4$) we obtain a minimal dependence between four or five vectors, contradicting the equality $\text{md } H = 2$. So, $-v^*$ coincides with one of the vectors $e_1, e_3, -e_1 - e_3$, and, similarly, $-v^{**}$ coincides with one of the vectors $e_2, e_4, -e_2 - e_4$ (up to positive coefficients). Thus, we have the following cases (α, β are positive):

$$v = -\alpha e_1 - \beta e_2, \quad v = -\alpha e_3 - \beta e_4, \quad v = \alpha(e_1 + e_3) + \beta(e_2 + e_4); \quad (18)$$

$$v = -\alpha e_1 - \beta e_4, \quad v = -\alpha e_3 - \beta e_2; \quad (19)$$

$$\begin{aligned} v &= \alpha(e_1 + e_3) - \beta e_2, & v &= \alpha(e_1 + e_3) - \beta e_4; \\ v &= \alpha(e_2 + e_4) - \beta e_1, & v &= \alpha(e_2 + e_4) - \beta e_3. \end{aligned} \quad (20)$$

In the first case of (18) we have

$$v + \alpha(e_1 + e_2 + e_3 + e_4) + (\alpha - \beta)(-e_2 - e_4) + \alpha(-e_3 - e_4) + (\alpha - \beta)e_4 = 0$$

(for $\alpha > \beta$),

$$v + \beta(e_1 + e_2 + e_3 + e_4) + (\beta - \alpha)(-e_1 - e_3) + \beta(-e_3 - e_4) + (\beta - \alpha)e_3 = 0$$

(for $\alpha < \beta$).

So, if $\alpha \neq \beta$, then $\text{md}(H \cup \{-\alpha e_1 - \beta e_2\}) \geq 4$. Consequently, $-\alpha e_1 - \beta e_2 \in H$ only in the case $\alpha = \beta = 1/|e_1 + e_2|$ (since $|v| = 1$). In other words, if $v = -\alpha e_1 - \beta e_2 \in H$, then $v \in H^{(0)}$. The two last cases of (18) may be considered similarly. As to (19) and (20), there exists no positive α, β , for which $v \in H$ in these cases. For example: if $v = -\alpha e_1 - \beta e_4$, then

$$v + \alpha(e_1 + e_2 + e_3 + e_4) + \alpha(-e_2 - e_4) + \alpha(-e_3 - e_4) + (\alpha + \beta)e_4 = 0;$$

if $v = \alpha(e_1 + e_3) - \beta e_2$, then

$$v + \alpha(-e_2 - e_3) + \alpha(-e_1 - e_4) + \alpha e_4 + (\alpha + \beta)e_2 = 0.$$

Thus, if $v \in H$, then $v \in H^{(0)}$, that is, $H \subset H^{(0)}$. This completes the proof. $\square$

Looking the proof through, we notice that we have used only the fact that if $e_0, e_1, e_2 \in H$ are minimally dependent vectors, then none of them is particular. So the following stronger result is true.

Theorem 8'. *Let $d \geq 3$ and let $H \subset S^{d-1}$ be an indecomposable 2-system, such that if $e_0, e_1, e_2 \in H$ are minimally dependent vectors, then none of them is particular with respect to H . Under these conditions, $d = 4$ and $H = H^{(0)}$.*

This theorem is used in the next section.

5. Proof of Theorem 9

Let $L = \text{lin } H^\Delta$ and let N be a complementary subspace for L , that is, $R^d = L \oplus N$. We denote by π the natural projection $R^d \rightarrow N$ with kernel L . We consider at first the case when $1 \leq \dim L < d$, that is $1 \leq \dim N < d$. Then $H^* = \text{norm } \pi(H)$ is a vector system in N which is, evidently, non-one-sided. We are going to show that $\text{md } H^* = 1$. This is obvious if $\dim N = 1$. So we must consider the case $\dim N \geq 2$. Let us suppose that $\text{md } H^* = m > 1$ and let $a_0^*, a_1^*, \dots, a_m^* \in H^*$ be minimally dependent vectors, and $a_0, a_1, \dots, a_m$ be their representatives, that is, $a_i \in \pi^{-1}(a_i^*) \cap H$ ($i = 0, \dots, m$). Let $\lambda_0 a_0^* + \dots + \lambda_m a_m^* = 0$ be the corresponding positive dependence. Then the vector $w = \lambda_0 a_0 + \dots + \lambda_m a_m$ belongs to the subspace L . If $w \neq 0$, then

$$\lambda_0 a_0 + \dots + \lambda_m a_m + \mu_1 b_1 + \dots + \mu_k b_k = 0,$$

where $b_1, \dots, b_k \in L \cap H$ are linearly independent and $\mu_i > 0$ ($k \geq 1$). We obtain $m + 1 + k \geq 4$ minimally dependent vectors of the system H , contradicting the relation $\text{md } H = 2$. Hence $w = 0$, that is, $\lambda_0 a_0 + \dots + \lambda_m a_m = 0$. Consequently, according to relations $m > 1$ and $\text{md } H = 2$, we have $m = 2$, that is, $\text{md } H^* = 2$. So the considered minimal dependence has the form $\lambda_0 a_0^* + \lambda_1 a_1^* + \lambda_2 a_2^* = 0$, and, moreover, $\lambda_0 a_0 + \lambda_1 a_1 + \lambda_2 a_2 = 0$ for any representatives $a_i \in \pi^{-1}(a_i^*) \cap H$, $i = 0, 1, 2$. This means that each of the sets $\pi^{-1}(a_i^*) \cap H$, $i = 0, 1, 2$, contains *only one* element. If now one of the vectors a_0^*, a_1^*, a_2^* (say, a_0^*) is particular with respect to the vector system H^* , then the corresponding vector a_0 is particular with respect to the vector system H , that is, $a_0 \in H^\Delta$. However, this contradicts the relations $H^\Delta \subset L$, $a_0 \notin L$. Thus if $a_0^*, a_1^*, a_2^* \in H^*$ are minimally dependent vectors, then these vectors are nonparticular with respect to H^* . Now we conclude from Theorem 8' that either (i) $\dim N = 2$ (and H^* contains no particular vector), or (ii) $\dim N = 4$ and $H^* = H^{(0)}$, or (iii) there is a direct sum decomposition $H^* = H_1 \cup \dots \cup H_l$, where each H_i is of type (i) or (ii).

In case (i) all the vectors $v \in H \sim L$ lie in one two-dimensional plane N' , and, consequently, $H \subset L \cup N'$, that is, H is decomposable, contradicting the condition of the theorem. In case (ii) the vector system $H \sim L$ lies in a four-dimensional subspace N' and is isomorphic to $H^{(0)}$. Consequently, $H \subset L \cup N'$, i.e., H is decomposable too, contradicting the condition of the theorem. The last case can be handled in the same way as cases (i) and (ii). This shows that in the case $\dim N \geq 2$ we have $\text{md } H^* = 1$.

It follows from the previous proofs that (in the case $1 \leq \dim L < d$, that is, $\dim N > 0$) $H^* = \{e_{k+1}^*, \dots, e_d^*, -e_{k+1}^*, \dots, -e_d^*\}$, where $e_{k+1}^*, \dots, e_d^*$ is a basis in N . Let us denote by G_l the subspace $\text{lin}(L \cup \pi^{-1}(e_l^*))$, $l = k+1, \dots, d$. Each subspace G_l has the dimension $k+1$, and the inclusion $H \subset G_{k+1} \cup \dots \cup G_d$ is true. Since e_i is a particular vector, then there exists a hyperplane $Q_i \subset R^d$ through the origin, such that $H \sim \{e_i\} \subset P_i$ and $e_i \notin P_i$, where P_i is a closed half-space with boundary hyperplane Q_i ($i = 1, \dots, k$). By virtue of Lemma 9, we have $e_j \in Q_i$ for $i \neq j$. The intersection $Q_i \cap G_l$ is a hyperplane in the space G_l , such that $e_j \in Q_i \cap G_l$ for $i \neq j$ and $e_i \notin Q_i \cap G_l$. Hence, the intersection $Q_1 \cap \dots \cap Q_k \cap G_l$ is a line which

is not contained in L . We denote by e_l a unit vector of this line. Then $e_1, \dots, e_k, e_l$ is a basis of the subspace G_l , and, consequently, $e_1, \dots, e_k, e_{k+1}, \dots, e_d$ is a basis in R^d .

Since $H \sim \{e_i\} \subset P_i$ ($i = 1, \dots, k$), then each vector $v \in (H \cap G_l) \sim \{e_i\}$ has the form

$$v = \alpha_1 e_1 + \dots + \alpha_k e_k + \beta e_l \quad (\alpha_i \leq 0).$$

Consequently, for each vector $v \in G_l \sim L$ we have

$$-v = x_1 e_1 + \dots + x_k e_k + x_l e_l \quad (x_i \geq 0 \text{ for all } i = 1, \dots, k). \quad (21)$$

Now we denote by $S_l^{(+)}, S_l^{(-)}$ the half-spaces of all vectors

$$x_1 e_1 + \dots + x_k e_k + x_{k+1} e_{k+1} + \dots + x_d e_d$$

for which $x_l \geq 0$ and $x_l \leq 0$, respectively, and by Q_l their intersection, that is, $Q_l = S_l^{(+)} \cap S_l^{(-)}$ is the $(d-1)$ -dimensional subspace spanning all vectors $e_1, \dots, e_d$ except e_l (l is one of the indices $k+1, \dots, d$). Then $G_l \cap S_l^{(+)}$ and $G_l \cap S_l^{(-)}$ are two closed half-spaces in G_l with the common boundary hyperplane $L \subset G_l$. It is easily shown that each of the sets $H_l^{(+)} = H \cap G_l \cap \text{int } S_l^{(+)}$ and $H_l^{(-)} = H \cap G_l \cap \text{int } S_l^{(-)}$ contains, at least, two elements. Indeed, if $H \cap \text{int } S_l^{(+)}$ contains no element of H , then the system H is one-sided (contradicting the condition of the theorem), and if $H \cap \text{int } S_l^{(+)}$ contains only one element e of H , then e is a particular element of H (contradicting the relation $e \notin L$). Hence, $H \cap \text{int } S_l^{(+)}$ contains, at least, two elements of H . Similarly, $H \cap \text{int } S_l^{(-)}$ contains, at least, two elements of H .

Let now $v \in H_l^{(+)}$, $v' \in H_l^{(-)}$, that is, $-\alpha v = x_1 e_1 + \dots + x_k e_k + e_l$, $-\alpha' v' = x'_1 e_1 + \dots + x'_k e_k - e_l$, where $\alpha > 0$, $\alpha' > 0$ and all the coefficients $x_1, \dots, x_k, x'_1, \dots, x'_k$ are nonnegative. Then

$$\alpha v + \alpha' v' + (x_1 + x'_1) e_1 + \dots + (x_k + x'_k) e_k = 0$$

and, consequently, at most, one of the sums $x_1 + x'_1, \dots, x_k + x'_k$ is positive (since $v, v', e_1, \dots, e_k \in H$ and $\text{md } H = 2$). It follows from this that each vector $v \in H_l^{(+)}$ has the form $-\alpha v = x_i e_i + e_l$, where i is one of the indices $1, \dots, k$ and $\alpha > 0$, $x_i \geq 0$. Moreover, there is at least one element $v \in H_l^{(+)}$ for which one of the coefficients $x_1, \dots, x_k$ is positive, say, $-\alpha v = x_i e_i + e_l$, $x_i > 0$ (since there are, at least, two elements of the set $H_l^{(+)}$, and if $x_1 = \dots = x_k = 0$ for one of them, then we may take another element). Similarly, each vector $v' \in H_l^{(-)}$ has the form $-\alpha' v' = x'_j e_j - e_l$; $x'_j \geq 0$, $\alpha' > 0$, and there is an element $v' \in H_l^{(-)}$ for which $x'_j > 0$. Finally, we conclude that $-\alpha v = x_i e_i + e_l$, $-\alpha' v' = x'_j e_j - e_l$, where the index $i = j$ is the same for all elements v, v' . We denote this index by $\varphi(l)$. So we have $-\gamma w = x_{\varphi(l)} e_{\varphi(l)} \pm e_l$ for each $w \in (H \cap G_l) \sim L$, where $\varphi(l)$ is one of the numbers $1, \dots, k$ and the coefficients $\gamma > 0$, $x_{\varphi(l)} \geq 0$ depend on w . This means that

$H \sim L \subset \text{norm} \left(\bigcup_{l>k} \sqcap_{l, \varphi(l)} \right)$ (see the statement of Theorem 9). More detailed,

$$\begin{aligned} H \sim L &\subset \text{norm} \bigcup_{l>1} \sqcap_{l,1} && \text{for } k=1, \\ H \sim L &\subset \text{norm} \bigcup_{l>k} \sqcap_{l, \varphi(l)} && \text{for } 2 \leq k \leq d-1, \\ H \sim L &= \emptyset && \text{for } k=d \end{aligned}$$

(see (1)–(4)).

It remains to investigate the part $H \cap L$ of the vector system H . We investigate this point in the general case, that is, for $\dim N = 0, 1, \dots, d-1$ (or, what is the same, for $k = \dim L = 1, \dots, d$). Since $H \sim \{e_i\} \subset P_i$ for $i = 1, \dots, k$, then

$$\begin{aligned} (H \cap L) &\sim \{e_1, \dots, e_k\} \subset P_1 \cap \dots \cap P_k \cap L \\ &= \{v: -v = x_1 e_1 + \dots + x_k e_k; x_1 \geq 0, \dots, x_k \geq 0\}. \end{aligned}$$

This means that

$$\begin{aligned} H \cap L &\subset \text{norm}\{e_1, -e_1\} && \text{for } k=1, \\ H \cap L &\subset \text{norm}\left(\{e_1, \dots, e_k\} \cup \left(\bigcup_{i<j \leq k} F_{ij}\right)\right) && \text{for } 2 \leq k \leq d. \end{aligned}$$

Uniting the results obtained, we have, for the set $H = (H \cap L) \cup (H \sim L)$,

$$\begin{aligned} H &\subset \text{norm}\left(\{e_i\} \cup \left(\bigcup_{l>1} \sqcap_{l,1}\right)\right) && \text{for } k=1, \\ H &\subset \text{norm}\left(\{e_1, \dots, e_k\} \cup \left(\bigcup_{i<j \leq k} F_{ij}\right) \cup \left(\bigcup_{l>k} \sqcap_{l, \varphi(l)}\right)\right) && \text{for } 2 \leq k \leq d-1, \\ H &\subset \text{norm}\left(\{e_1, \dots, e_d\} \cup \left(\bigcup_{i<j \leq d} F_{ij}\right)\right) && \text{for } k=d. \end{aligned}$$

We notice, furthermore, that in the case $k=2$, if φ is not onto, say, $\varphi(l) = 1$ for all $l = 3, \dots, d$, then

$$H \subset \text{norm}\left(\{e_1, e_2\} \cup F_{12} \cup \left(\bigcup_{l>2} \sqcap_{l,1}\right)\right) \subset \text{norm}\left(\{e_1\} \cup \left(\bigcup_{l>1} \sqcap_{l,1}\right)\right)$$

(since $\{e_2\} \cup F_{12} \subset \sqcap_{2,1}$). This means that for $k=2$ it is sufficient to consider the case when φ is onto (see (3)).

It remains to prove that (i) $\text{md } H' = 2$ for every vector system H' indicated on the right-hand side of relations (1)–(4); (ii) all the 2-systems indicated on the

right-hand sides of relations (1)–(4) are maximal; and (iii) all the 2-systems on the right-hand sides of relations (1)–(4) are indecomposable.

Let us prove assertion (i). Let $a_0, \dots, a_m \in H'$ be minimally dependent vectors. First we remark that for every $j = 1, \dots, d$ the intersection $H' \cap Q_j$ is a vector system which is contained in a vector system of the same type as (1)–(4), but situated in $(d-1)$ -dimensional space Q_j (we write “is contained” instead of “coincides” because, for example, in the case $k=1$ the intersection $H' \cap Q_1$ has the form $\{e_2, \dots, e_d, -e_2, \dots, -e_d\}$, that is, $\text{md}(H' \cap Q_1) < 2$). Thus, by an obvious induction, we conclude that if $\{a_0, \dots, a_m\} \subset Q_j$, then $\text{md}\{a_0, \dots, a_m\} \leq 2$ (that is, $m \leq 2$).

Furthermore, in the case $k < d$ we may suppose that each intersection $\{a_0, \dots, a_m\} \cap \text{int } S_d^{(+)}$, $\{a_0, \dots, a_m\} \cap \text{int } S_d^{(-)}$ is nonempty (otherwise we have $\{a_0, \dots, a_m\} \subset Q_d$ and, consequently, $m \leq 2$). Without loss of generality we may suppose that $a_0 \in \text{int } S_d^{(+)}$, $a_1 \in \text{int } S_d^{(-)}$ (and hence $a_0, a_1 \in \cap_{d, \varphi(d)}$). Moreover, $e_{\varphi(d)} \in \{a_2, \dots, a_m\}$ (otherwise $\{a_0, \dots, a_m\} \subset P_{\varphi(d)}$ and, consequently, $\{a_0, \dots, a_m\} \subset Q_{\varphi(d)}$, that is, $m \leq 2$). However, in this case the vectors $a_0, a_1, e_{\varphi(d)}$ are linearly dependent, and, consequently, $m \leq 2$ too.

Finally, in the case $k = d$ there exist indices i, j , such that $\{a_0, \dots, a_m\} \cap F_{ij} \neq \emptyset$ (otherwise $\{a_0, \dots, a_m\} \subset \{e_1, \dots, e_d\}$, contradicting the minimal dependence of the vectors $a_0, \dots, a_m$). Without loss of generality we may suppose that $\{a_0, \dots, a_m\} \cap F_{12} \neq \emptyset$ and $a_0 \in F_{12}$. Then $e_1 \in \{a_0, \dots, a_m\}$ (otherwise $\{a_0, \dots, a_m\} \subset Q_1$ and $m \leq 2$); similarly, $e_2 \in \{a_0, \dots, a_m\}$. However, in this case the vectors a_0, e_1, e_2 are linearly dependent and, consequently, $m \leq 2$.

So, $m \leq 2$ in any case, that is, $\text{md } H' \leq 2$ (the inequality $\text{md } H' \geq 2$ is obvious).

Now we prove that all the 2-systems H' indicated on the right-hand sides of relations (1)–(4) are maximal. Thus we must prove that if $w \notin H'$, where $w \in R^d$ is a unit vector, then $\text{md}(H' \cup \{w\}) > 2$. We have $-w = \alpha_1 e_1 + \dots + \alpha_d e_d$ and, consequently, $w + |\alpha_1| f_1 + \dots + |\alpha_d| f_d = 0$, where $f_i = \pm e_i$. Hence, no more than two of the coefficients $\alpha_1, \dots, \alpha_d$ are nonzero (because $f_i \in H'$ for all $i = 1, \dots, d$). So, $w = \alpha_i e_i + \alpha_j e_j$, where $i < j$ and $\alpha_i \neq 0, \alpha_j \neq 0$. We consider three cases:

- (i) $i < j \leq k$. In this case at least one of the coefficients α_i, α_j (say, α_i) is positive (since $F_{ij} \subset H'$), that is, $w = \alpha_i e_i + |\alpha_j| f_j$ where $\alpha_i > 0$ and $f_j = \pm e_j$. If now $k > 2$ and $\mu \leq k$ is an index distinct from i and j , then the vectors $w, -f_j, e_\mu, -e_i - e_\mu$ are minimally dependent. Furthermore, if $k = 2$, then there exists an index $l = k + 1, \dots, d$, such that $\varphi(l) = i$ (see (3)). Then the vectors $w, -f_j, e_l, -e_i - e_l$ are minimally dependent.
- (ii) $i \leq k < j$. If $\varphi(j) \neq i$, then $w = |\alpha_i| f_i + |\alpha_j| f_j$ where $f_i = \pm e_i$ and $f_j = \pm e_j$. Consequently, the vectors $w, -f_j, e_{\varphi(j)}, -f_i - e_{\varphi(j)}$ are minimally dependent. Furthermore, if $\varphi(j) = i$ and $k \geq 2$, then there is an index $\mu \leq k$, such that $\mu \neq i$. In this case $w = \alpha_i e_i + |\alpha_j| f_j$, where $\alpha_i > 0$ and $f_j = \pm e_j$. Consequently, the vectors $w, -f_j, e_\mu, -e_i - e_\mu$ are minimally dependent. Finally, if $\varphi(j) = i$ and $k = 1$, then $d - k \geq 2$ (we recall that $d \geq 3$), and, consequently, there exists an index $v > k$, such that $v \neq j$ and $\varphi(v) = \varphi(j) = i = 1$. In this case $w = \alpha_i e_i + |\alpha_j| f_j$, where $\alpha_i > 0$ and $f_j = \pm e_j$. Hence, the vectors $w, -f_j, -e_v, e_v - e_{\varphi(v)}$ are minimally dependent.

- (iii) $k < i < j$. In this case $w = |\alpha_i|f_i + |\alpha_j|f_j$, where $f_i = \pm e_i$ and $f_j = \pm e_j$. If $\varphi(i) \neq \varphi(j)$, then the vectors $w, -f_i - e_{\varphi(i)}, -f_j - e_{\varphi(j)}, e_{\varphi(i)}, e_{\varphi(j)}$ are minimally dependent. If $\varphi(i) = \varphi(j)$, then the vectors $w, -f_i - e_{\varphi(i)}, -f_j - e_{\varphi(j)}, e_{\varphi(i)}$ are minimally dependent.

Thus in any case we have $\text{md}(H' \cup \{w\}) > 2$, contradicting the assumption.

Finally, every two elements of these systems are contained in a minimal dependent system. This proves that all the four right-hand sides are really indecomposable. This completes the proof. $\square$

6. Proof of Theorem 10

The theorem is deduced from the following lemmas. One of them (Lemma 15) is evident, and the others are known in the theory of convex sets. A similar construction is contained in Kincses' paper [9].

Lemma 12. *Let $M \subset R^d$ be a compact convex body and let $H(M)$ be the set of the outward normals at all its regular boundary points. For every $u \in H(M)$ we denote by $P(u)$ the closed half-space with outward normal u , such that $P(u)$ contains M and $\text{bd } P(u)$ is a support hyperplane of M . Then the intersection $\bigcap_{u \in H(M)} P(u)$ coincides with M .*

Lemma 13. *Let $M \subset R^d$ be a compact convex body and let $H(M) = H_1 \cup \dots \cup H_k$. We denote by N_i the intersection $\bigcap_{u \in H_i} P(u)$. Then all the bodies N_i are closed (maybe noncompact) and $M = N_1 \cap \dots \cap N_k$.*

Lemma 14. *If a compact convex body $M \subset R^d$ is represented as the intersection $N_1 \cap \dots \cap N_k$, where the convex bodies $N_1, \dots, N_k$ are closed and $N_1^*, \dots, N_k^*$ are compact, then $M^* = \text{conv}(N_1^* \cup \dots \cup N_k^*)$.*

Lemma 15. *If $H = H(M)$ satisfies conditions (1), (2), (3), or (4), then the body M satisfies conditions (A), (B), (C; $k = 2$), and (C; $k > 2$) of Theorem 10, respectively.*

Now the "only in" part of Theorem 10 follows immediately from Lemmas 12–15. Indeed, if $H(M)$ contains at least one particular vector, then

$$H(M) = \bigcup_{i < j \leq k} (H(M) \cap (\{e_i, e_j\} \cup F_{ij})) \cup \bigcup_{l > k} (H(M) \cap (\{e_l, e_{\varphi(l)}\} \cup \bigcap_{l, \varphi(l)})).$$

The "in" part of Theorem 10 follows from the next evident lemma and Theorem 6.

Lemma 16. *Let $N_1, \dots, N_k$ be compact convex sets in R^d . Then*

$$\text{ext conv}(N_1 \cup \dots \cup N_k) \subseteq (\text{ext } N_1) \cup \dots \cup (\text{ext } N_k).$$

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