Corrections to the paper

On the Numerical Solution of Higher Order Nonlinear Parabolic Equations

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The indicated proof of Theorem 3 is incorrect since it suffers from the same inaccuracies which BRAMBLE [MR 36 # 1121] pointed out in the case of the analogous theorem in [2]. The result is, however, true provided the difference scheme is appropriately modified. The operators p_N and φ_N must then be redefined and the proof altered as BRAMBLE has suggested. The details are outlined below.

First, add to the assumption (H 2), the assumption that the function $f(x, t, D^e u)$ be continuous in the variables involving $D^e u$.

Next we modify the difference scheme (3). Let $D_h = \bigcup \overline{Q_x(h)}$, the union taken over $\bar{x} \in R_n$, $Q_{\bar{x}}(h) \subset \bar{\Omega}$. In the definition of Ω_h , replace $\bar{\Omega}$ by D_h . Extend the domains of definition of the functions $a_{\alpha\beta}$, φ , f , etc., by simply setting them equal to zero for (x, t) outside $\bar{Q}_T$ or $\bar{\Omega}$ as the case may be. For any (x, t) , let $S(x, t) = \left\{ (\xi, \tau) \mid x_i - \frac{h}{2} < \xi_i < x_i + \frac{h}{2}, i = 1, \dots, n, t - k < \tau < t \right\}$ and $S(x) = \left\{ \xi \mid x_i - \frac{h}{2} < \xi_i < x_i + \frac{h}{2}, i = 1, \dots, n \right\}$. Set $\bar{a}_{\alpha\beta}(x, t) = |S(x, t)|^{-1} \int_{S(x, t)} a_{\alpha\beta}(\xi, \tau) d\xi d\tau$, $\bar{\varphi}(x) = |S(x)|^{-1} \int_{S(x)} \varphi(\xi) d\xi$, and $\bar{f}(x, t, q) = |S(x, t)|^{-1} \int_{S(x, t)} f(\xi, \tau, q) d\xi d\tau$,

where $|S(x, t)|$ and $|S(x)|$ are the volumes of $S(x, t)$ and $S(x)$ and q is a vector of appropriate length. In the difference scheme (3), simply write a bar over the $a_{\alpha\beta}$, f , and φ . Observe also that the $\bar{a}_{\alpha\beta}$, $\bar{f}$, and $\bar{\varphi}$ satisfy the same bounds as $a_{\alpha\beta}$, f , and φ .

Let B_N be the finite dimensional BANACH space of functions $U \in W_{h,k}$ with the norm $|U|_{\bar{Q}_{h,k}}^2 = \sum_{|\alpha| \leq p} \left\{ \sum_{|\alpha| \leq p} (\Delta^\alpha U)^2 + (\nabla_i U)^2 \right\} h^n k$. Let B be the SOBOLEV space $\overset{\circ}{W}_{2^{p-1},0}^{p-1,0}(Q_T)$. Then for any $v \in B$ and $(x, t) \in \bar{Q}_{h,k}$,

define $p_N: B \rightarrow B_N$ by setting $(p_N v)(x, t) = \bar{v}(x, t)$. For any $U \in B_N$, let the mapping S_N be defined by $(S_N U)(x, t) = U(\bar{x}, t)$ for $x \in S(\bar{x})$, $\bar{x} \in \Omega_h$, $t_{j-1} < t \leq t_j$, $j = 1, \dots, m$, $(S_N U)(x, t) = \varphi(\bar{x})$ for $x \in S(\bar{x})$, $\bar{x} \in \Omega_h$, $-k < t \leq 0$ and for all other $(x, t) \in R_{n+1}$, set $(S_N U)(x, t) = 0$. For any $U \in B_N$, $S_N U$ is simply a step function. Let $h' > 0$ be fixed and let $K = \bigcup_{\bar{x} \in \Omega_{h'}} S(\bar{x})$, where h' is chosen sufficiently small that $K \subset \Omega$

and K is not empty. In the sequel we take $h \leq h'$. Now define $J_N(S_N U) = \varepsilon^{-n} k^{-1} \int_{R_{n+1}} j\left(\frac{x-\xi}{\varepsilon}, \frac{t-\tau}{k}\right) g(\xi, \tau) d\xi d\tau$, where $j(x, t) \geq 0$ is infinitely

often differentiable, $\int_{R_{n+1}} j(x, t) dx dt = 1$, the support of j in the x

variable lies in K and in the t variable contains the interval $0 \leq t \leq T$, and where $\varepsilon = \frac{h}{2}$. Let $\psi_N \equiv J_N \circ S_N: B_N \rightarrow \mathring{W}_2^{p,1}(Q_T)$. It may be observed that for $v \in \mathring{W}_2^{p,1}(Q_T)$, $\psi_N \circ p_N v \rightarrow v$, $N \rightarrow \infty$. Finally, let i be the operator embedding $\mathring{W}_2^{p,1}(Q_T)$ into $\mathring{W}_2^{p-1,0}(Q_T)$ and recall that i is compact. Define $\varphi_N \equiv i \circ \psi_N: B_N \rightarrow B$.

To verify (A 4), it suffices to show that $\varphi_N T_N p_N v - T \varphi_N p_N v \rightarrow 0$, as $N \rightarrow \infty$. The argument for this is similar to an argument given by CEA [Approximation Variationnelle des Problèmes aux Limites. Ann. Inst. Fourier, Grenoble, 14 (2) 345–444 (1964)] for an analogous result for elliptic equations.

To verify (A 5), it suffices to show that $\{\psi_N T_N V_N\}$ is uniformly bounded in $\mathring{W}_2^{p,1}(Q_T)$ from which it will follow that $\{\varphi_N T_N V_N\}$ is compact in B , and the necessary modifications of the argument given in [2] will be apparent. First, let $W_N \equiv T_N V_N$ and observe that $|W_N| B_N \leq C$, C a constant independent of N . Next, setting $w_N = S_N W_N$, we find $J_N w_N(x, t) = \int_{R_{n+1}} j(\xi, \tau) w_N(x - \varepsilon \xi, t - k \tau) d\tau$. Thus,

$$\begin{aligned} & \int_{R_{n+1}} (\psi_N W_N)^2 dx dt = \\ &= \int_{R_{n+1}} \left\{ \int_{R_{n+1}} j^{1/2}(\xi, \tau) j^{1/2}(\xi, \tau) w_N(x - \varepsilon \xi, t - k \tau) d\xi d\tau \right\}^2 dx dt \leq \\ &\leq \int_{R_{n+1}} dx dt \int_{R_{n+1}} j(\xi, \tau) w_N^2(x - \varepsilon \xi, t - k \tau) d\xi d\tau = \\ &= \int_{R_{n+1}} j(\xi, \tau) d\xi d\tau \int_{R_{n+1}} w_N^2(x - \varepsilon \xi, t - k \tau) dx dt \leq \\ &\leq C \int_{R_{n+1}} j(\xi, \tau) d\xi d\tau = C, \end{aligned}$$

by Theorem 1 and the way in which w_N was defined. Similar bounds independent of N may be obtained for the differences $\Delta^\alpha w_N$, $|\alpha| \leq p$, and $\nabla_t w_N$ in the same way. To obtain integral bounds on the squares of the derivatives of $\psi_N W_N$, we write

$$\frac{\partial}{\partial x_i} \psi_N W_N = \Delta_i \psi_N W_N + \left[\frac{\partial}{\partial x_i} \psi_N W_N - \Delta_i \psi_N W_N \right].$$

It remains to derive an integral bound on the term in brackets. But, using the mean value theorem yields

$$\begin{aligned} \left[\frac{\partial}{\partial x_i} \psi_N W_N - \Delta_i \psi_N W_N \right] &= \varepsilon^{-n} k^{-1} \int_{R_{n+1}} \left[\varepsilon^{-1} j_{x_i} \left(\frac{x - \xi}{\varepsilon}, \frac{t - \tau}{k} \right) - \right. \\ &\quad \left. - \Delta_i j \left(\frac{x - \xi}{\varepsilon}, \frac{t - \tau}{k} \right) \right] w_N d\xi d\tau = \\ &= \varepsilon^{-n-1} k^{-1} \int_{R_{n+1}} \left[j_{x_i} \left(\frac{x - \xi}{\varepsilon}, \frac{t - \tau}{k} \right) - \right. \\ &\quad \left. - j_{x_i} \left(\frac{x + \vartheta h e_i - \xi}{\varepsilon}, \frac{t - \tau}{k} \right) \right] w_N(\xi, \tau) d\xi d\tau = \\ &= 2 \varepsilon^{-n} k^{-1} \int_{R_{n+1}} j_{x_i}(\xi, \tau) \left[\frac{w_N(x - \varepsilon \xi, t - k\tau) - w(x + \vartheta h e_i - \varepsilon \xi, t - k\tau)}{h} \right] d\xi d\tau, \end{aligned}$$

where $0 < \vartheta < 1$. Now square both sides and apply the SCHWARZ inequality as before with $|j_{x_i}|^{1/2}$ instead of $j^{1/2}$. Then integrate with respect to x, t over R_{n+1} , interchange the order of integration, observe that either $w_N(x + \vartheta h e_i, t) - w_N(x, t)$ equals $w_N(x + h, t) - w_N(x, t)$ or is zero and use the bound already obtained on $\int_{R_{n+1}} (\Delta_i w_N)^2 dx dt$ to obtain

the desired bound. The higher order derivatives and the derivative with respect to t are estimated similarly.

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