On a New Method for Constructing Good Point Sets on Spheres*

Gerold Wagner

Mathematisches Institut A, Postfach 801140,
7000 Stuttgart 80, Federal Republic of Germany

Abstract. We use quadrature formulas with equal weights in order to construct N point sets on spheres in d -space ($d \geq 3$) which are almost optimal with respect to a discrepancy concept, based on distance functions (potentials) and distance functionals (energies). By combining this approach with the probabilistic method, we obtain almost best possible approximations of balls by zonotopes, generated by N segments of equal length.

1. Introduction

The problem of distributing a large number N of points $x_1, x_2, \dots, x_N$ uniformly over the sphere S^{d-1} ($d \geq 3$) in Euclidean space E^d arises in various branches of mathematics. There is no generally valid concept of uniformity: a point set which is optimal for one purpose may not be so for another one. Most of these concepts, however (here we quote from the introductory remarks in [7]), are based on making

$$\sup_{f \in \Phi} \left\| \frac{1}{N} \sum_{j=1}^N f(x_j) - \int f d\sigma \right\|, \quad \sigma = \text{normalized Lebesgue measure on } S^{d-1}, \quad (1)$$

small in an appropriate norm and for a suitable class Φ of functions f . If we choose the maximum norm and the class of indicator functions of spherical caps of all

* Editors' note: We learned with sadness of Gerold Wagner's untimely death as a result of an avalanche in the Alps shortly after the submission of this paper. When one of the referees, Joram Lindenstrauss, suggested that Wagner's results might be extended to dimensions > 6 , we invited Professor Lindenstrauss to submit a paper containing that extension which we would publish alongside the Wagner paper. The result is the paper by Bourgain and Lindenstrauss that follows the present one.

kinds, we obtain the *spherical cap discrepancy*. Using an ingenious variant of the probabilistic method, Beck [1] got almost optimal sets for arbitrary d and N .

If we choose the class $\Phi = \{f \in L^2(S^{d-1}): \|f\|_2 = 1\}$, and a set of N rotations $\gamma_1, \gamma_2, \dots, \gamma_N$ on S^{d-1} , we obtain the so-called *operator discrepancy*

$$\sup_{f \in \Phi} \left\| \frac{1}{N} \sum_{j=1}^N f(\gamma_j x) - \int f d\sigma \right\|_2.$$

Using deep methods from number theory, the authors in [7] succeeded in constructing optimal sets of rotations for $d = 3$, $N = p + 1$, p prime. When applied to a fixed point x on S^2 , this set of rotations generates a good point set.

In this paper we present a method which enables us to construct point sets which are essentially optimal for a discrepancy concept, defined by means of *distance functions* ("potentials") and *distance functionals* ("energies"). By combining our constructive method with Beck's probabilistic approach, we obtain almost best possible approximations of balls B^d ($3 \leq d \leq 6$) by zonotopes, generated by N line segments of *equal length*.

In order to give a brief description of the method, let us introduce a few notations. The unit sphere S^{d-1} is endowed with the normalized surface measure σ , $\sigma(S^{d-1}) = 1$, and a system of spherical coordinates $\theta_1, \theta_2, \dots, \theta_{d-2}, \varphi$. Given the number N of points to be distributed, we cut the surface S^{d-1} into a certain number $r = r(N) \sim N$ of coordinate boxes $D_1, \dots, D_r$, i.e., subdomains of S of the form

$$D_\rho = \{(\theta_1, \dots, \theta_{d-2}, \varphi): \alpha_{\rho,i} \leq \theta_i \leq \beta_{\rho,i}; \alpha_\rho \leq \varphi \leq \beta_\rho; 1 \leq i \leq d-2\},$$

$$\rho = 1, 2, \dots, r,$$

where the numbers $N_\rho := N \cdot \sigma(D_\rho)$, $\rho = 1, 2, \dots, r$, are positive integers. On each subdomain D_ρ we distribute N_ρ points $X_1^{(\rho)}, X_2^{(\rho)}, \dots, X_{N_\rho}^{(\rho)}$ in such a way that the relation

$$\frac{1}{N_\rho} \sum_{j=1}^{N_\rho} p(X_j^{(\rho)}) = \frac{1}{\sigma(D_\rho)} \int_{D_\rho} p(x) d\sigma(x) \quad (2)$$

holds true for all polynomials $p(x) = p(x_1, x_2, \dots, x_d)$ of degree less than or equal to a given bound $t \geq 1$. The crucial part of the proof consists in the observation that relation (2) can be satisfied as soon as $N_\rho \geq n_0(t, d)$ holds, where the bound $n_0(t, d)$ does not depend on the size or the location of the subdomain D_ρ . The main tool for proving this assertion is a quantitative result on the existence of *quadrature formulas with equal weights*, proved by the same author in [12].

The method sketched above could be made even more effective if it were possible to choose the points $X_1^{(\rho)}, \dots, X_{N_\rho}^{(\rho)}$ as *random points*. More precisely, suppose there is a probability space (Ω, P) with random vectors $X_1^{(\rho)}(\omega), \dots, X_{N_\rho}^{(\rho)}(\omega)$, $\omega \in \Omega$, such that:

- (1) All the $X_j^{(\rho)}(\omega)$, $j = 1, 2, \dots, N_\rho$, are uniformly distributed over the domain D_ρ .

(2) The relation

$$\frac{1}{N_\rho} \sum_{j=1}^{N_\rho} p(X_j^{(\rho)}(\omega)) = \frac{1}{\sigma(D_\rho)} \int_{D_\rho} p(x) d\sigma(x) \quad (3)$$

holds true for all polynomials $p(x)$ of degree $\leq t$ and for all $\omega \in \Omega$.

We did not succeed in proving the existence (for suitable N_ρ) of such a set of random vectors under these general conditions. This seems to be a rather difficult problem. However, it is possible to approximate the measure σ on D_ρ by a simpler measure for which (3) can indeed be satisfied with $t = 1$ and $N_\rho = 3^{d-1}$, independent of the size and location of D_ρ . This partial result will be sufficient for our purpose.

2. Distance Functions and Distance Functionals

On the unit sphere $S = S^{d-1}$ in E^d ($d \geq 3$) we consider a certain class of distance functions and distance functionals, associated with a given N point set $\omega_N = \{x_1, \dots, x_N\}$ on S . Denote by $|x - y|$ the Euclidean distance between two points x and y in E^d . Let $x \in S^{d-1}$ be a variable point. For each value of a real parameter α , $1 - d < \alpha < \infty$, consider the distance function $U_\alpha(x, \omega_N)$ which is defined as follows:

$$U_\alpha(x, \omega_N) = \begin{cases} \sum_{j=1}^N (|x - x_j|^\alpha - m(\alpha, d)) & \text{for } \alpha \neq 0, 2, 4, \dots, \\ - \sum_{j=1}^N (|x - x_j|^\alpha \cdot \log|x - x_j| - m(\alpha, d)) & \text{for } \alpha = 0, 2, 4, \dots \end{cases}$$

Here $m(\alpha, d)$ denotes the mean value of the distance kernel on S^{d-1} , that means

$$m(\alpha, d) = \begin{cases} \int_S |x - x_1|^\alpha d\sigma(x) & \text{for } \alpha \neq 0, 2, 4, \dots, \\ - \int_S |x - x_1|^\alpha \cdot \log|x - x_1| d\sigma(x) & \text{for } \alpha = 0, 2, 4, \dots, \end{cases}$$

σ = normalized Lebesgue measure on S , $\sigma(S) = 1$.

In a previous paper [11] the author proved certain lower bounds for the L^1 -norms of the functions $U_\alpha(x, \omega_N)$. The existence of such "natural" lower bounds is due to the fact that uniform distribution on S can be approximated by an N point distribution to a certain degree of accuracy only. We show that the lower bounds obtained in [11] are best possible, apart from the values of the constants.

For convenience, put

$$\|U_\alpha(x, \omega_N)\|_{(\infty)} = \begin{cases} \max_{x \in S} |U_\alpha(x, \omega_N)| & \text{for } \alpha > 0, \\ -\min_{x \in S} U_\alpha(x, \omega_N) & \text{for } 1 - d < \alpha \leq 0. \end{cases}$$

We prove

Theorem A. *For each $N \geq 1$ and each $\alpha > 1 - d$, there is an N point set ω_N on S^{d-1} such that the following inequality holds:*

$$\|U_\alpha(x, \omega_N)\|_{(\infty)} \leq c(\alpha, d) \cdot N^{-\alpha/(d-1)}. \quad (4)$$

Here $c(\alpha, d)$ is a positive constant independent of N .

Note that in view of the relation $\int_S U_\alpha(x, \omega_N) d\sigma(x) = 0$, the upper bound (4) is also an upper bound for the L^1 -norm of $U_\alpha(x, \omega_N)$. (The reader should compare Theorem A with Theorem 1 in [11].)

In [11] we also considered distance functionals $E_\alpha(\omega_N)$, defined by

$$E_\alpha(\omega_N) = \begin{cases} \sum_{\substack{j=1 \\ j \neq k}}^N \sum_{k=1}^N (|x_j - x_k|^\alpha - m(\alpha, d)) & \text{for } 1 - d < \alpha < 0, \\ -\sum_{\substack{j=1 \\ j \neq k}}^N \sum_{k=1}^N (\log|x_j - x_k| - m(0, d)) & \text{for } \alpha = 0, \\ \sum_{j=1}^N \sum_{k=1}^N (|x_j - x_k|^\alpha - m(\alpha, d)) & \text{for } 0 < \alpha < 2. \end{cases}$$

For $0 < \alpha < 2$ and $N \geq 2$, the sum $E_\alpha(\omega_N)$ is known to be negative (see Theorem 2 in [11]). Application of Theorem A immediately yields

Corollary. *For each $N \geq 2$ and for each α , $0 < \alpha < 2$, there exists an N point set ω_N such that the following inequality holds:*

$$-E_\alpha(\omega_N) \leq c(\alpha, d) \cdot N^{1-\alpha/(d-1)}.$$

Here $c(\alpha, d)$ is a positive constant independent of N .

We remark that the special case $\alpha = 1$ of the corollary has already been established by Stolarsky [9].

In the unbounded case $1 - d < \alpha \leq 0$ the situation is more complicated. The present method of construction seems to apply only to the logarithmic case $\alpha = 0$. (For $d = 3$, a different approach shows that for $-2 < \alpha < 0$ also

the bounds from Theorem 2 in [11] are essentially best possible. As the proof is highly computational, we do not present it here.)

We prove

Theorem B. *For each $N \geq 2$ there exists an N point set ω_N on S^{d-1} , satisfying the relation*

$$E_0(\omega_N) \leq -\frac{N}{d-1} \log N + c(d) \cdot N. \quad (5)$$

Here $c(d)$ is a positive constant independent of N .

Note that in the cases $d = 2$ and $d = 3$ (and probably also for all $d \geq 4$) the constant $-1/(d-1)$ cannot be improved. The reverse inequality for $d = 3$ has been obtained independently by N. Elkies (see the remark on p. 183 in [7]; the definition of $E_0(\omega_N)$ differs from ours by a factor $1/(2\pi)$) and the author [10].

Finally, let us remark that the results of this section can be extended to a large class of (sufficiently smooth) surfaces S^* , provided that the equilibrium distribution γ (with respect to the potential kernel $|x - y|^\alpha$) has a smooth surface density $(d\gamma/d\sigma)$, $\sigma =$ Lebesgue measure on S^* , satisfying $0 < c_1 \leq (d\gamma/d\sigma) \leq c_2$. This property is known to be true for the Newtonian case $\alpha = 2 - d$ (see Lemma 3 in [8], p. 67). Hence Theorem A remains valid for electron distributions on conductors such as ellipsoids, tori, and so on.

3. Approximating Balls by Zonotopes

A zonotope $Z_N \subset E^d$ is defined as the sum (Minkowski sum) of N line segments $I_1, I_2, \dots, I_N$ from E^d (see [3]). In connection with the so-called shadow problem (see, e.g., [6]) and the almost isometric embedding of the Euclidean space l_2^n into l_1^m (see [5]), the following problem arises: given a ball $B^d \subset E^d$ and a number $\varepsilon > 0$, can we find a zonotope Z_N such that

$$B^d \subset Z_N \subset (1 + \varepsilon) \cdot B^d \quad (6)$$

holds? The answer being yes, we may ask for the smallest N for which (6) is possible. It was proved by Betke and McMullen [2] that necessarily

$$N > c_d \cdot \varepsilon^{-2(d-1)/(d+2)}$$

holds. [Editors' note: Although the problem was previously studied in [2], this result was proved in [5].] Linhart [6], using a result due to Beck [1], proved that (6) can be satisfied whenever $N > c_d \cdot (\varepsilon^2 \cdot \log \varepsilon)^{-(d-1)/d}$. Later, Bourgain and Lindenstrauss [4] almost settled the problem by showing that (6) can be satisfied for all $N > c_d \cdot (\varepsilon^2 \cdot \log \varepsilon)^{-(d-1)/(d+2)}$.

However, if we require the generating segments $I_1, I_2, \dots, I_N$ to be congruent (zonotopes of this kind are denoted by Z_N^*), the results are less complete. Bourgain *et al.* [5] proved that

$$B^d \subset Z_N^* \subset (1 + \varepsilon)B^d$$

can be satisfied whenever $N > c_d \cdot (\varepsilon^2 \cdot \log \varepsilon)^{-1}$, and Linhart [6] showed that this is even true for all $N > c_d \cdot (\varepsilon^2 \cdot \log \varepsilon)^{-(d-1)/d}$.

We prove

Theorem C. *Let $B^d \subset E^d$ be a ball, $3 \leq d \leq 6$. Then for each $N > c_d \cdot (\varepsilon^2 \cdot \log \varepsilon)^{-(d-1)/(d+2)}$ there exists a zonotope Z_N such that the relation $B^d \subset Z_N^* \subset (1 + \varepsilon)B^d$ is true.*

Theorem C shows that, for $3 \leq d \leq 6$, the Bourgain–Lindenstrauss bound is not affected if we restrict ourselves to zonotopes which are generated by congruent line segments. Theorem C should be true for all dimensions. Furthermore, Theorem C remains true not only for balls, but for a whole class of so-called zonoids (see [3] and [4]), generated by a measure μ on the unit sphere S^{d-1} which satisfies the conditions $(d\mu/d\sigma) \in C^3(S^{d-1})$ and $0 < K_1 \leq (d\mu/d\sigma) \leq K_2$.

4. The Main Lemma

In order to prove Theorems A and B, we need a result on numerical integration with equal weights.

Lemma 1 (see [12]). *Let $w(x)$ be an integrable weight function on the interval $-1 \leq x \leq 1$, satisfying the relations $\int_{-1}^1 w(x) dx = 1$ and $L_2 \cdot (1 - |x|)^\beta \leq w(x) \leq L_1$ for some positive constants L_1, L_2 , and β . Let $\Phi = \{f_1, f_2, \dots, f_s\} \in C^3[-1, 1]$ be a system of functions such that the derivatives $f'_1, \dots, f'_s$ form an orthonormal system with respect to w . Put*

$$K = \max_{-1 \leq x \leq 1} \max_{\mu=1, \dots, s} (|f'_\mu(x)|, |f''_\mu(x)|, |f'''_\mu(x)|).$$

Then the following assertions are true:

- (a) *For each $n > n_0(s, K, L_1, L_2, \beta)$ there are points $-1 < \xi_1 < \dots < \xi_n < 1$ such that*

$$\frac{1}{n} \sum_{j=1}^n f_\mu(\xi_j) = \int_{-1}^1 f_\mu(x) w(x) dx$$

holds true for $\mu = 1, 2, \dots, s$.

- (b) Defining x_j ($j = 0, 1, \dots, n$) by the relation $\int_{x_{j-1}}^{x_j} w(x) dx = j/n$, we may assume that $x_{j-1} < \xi_j < x_j$ ($j = 1, \dots, n$), and that

$$\min(\xi_j - x_{j-1}, x_j - \xi_j) > \frac{1}{3}(x_j - x_{j-1})$$

holds for $j = 1, 2, \dots, n$.

Assertion (a) of Lemma 1 is a weaker form of the main result in [12]. Assertion (b) is not stated explicitly in [12], but is contained in the very last inequality of the proof.

We also need

Lemma 2. Let $w(x)$ be a weight function on the interval $[-1, 1]$, subject to the conditions imposed in Lemma 1. Let $\Phi = \{T_0, T_1, \dots, T_{2t}\}$ be the system of functions on $[-1, 1]$ defined by

$$T_{2j}(x) = \left(\frac{1 - \cos \varepsilon x}{\varepsilon^2/2} \right)^j \quad (j = 0, 1, \dots, t)$$

and

$$T_{2j+1}(x) = T_{2j}(x) \cdot \frac{\sin \varepsilon x}{\varepsilon} \quad (j = 0, 1, \dots, t-1).$$

Here ε is a positive real parameter. Denote by $\Psi = \{g_0, g_1, \dots, g_{2t}\}$ the new system growing out of Φ by orthonormalizing the functions T_k with respect to $w(x)$ according to the Gram-Schmidt process. Write $g_j(x) = \sum_{k=0}^j b_{jk} T_k(x)$ ($j = 0, \dots, 2t$). Then, for all $\varepsilon < \varepsilon_0(t, L_2, \beta)$ and $j = 0, 1, \dots, 2t$; $k = 0, \dots, j$, the inequality

$$|b_{jk}| \leq K$$

holds true, where the positive bound K depends on t, L_2 , and β , but not on ε .

Proof. The assertion is proved by induction. Assume that, for some $s \geq 0$, the following inequality holds:

$$|b_{jk}| \leq K(s, L_2, \beta) =: K_s \quad (j = 0, 1, \dots, s; k = 0, 1, \dots, j). \quad (7)$$

For $s = 0$, (7) is true with $K_0 = 1$.

We orthogonalize the function T_{s+1} with respect to $g_0, g_1, \dots, g_s$ by setting

$$f_{s+1}(x) = T_{s+1}(x) - \sum_{j=0}^s \langle g_j, T_{s+1} \rangle \cdot g_j(x) = \sum_{j=0}^{s+1} a_j T_j(x), \quad (8)$$

and $g_{s+1}(x) = f_{s+1}(x)/\|f_{s+1}\|_2$. Here, as usual, we define

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x)w(x) dx$$

and $\|f\|_2^2 = \langle f, f \rangle$.

From (7) and (8) it follows that

$$|a_j| \leq (s+1) \cdot K_s \quad \text{for } j = 0, 1, \dots, s+1.$$

All we have to prove is that $\|f_{s+1}\|_2$ is bounded from below.

From (8) and the inequality

$$|T_j(x) - x^j| \leq c_1(s) \cdot \varepsilon^2,$$

valid for all $x \in [-1, 1]$, $j = 0, 1, \dots, s+1$, and $\varepsilon < 1$, it follows that f_{s+1} admits an approximation by a polynomial with leading coefficient 1, that is,

$$f_{s+1}(x) = x^{s+1} + d_s x^s + \dots + d_0 + R(x) = p_{s+1}(x) + R(x),$$

where $|R(x)| < \varepsilon^2 \cdot c_2(s, L_2, \beta)$.

Using the expansion of p_{s+1} into Legendre polynomials, e.g., we easily obtain

$$\max_{-1 \leq x \leq 1} |f_{s+1}(x)| \geq c_3(s) - c_2(s, L_2, \beta) \cdot \varepsilon^2,$$

where $c_3(s) > 0$. Furthermore, using

$$|f'_{s+1}(x)| \leq \sum_{j=0}^{s+1} |a_j| \cdot |T'_j(x)| < c_4(s, L_2, \beta),$$

we find that

$$|f_{s+1}(x)| > \frac{1}{2}c_3(s) \tag{9}$$

holds true on an interval of length $\geq \delta(s, L_2, \beta) > 0$, provided that $\varepsilon < \varepsilon_0(s, L_2, \beta)$.

From (9) and the assumption $w(x) \geq L_2 \cdot (1 - |x|)^\beta$, we obtain the desired estimate

$$\|f_{s+1}\|_2 \geq c_5(s, L_2, \beta) > 0.$$

This proves the assertion. □

Now we formulate the Main Lemma. For convenience, denote the spherical coordinates on S^{d-1} by $\theta_1, \theta_2, \dots, \theta_{d-2}$, $\varphi =: \theta_{d-1}$, $0 \leq \theta_\mu \leq \pi$ for $\mu = 1, 2, \dots, d-2$; $0 \leq \theta_{d-1} < 2\pi$. Furthermore, denote by Ω_t ($t \in \mathbb{N}$) the set of trigonometric

polynomials in the variables $\theta_1, \dots, \theta_{d-1}$, of degree not exceeding t , i.e., polynomials of the form

$$p(\theta_1, \dots, \theta_{d-1}) = \sum_{|j_\mu| \leq t} a(j_1, \dots, j_{d-1}) \cdot \exp\left(i \sum_{\mu=1}^{d-1} j_\mu \theta_\mu\right),$$

where $j_1, \dots, j_{d-1}$ are integers and $a(j_1, \dots, j_{d-1})$ are arbitrary complex coefficients.

Lemma 3 (Main Lemma). *Let $t \geq 1$ be a fixed integer, and let*

$$D = \{\beta_{1\mu} \leq \theta_\mu \leq \beta_{2\mu}; \mu = 1, 2, \dots, d-1\}$$

be a domain on S^{d-1} (D may coincide with S). Then there exists a natural number $n_0(t, d)$, independent of D , such that the following is true: for each $(d-1)$ -tuple of integers $(m_1, \dots, m_{d-1})$ with $m_j \geq n_0$ for $j = 1, 2, \dots, d-1$, there is a set of $\prod_{j=1}^{d-1} m_j$ points $(\theta_1^{(\mu_1)}, \dots, \theta_{d-1}^{(\mu_{d-1})})$ ($1 \leq \mu_j \leq m_j$) on D with the property that

$$\left(\prod_{j=1}^{d-1} m_j\right)^{-1} \sum_{\mu_1, \dots, \mu_{d-1}} p(\theta_1^{(\mu_1)}, \dots, \theta_{d-1}^{(\mu_{d-1})}) = \frac{1}{\sigma(D)} \int_D p(x) d\sigma(x) \quad (10)$$

holds for all polynomials p in Ω_t .

Proof. We begin by noting that it is sufficient to prove Lemma 3 for domains D for which the differences $\beta_{2\mu} - \beta_{1\mu}$ ($\mu = 1, 2, \dots, d-1$) are sufficiently small, $< \varepsilon_0 = \varepsilon_0(d, t)$ say. In order to obtain the assertion for domains of arbitrary size, we only have to stick together a bounded number of “small” D ’s. Furthermore, by separating variables, we may replace (10) by the $t \cdot (d-1)$ conditions

$$\frac{1}{m_j} \sum_{\mu=1}^{m_j} \exp(i\tau\theta_j^{(\mu)}) = \frac{\int_{\beta_{1j}}^{\beta_{2j}} \exp(i\tau\theta_j) \cdot \sin^{d-1-j}\theta_j d\theta_j}{\int_{\beta_{1j}}^{\beta_{2j}} \sin^{d-1-j}\theta_j d\theta_j}, \quad (j = 1, 2, \dots, d-1; \tau = 1, 2, \dots, t). \quad (11)$$

we fix j and replace the variable θ_j by x . After a suitable linear transformation, we obtain the interval of integration $-1 \leq x \leq 1$, the weight function

$$w(x) = \sin^{d-1-j} \varepsilon(x - x_0) \Big/ \int_{-1}^1 \sin^{d-1-j} \varepsilon(x - x_0) dx,$$

where $\varepsilon = \frac{1}{2}(\beta_{2j} - \beta_{1j})$ and $x_0 = (\beta_{2j} + \beta_{1j})/(\beta_{2j} - \beta_{1j})$, and the system of functions $\Omega'_t = \{1, \cos \varepsilon x, \cos 2\varepsilon x, \dots, \cos t\varepsilon x, \sin \varepsilon x, \dots, \sin t\varepsilon x\}$. We replace the system Ω'_t by the equivalent system $\Omega''_t = \{1, G_0, G_1, \dots, G_{2t}\}$, where $G_0, \dots, G_{2t}$ are arbitrary primitives of the functions $g_0, g_1, \dots, g_{2t}$ as defined in Lemma 2.

By Lemma 2, the derivatives up to the third order of the functions G_τ ($\tau = 0, \dots, 2t$) are bounded in absolute value by a constant $K_j = K_j(d, t)$, as soon as $\varepsilon < \varepsilon_j(d, t)$. An application of Lemma 1 finishes the proof. $\square$

5. Proof of Theorems A and B

The proof of Theorem A splits into several cases, according to the value of the parameter α .

5.1. The Case $0 < \alpha < 2$.

Let α be fixed, and let $n_0 = n_0(t = d, d)$ be the number the existence of which is guaranteed by Lemma 3. Let N be a sufficiently large natural number, $N = k \cdot n_0^{d-1} + l$, where $0 \leq l < n_0^{d-1}$ and $k = [N/n_0^{d-1}]$. By cutting the coordinate intervals $0 \leq \theta_\mu \leq \pi (\mu = 1, \dots, d-2)$ and $0 \leq \theta_{d-1} < 2\pi$ into pieces appropriately, the surface S^{d-1} can be divided into subdomains $D_1, \dots, D_r$, $r = r(N)$, which are boxes in the system of spherical coordinates, and which possess the following basic properties:

$$(a) \quad \sigma(D_1) = \sigma(D_2) = \dots = \sigma(D_{r-1}) = N^{-1} \cdot n_0^{d-1}, \sigma(D_r) = N^{-1} \cdot n_0^{d-1} \text{ if } l = 0, \text{ and} \\ \sigma(D_r) = N^{-1} \cdot l \cdot (1 + n_0^{d-1})^{d-1} \text{ if } l > 0.$$

$$(b) \quad \sup_{x, y \in D_\rho} |x - y| \leq c_1 \cdot N^{-1/(d-1)}, \quad (12)$$

where c_1 is a positive constant depending on the dimension d only.

We apply Lemma 3 to each of the domains D_ρ . We choose $m_1 = \dots = m_{d-1} = n_0$ if $\sigma(D_\rho) = N^{-1} \cdot n_0^{d-1}$, and $m_1 = \dots = m_{d-2} = (1 + n_0^{d-1})$, $m_{d-1} = l \cdot (1 + n_0^{d-1})$ for D_r if $l > 0$. The interpolation points, distributed over each D_ρ according to the Main Lemma, are denoted by $x_\mu^{(\rho)}$, where $\mu = 1, 2, \dots, \prod_{j=1}^{d-2} m_j$. Let

$$z_\rho = (\tfrac{1}{2}(\beta_1 + \beta'_1), \dots, \tfrac{1}{2}(\beta_{d-1} + \beta'_{d-1}))$$

be the "midpoint" of the domain $D_\rho = \{\beta_\mu \leq \theta_\mu \leq \beta'_\mu\}$. (The choice of z_ρ on D_ρ is rather arbitrary.) Fix $x \in S^{d-1}$. Denote by $\bar{D}_\rho$ the convex hull of D_ρ in E^d .

We divide the domains D_ρ into disjoint subclasses in the following way. Let $M_q = M_{q,x}$ ($q = 0, 1, \dots$) be the class of domains D_ρ such that

$$q \cdot c_1 \cdot N^{-1/(d-1)} \leq \min_{y \in \bar{D}_\rho} |x - y| < (q+1) \cdot c_1 \cdot N^{-1/(d-1)}.$$

Each class M_q contains at most $\ll (q+1)^{d-2}$ domains, according to (12). Here, as in what follows, all the constants implicit in the Vinogradov symbols may depend on α and d , but not on q , x , ρ , or N .

For each domain D_ρ in M_0 , the following inequality is true:

$$\left| N \int_{D_\rho} |x - y|^\alpha d\sigma(y) - \sum_\mu |x - x_\mu^{(\rho)}|^\alpha \right| \\ \leq N \cdot \sigma(D_\rho) \cdot (2c_1 \cdot N^{-1/(d-1)})^\alpha + \sum_\mu (2c_1 \cdot N^{-1/(d-1)})^\alpha \\ \ll N^{-\alpha/(d-1)}. \quad (13)$$

For each domain D_ρ in M_q , $q \geq 1$, consider the Taylor expansion

$$\begin{aligned} |x - y|^\alpha &= |x - z_\rho|^\alpha + \sum_{m=1}^d \frac{1}{m!} ((y - z_\rho) \operatorname{grad}_w)^m |x - w|^\alpha_{w=z_\rho} + R(x, y) \\ &= T_d(x, y) + R(x, y), \end{aligned}$$

where

$$R(x, y) = \frac{1}{(d+1)!} ((y - z_\rho) \operatorname{grad}_w)^{d+1} |x - w|^\alpha_{w=z_\rho + \delta(y - z_\rho)}, \quad 0 < \delta < 1.$$

The remainder term can be estimated as

$$|R(x, y)| \ll N^{-(d+1)/(d-1)} \cdot (q \cdot N^{-1/(d-1)})^{\alpha-d-1} \ll q^{\alpha-d-1} \cdot N^{-\alpha/(d-1)}. \quad (14)$$

The main term $T_d(x, y)$ is a polynomial in the cartesian coordinates of y of degree $\leq d$. By the choice of the point set $\{x_\mu^{(\rho)}\}$, we have

$$N \int_{D_\rho} T_d(x, y) d\sigma(x, y) = \sum_\mu T_d(x, x_\mu^{(\rho)}).$$

Hence, for each D_ρ in M_q , $q \geq 1$, noting (14), we have the inequality

$$\left| N \int_{D_\rho} |x - y|^\alpha d\sigma(y) - \sum_\mu |x - x_\mu^{(\rho)}|^\alpha \right| \ll q^{\alpha-d-1} \cdot N^{-\alpha/(d-1)}. \quad (15)$$

Summing up (13) and (15), and noting that $\#M_q \ll (q+1)^{d-2}$, we finally obtain

$$\begin{aligned} &\left| N \int_{S^{d-1}} |x - y|^\alpha d\sigma(y) - \sum_{\rho=1}^r \sum_\mu |x - x_\mu^{(\rho)}|^\alpha \right| \\ &\ll N^{-\alpha/(d-1)} + \sum_{q=1}^{\infty} q^{d-2} \cdot q^{\alpha-d-1} \cdot N^{-\alpha/(d-1)} \\ &\ll N^{-\alpha/(d-1)}. \end{aligned}$$

This proves Theorem A in the case $0 < \alpha < 2$.

5.2. The Case $2 \leq \alpha < \infty$.

In the cases $2 \leq \alpha < 4$, $4 \leq \alpha < 6$, ..., we proceed as before, choosing successively $t = d+2$, $d+4$, ..., and approximating $|x - y|^\alpha$ by a Taylor polynomial of degree t .

5.3. The Case $1 - d < \alpha \leq 0$.

We proceed as in the case $0 < \alpha < 2$, choosing $t = d$ in the Main Lemma. The only difference in the argument concerns the derivation of the estimate (13), which has to be replaced by a *one-sided* estimate. For fixed x on S^{d-1} , consider again those domains D_ρ for which $|x - y| \leq c_1 \cdot N^{-1/(d-1)}$ holds for some point y in the convex hull of D_ρ . Then the following estimate is true for $\alpha < 0$:

$$\sum_{\mu} |x - x_{\mu}^{(\rho)}|^{\alpha} - N \int_{D_\rho} |x - y|^{\alpha} d\sigma(y) \geq -c_2 \cdot N^{-\alpha/(d-1)},$$

$c_2 = c_2(\alpha, d)$ a positive constant. (16)

In order to prove (16), we simply omit the sum and estimate the integral, using relation (12). In the logarithmic case $\alpha = 0$, the corresponding inequality is

$$\sum_{\mu} \log|x - x_{\mu}^{(\rho)}| - N \int_{D_\rho} \log|x - y| d\sigma(y) \leq c_3(d).$$

Here the sum cancels the logarithmic part of the integral, leaving a remainder which is bounded from above.

This finishes the proof of Theorem A.

5.4. Proof of Theorem B

The proof of Theorem B requires only minor modifications. We consider the same subdivision of S^{d-1} into domains $D_1, D_2, \dots, D_r$ as in Section 5.1. From assertion (b) of Lemma 1 it follows that Lemma 3 can be strengthened in the following way. Put $N_\rho = N \cdot \sigma(D_\rho)$ ($\rho = 1, 2, \dots, r$). Then each domain D_ρ may be further subdivided into N_ρ coordinate boxes $D_{\rho 1}, \dots, D_{\rho N_\rho}$ such that:

- (a) $\sigma(D_{\rho\mu}) = 1/N$ holds for $\mu = 1, 2, \dots, N_\rho$.
- (b) Each interpolation point $x_{\mu}^{(\rho)}$ belongs to exactly one $D_{\rho\mu}$.
- (c) Each $x_{\mu}^{(\rho)}$ is bounded away from the boundary of $D_{\rho\mu}$, that means

$$|x_{\mu}^{(\rho)} - y| \geq c_4(d) \cdot N^{-1/(d-1)} \quad \text{for each } y \in S, \quad y \notin D_{\rho\mu}.$$

As in Section 5.1, we fix a point $x = x_v^{(\tau)}$, and divide the domains D_ρ into classes $M_q = M_{q, x_v^{(\tau)}}(q = 0, 1, \dots)$.

For each domain $D_\rho \in M_0$, and each point $x_{\mu}^{(\rho)} \neq x_v^{(\tau)}$, we have

$$\begin{aligned} & \left| \log|x_{\mu}^{(\rho)} - x_v^{(\tau)}| - N \int_{D_{\rho\mu}} \log|x_v^{(\tau)} - y| d\sigma(y) \right| \\ &= N \left| \int_{D_{\rho\mu}} \log\left(\frac{|x_{\mu}^{(\rho)} - x_v^{(\tau)}|}{|x_v^{(\tau)} - y|}\right) d\sigma(y) \right| \\ &\leq \log\left(\frac{\max_{y \in D_{\rho\mu}} |x_v^{(\tau)} - y|}{\min_{y \in D_{\rho\mu}} |x_v^{(\tau)} - y|}\right) \\ &\leq c_5(d). \end{aligned}$$

Adding up these inequalities for all $D_\rho \in M_0$, and $\mu = 1, 2, \dots, N_\rho$, $(\rho, \mu) \neq (\tau, \nu)$, we obtain

$$\begin{aligned} & \sum_{\substack{(\rho, \mu) \neq (\tau, \nu) \\ D_\rho \in M_0}} \log |x_\mu^{(\rho)} - x_\nu^{(\tau)}| \\ &= N \sum_{D_\rho \in M_0} \int_{D_\rho} \log |x_\nu^{(\tau)} - y| d\sigma(y) + O(1) - N \int_{D_{\tau\nu}} \log |x_\nu^{(\tau)} - y| d\sigma(y) \\ &= N \sum_{D_\rho \in M_0} \int_{D_\rho} \log |x_\nu^{(\tau)} - y| d\sigma(y) + O(1) + \frac{1}{d-1} \log N. \end{aligned} \quad (17)$$

As for the remaining classes $M_1, M_2, \dots$, we obtain, as in Section 5.1,

$$\begin{aligned} \sum_{q=1}^{\infty} \sum_{\substack{(\rho, \mu) \\ D_\rho \in M_q}} \log |x_\mu^{(\rho)} - x_\nu^{(\tau)}| &= N \sum_{q=1}^{\infty} \sum_{D_\rho \in M_q} \int_{D_\rho} \log |x_\nu^{(\tau)} - y| d\sigma(y) \\ &\quad + O\left(\sum_{q=1}^{\infty} q^{d-2} \cdot q^{-d-1}\right). \end{aligned} \quad (18)$$

Here the constants in the O -symbols depend on the dimension d only.

If we add (17) and (18) over all points $x_\nu^{(\tau)}$, $\tau = 1, 2, \dots, r$; $\nu = 1, 2, \dots, N_\tau$, the assertion of Theorem B follows.

6. Proof of Theorem C

According to [4] or [6], it is sufficient to prove that, for $3 \leq d \leq 6$, it is possible to distribute N points $x_1, x_2, \dots, x_N$ on S^{d-1} such that

$$\max_{x \in S^{d-1}} \left| \sum_{j=1}^N |\langle x, x_j \rangle| - N \cdot m_d \right| \leq c(d) \cdot N^{(1/2)((d-4)/(d-1))} \cdot \sqrt{\log N} \quad (19)$$

holds true for some positive constant $c = c(d)$, and for all $N \geq 2$. Here $\langle x, x_j \rangle$ denotes the scalar product of the unit vectors x, x_j , and m_d is the mean value

$$m_d = \int_{S^{d-1}} |\langle x, x_1 \rangle| d\sigma(x).$$

Let us first exclude the case $d = 3$. For $d \geq 4$, it is sufficient to prove (19) for all N of the form $N = 2d \cdot 3^{d-1} \cdot n$, $n \in \mathbb{N}$.

We project the points of S^{d-1} from the origin onto the surface W^{d-1} of the hypercube $[-1, 1]^d$. Denoting the faces of W^{d-1} by $F_1, F_2, \dots, F_{2d}$, and the projection by π , we obtain the following equivalent problem:

Let $\mu = \sigma \circ \pi^{-1}$ be the projection of σ from S^{d-1} onto W^{d-1} . On each face F_k ,

μ has a surface density $d\mu(\xi) = p(\xi) d\sigma_1(\xi)$, $\sigma_1 =$ Lebesgue measure on W^{d-1} , where

$$p(\xi) \in C^3(F_k) \quad (k = 1, 2, \dots, 2d)$$

and

$$0 < c_1(d) \leq p(\xi) \leq c_2(d), \quad \xi \in W^{d-1}. \quad (20)$$

For $\xi, \eta \in W$, let $g(\xi, \eta) = |\langle \pi^{-1}\xi, \pi^{-1}\eta \rangle| - m_d$. The function g has the following properties:

- (1) $\int_W g(\xi, \eta) d\mu(\xi) = 0$ for all $\eta \in W$.
- (2) On each F_k , g satisfies the Lipschitz condition

$$|g(\xi, \eta) - g(\xi', \eta)| \leq c_3(d) \cdot |\xi - \xi'|, \quad \eta \in W^{d-1}. \quad (21)$$

- (3) On each $F_k \setminus M_\eta$, $M_\eta = \{\xi \in W^{d-1} : g(\xi, \eta) = -m_d\}$, g has continuous partial derivatives of all orders, uniformly bounded in η .

We have to find N points $\xi_1, \xi_2, \dots, \xi_N$ on W^{d-1} such that

$$\max_{\xi \in W^{d-1}} \left| \sum_{j=1}^N g(\xi, \xi_j) \right| \leq c(d) \cdot N^{(1/2)((d-4)/(d-1))} \cdot \sqrt{\log N}.$$

In view of (20), it is possible to divide each face F_k into n boxes

$$B_\lambda (\lambda = (k-1)n + 1, \dots, k \cdot n; k = 1, 2, \dots, 2d),$$

with edges parallel to the edges of F_k , each with μ -measure $\mu(B_\lambda) = (2dn)^{-1}$, such that

$$\frac{(\text{diam } B_\lambda)^{d-1}}{\mu(B_\lambda)} \ll_d 1. \quad (22)$$

On each box B_λ , we replace the μ -density $p(\xi)$ by the approximation

$$p_\lambda^*(\xi) = u_\lambda \prod_{j=1}^{d-1} (1 + a_{\lambda j}(\xi_j - \xi_{\lambda j})).$$

Here ξ_j and $\xi_{\lambda j}$ ($j = 1, 2, \dots, d-1$) denote the cartesian coordinates of the point ξ and the midpoint ξ_λ of B_λ , respectively. The factor u_λ is chosen such that

$$\int_{B_\lambda} p_\lambda^*(\xi) d\xi = \mu(B_\lambda) = (2dn)^{-1}, \quad (23)$$

and the $a_{\lambda j}$ are defined by $a_{\lambda j} = (1/u_\lambda)(\partial/\partial \xi_j)p(\xi)|_{\xi=\xi_\lambda}$. Note that

$$|u_\lambda - p(\xi_\lambda)| \ll_d (\text{diam } B_\lambda)^2 \ll_d N^{-2/(d-1)}. \quad (24)$$

Now let μ_λ^* be the probability measure on B_λ , defined by

$$\mu_\lambda^*(A) = 2dn \cdot \int_A p_\lambda^*(\xi) d\xi, \quad A \subset B_\lambda.$$

Choose a probability space Ω and random points $X_1^{(\lambda)}(\omega), \dots, X_{3^d-1}^{(\lambda)}(\omega)$, $\omega \in \Omega$, such that:

- (1) All the $X_\rho^{(\lambda)}$ ($\rho = 1, 2, \dots, 3^{d-1}$; $\lambda = 1, 2, \dots, 2dn$) are μ_λ^* -distributed.
- (2) The relation

$$\frac{1}{3^{d-1}} \sum_{\rho=1}^{3^{d-1}} X_\rho^{(\lambda)}(\omega) = \int_{B_\lambda} \xi \cdot d\mu_\lambda^*(\xi) \quad (25)$$

holds true for all $\omega \in \Omega$.

- (3) The random points are independent for distinct values of λ .

The existence of such a set of random points is proved below.

The sum

$$\sum_{\lambda=1}^{2dn} \sum_{\rho=1}^{3^{d-1}} g(X_\rho^{(\lambda)}(\omega), \eta) =: Y_\eta(\omega)$$

is a random variable which depends on η . We estimate the expected value $E(Y_\eta)$ and the variance $\sigma^2(Y_\eta)$. We have

$$\begin{aligned} |E(Y_\eta)| &= 3^{d-1} \left| \sum_{\lambda=1}^{2dn} \int_{B_\lambda} g(\xi, \eta) d\mu_\lambda^*(\xi) \right| \\ &= N \left| \sum_{\lambda=1}^{2dn} \int_{B_\lambda} g(\xi, \eta) \cdot (p_\lambda^* - p)(\xi) d\xi \right| \\ &\leq N \left| \sum_{\lambda \in \Lambda_1} \right| + N \left| \sum_{\lambda \in \Lambda_2} \right|, \end{aligned}$$

where $\Lambda_1 = \{\lambda: B_\lambda \cap M_\eta = \emptyset\}$ and $\Lambda_2 = \{\lambda: B_\lambda \cap M_\eta \neq \emptyset\}$.

In order to estimate the terms in the two sums, we use the Taylor expansion of g and $(p_\lambda^* - p)$ about the midpoints $\xi_\lambda \in B_\lambda$. We have

$$p_\lambda^*(\xi) - p(\xi) = u_\lambda - p(\xi_\lambda) + b_2(\xi - \xi_\lambda)^2 + O(|\xi - \xi_\lambda|^3)$$

and, for $\lambda \in \Lambda_1$,

$$g(\xi, \eta) = a_0 + a_1(\xi - \xi_\lambda) + O(|\xi - \xi_\lambda|^2).$$

Note that the Taylor coefficients and the constants in the O -symbols are uniformly bounded by a constant $c = c(d)$.

For $\lambda \in \Lambda_1$, we obtain, from (23) and (24),

$$\left| \int_{B_\lambda} g(\xi, \eta)(p_\lambda^* - p)(\xi) d\xi \right| \ll_d \sigma_1(B_\lambda) \cdot N^{-4/(d-1)}. \quad (26)$$

For $\lambda \in \Lambda_2$, we obtain, from (21), (23), and (24),

$$\begin{aligned} \left| \int_{B_\lambda} g(\xi, \eta)(p_\lambda^* - p)(\xi) d\xi \right| &\ll_d \sigma_1(B_\lambda) \cdot (\text{diam } B_\lambda) \cdot \sup_{\xi \in B_\lambda} |p_\lambda^*(\xi) - p(\xi)| \\ &\ll_d \sigma_1(B_\lambda) \cdot N^{-3/(d-1)}. \end{aligned} \quad (27)$$

Adding (26) and (27), and noting that $|\Lambda_2| \ll_d N^{(d-2)/(d-1)}$, we obtain

$$|E(Y_\eta)| \ll_d N \cdot N^{-4/(d-1)} + N^{(d-2)/(d-1)} \cdot N^{-3/(d-1)} \ll N^{(d-5)/(d-1)}. \quad (28)$$

For the variance $\sigma^2(Y_\eta)$ we obtain

$$\begin{aligned} \sigma^2(Y_\eta) &= \sum_{\lambda=1}^{2dn} \sigma^2 \left(\sum_{\rho=1}^{3^{d-1}} g(X_\rho^{(\lambda)}(\omega), \eta) \right) = \sum_{\lambda \in \Lambda_1} + \sum_{\lambda \in \Lambda_2} \\ &\ll_d \sum_{\lambda \in \Lambda_1} (\text{diam } B_\lambda)^4 + \sum_{\lambda \in \Lambda_2} (\text{diam } B_\lambda)^2 \ll_d N^{(d-4)/(d-1)}. \end{aligned} \quad (29)$$

It is here where we make essential use of relation (25).

Now we follow the proof in [1] or [4]. From (28) and (29), and from Bernstein's inequality, it follows that the relation

$$\left| \sum_{\lambda=1}^{2dn} \sum_{\rho=1}^{3^{d-1}} g(X_\rho^{(\lambda)}(\omega), \eta) \right| \leq c_4(d) \cdot N^{(1/2)((d-4)/(d-1))} \cdot \sqrt{\log N} + c_5(d) \cdot N^{(d-5)/(d-1)}$$

holds with probability >0 for all $\eta \in W$ simultaneously. Noting that

$$\frac{d-5}{d-1} \leq \frac{1}{2} \frac{d-4}{d-1} \quad \text{for } d \leq 6,$$

the assertion follows.

The case $d=3$ needs only minor modifications. Let $N = 6 \times 3^2 \cdot n + m$, $0 < m < 6 \times 3^2$. Replace the unit cube by a box $[-1, 1]^2 \times [a, b]$ with faces $F_1, \dots, F_6$, where the interval $[a, b]$ is chosen such that

$$\sigma(\pi^{-1}F_1) : \dots : \sigma(\pi^{-1}F_5) : \sigma(\pi^{-1}F_6) = 3^2 \cdot n : \dots : 3^2 \cdot n : (3^2 \cdot n + m).$$

Divide each face F_j into n boxes B_λ ($\lambda = (j-1)n + 1, \dots, j \cdot n$; $j = 1, 2, \dots, 6$), subject to condition (22), such that $\mu(B_\lambda) = 3^2/N$ ($\lambda = 1, 2, \dots, 6n-1$) and

$\mu(B_{6n}) = (3^2 + m)/N$. On each B_λ ($\lambda = 1, 2, \dots, 6n - 1$) we choose 3^2 random points as before, whereas on B_{6n} we choose an arbitrary set of $(3^2 + m)$ μ -distributed random points. The contribution of the latter set of random points to the variance $\sigma^2(Y_\eta)$ is $\ll N^{-1}$. This finishes the proof in the case $d = 3$.

It remains to prove the following:

Lemma 4. Let μ_m be the measure on the interval $[-1, 1]$ with density $p_m(x) = \frac{1}{2} + mx$, $|m| \leq \frac{1}{2}$. Then there exist a probability space Ω and three random variables $g_1(\omega)$, $g_2(\omega)$, $g_3(\omega)$ on Ω such that

- (a) $g_j(\omega)$ ($j = 1, 2, 3$) is μ_m -distributed on $[-1, 1]$, and
- (b) $\frac{1}{3}(g_1(\omega) + g_2(\omega) + g_3(\omega)) = \int_{-1}^1 x \cdot p_m(x) dx$ for all $\omega \in \Omega$.

Proof. Note that $\int_{-1}^1 x p_m(x) dx = \frac{2}{3}m$. Denote by Ω_1 the intersection of the unit cube $[-1, 1]^3$ with the plane $x + y + z = 2m$. Ω_1 is a hexagon. Note the special cases $m = 0$ (regular hexagon) and $|m| = \frac{1}{2}$ (equilateral triangle). For the proof of Lemma 4 it is sufficient to find a probability measure P_1 on Ω_1 such that its orthogonal projections onto each of the three coordinate axes coincides with the measure μ_m . Equivalently, we have to find a probability measure P on the domain Ω (see Fig. 1 below) such that the projections in the three indicated directions onto straight lines perpendicular to these directions yield the same measure with linear density.

Let v_1 be the measure with constant density $p_1(\xi) = 1$, $\xi \in \Omega$. Let v_2 be the measure with density $p_2(\xi) = 1 - (2/a\sqrt{3}) \cdot \min(\|\xi\|, a\sqrt{3}/2)$, where $\|\xi\|$ denotes the distance from ξ to the boundary of Ω . (Hence v_2 vanishes on the inner triangle.) Put $P = ((b - a) \cdot v_1 + 4a \cdot v_2) / ((b - a) \cdot v_1(\Omega) + 4a \cdot v_2(\Omega))$. Then P satisfies the required conditions. $\square$

Fig. 1. The domain Ω .

Now it is easy to prove the existence of a set of 3^{d-1} random points on B_λ , subject to condition (25). Denote by $g_1^{(j)}(\omega), g_2^{(j)}(\omega), g_3^{(j)}(\omega)$ ($j = 1, 2, \dots, d-1$) three random points according to Lemma 4 with parameter $m = a_j$, $|a_j| \leq \frac{1}{2}$. Random points that belong to distinct values of j are assumed to be independent. Consider the set of random points on $[-1, 1]^{d-1}$ defined by

$$X_{i_1 i_2 \dots i_{d-1}}(\omega) = (g_{i_1}^{(1)}(\omega), g_{i_2}^{(2)}(\omega), \dots, g_{i_{d-1}}^{(d-1)}(\omega)), \quad i_j \in \{1, 2, 3\}.$$

Then all these points are μ -distributed, where

$$\left(\frac{d\mu}{d\xi}\right) = 2^{-(d-1)} \prod_{j=1}^{d-1} (1 + 2a_j \xi_j),$$

$$\xi = (\xi_1, \xi_2, \dots, \xi_{d-1}) \in [-1, 1]^{d-1},$$

and

$$\begin{aligned} 3^{-(d-1)} \sum X_{i_1 \dots i_{d-1}}(\omega) &= \left(\int_{-1}^1 \xi_1 \cdot \left(\frac{1}{2} + a_1 \xi_1\right) d\xi_1, \dots, \int_{-1}^1 \xi_{d-1} \cdot \left(\frac{1}{2} + a_{d-1} \xi_{d-1}\right) d\xi_{d-1} \right) \\ &= \int_{[-1, 1]^{d-1}} \xi d\mu(\xi). \end{aligned}$$

By a suitable linear transformation, we obtain the required set of random points for any subdomain B_λ .

The proof of Theorem C shows that we may replace uniform distribution σ on S^{d-1} by any measure σ' with $0 < K_1 \leq (d\sigma'/d\sigma) \leq K_2$ and $(d\sigma'/d\sigma) \in C^3(S^{d-1})$. Hence Theorem C not only holds for balls, but for a whole class of zonoids, generated by such measures σ' .

References

1. J. Beck, Some upper bounds in the theory of irregularities of distribution. *Acta Arith.* **43** (1984), 115–130.
2. U. Betke and P. McMullen, Estimating the sizes of convex bodies by projections. *J. London Math. Soc.* (2) **27** (1983), 525–538.
3. E. D. Bolker, A class of convex bodies. *Trans. Amer. Math. Soc.* **145** (1969), 323–345.
4. J. Bourgain and J. Lindenstrauss, Distribution of points on spheres and approximation by zonotopes. *Israel J. Math.* **64** (1988), 25–32.
5. J. Bourgain, J. Lindenstrauss, and V. Milman, Approximation of zonoids by zonotopes. *Acta Math.* **162** (1989), 73–141.
6. J. Linhart, Approximation of a ball by zonotopes using uniform distribution on the sphere. *Arch. Math.* **53** (1989), 82–87.
7. A. Lubotzky, R. Phillips, and P. Sarnak, Hecke operators and distributing points on the sphere. *I. Comm. Pure Appl. Math.* **39** (1986), 149–186.
8. P. Sjögren, Estimates of mass distributions from their potentials and energies. *Ark. Mat.* **10** (1972), 59–77.

9. K. B. Stolarsky, Sums of distances between points on a sphere II. *Proc. Amer. Math. Soc.* **41** (1973), 575–582.
10. G. Wagner, On the product of distances to a point set on the sphere. *J. Austral. Math. Soc. Ser. A* **47** (1989), 466–482.
11. G. Wagner, On means of distances on the surface of a sphere. I. Lower bounds. *Pacific J. Math.* **144** (1990), 389–398. II. Upper bounds. *Pacific J. Math.* **154** (1992), 381–396.
12. G. Wagner, On averaging sets. *Monatsh. Math.* **111** (1991), 69–78.

Received February 20, 1990.