Flight in a Packing of Disks

L. Fejes Tóth

Mathematical Institute of the Hungarian Academy of Sciences,
P.O. Box 127, H-1364 Budapest, Hungary

With sincere friendship dedicated to August Florian

Abstract. For a packing in the plane consisting of open convex domains, none of which contains the origin O , let $\lambda(r)$ be the length of the shortest path which evades the domains and connects O with a point at a distance r from O . Upper and lower bounds are given for the supremum of $\lambda(r)$ taken over all packings whose members are domains of diameter at least 2.

Introduction

By a packing of disks we mean a set of pairwise disjoint open convex domains. Let F be a family of disks. We consider a packing of disks from F none of which contains the origin O . Let $\lambda(r)$ be the length of a shortest path which evades the disks and connects O with a point of the circle of radius r centered at O . Let $\Lambda(F, r)$ be the supremum of $\lambda(r)$ extended over all packings defined above.

In this paper we consider the case when F consists of all convex domains of diameter at least 2. In Section 1 we give an upper bound and in Section 2 a lower bound for $\Lambda(r) = \Lambda(F, r)$. Section 3 contains some remarks.

1. Upper Bound

We claim that there is a constant c such that, for $r \geq 2$,

$$\Lambda(r) \leq r^2 + \frac{1}{2} \ln r + c. \quad (1)$$

We say that a boundary point P of a disk d is on the *arrival side* or on the *departure*

side of d according to whether the elongation of the segment OP does or does not intersect d .

Let P move on bd d into a new position P' . We write $\Delta r = \overline{OP'} - \overline{OP}$ and $\Delta \lambda$ for the length of the arc PP' . The proof of (1) is based on the following.

Lemma. *Let P be a point on the arrival side of a disk d of diameter at least 2. Assuming that $\overline{OP} = r \geq 2$, there is a point P' on the departure side of d for which*

$$\frac{\Delta r}{\Delta \lambda} \geq \sqrt{1 + r^2} - r. \quad (1.1)$$

Proof. Let AB be a segment of length 2 containing P . We write $\overline{AP} = a$, $\overline{BP} = b$, $\angle OPA = \varphi + 90^\circ$, and $\sqrt{1 + r^2} - r = \delta = \delta(r)$. Let AB be in a position such that $(\overline{OA} - r)/a = \delta$. Then

$$\overline{OA}^2 = r^2 + a^2 + 2ra \sin \varphi = (r + a\delta)^2,$$

whence, using the identity $(1 - \delta^2)/2r = \delta$, we obtain

$$\sin \varphi = (1 - a)\delta. \quad (1.2)$$

Applying this formula for $-\varphi$ instead of φ , we have $-\sin \varphi = (1 - b)\delta$ which shows that we have automatically $(\overline{OB} - r)/b = \delta$.

Let ψ be the obtuse or right angle made by a support line s of d through P and the line PO . Assuming that $\psi \geq 90^\circ + \arcsin \delta$, (1.2) implies that all points P' on the arrival side of d which arise by continuously increasing Δr satisfy (1.1). So does P' in the limiting case when OP' touches d , proving the lemma in the case $\psi \geq 90^\circ + \arcsin \delta$. Thus we may assume that $\psi < 90^\circ + \arcsin \delta$ which allows us to choose the segment AB on s so as to satisfy (1.2).

Now we introduce an oblique coordinate system xy so that $P = (0, 0)$, $O = (0, -r)$, $A = (a, 0)$, $B = (-b, 0)$, and consider the locus l_a of those points $Q = Q(x, y)$ for which

$$\overline{OQ} - r = (2a - x + y)\delta. \quad (1.3)$$

Similarly, we consider the locus l_b of the points Q defined by

$$\overline{OQ} - r = (2b + x + y)\delta. \quad (1.4)$$

Let ω be the angle between the x - and y -axis. Then by

$$\overline{OQ}^2 = x^2 + (y + r)^2 + 2x(y + r) \cos \omega, \quad \cos \omega = (1 - a)\delta,$$

we can write (1.3) and (1.4) in the form

$$s(x^2 + y^2) + txy + ux + vy + w = 0 \quad (1.5)$$

with $s = 1 - \delta^2$, $t = 2(1 - a)\delta + 2\delta^2$, $u = 2r(2 - a)\delta + 4a\delta^2$, $v = 2r - 2r\delta - 4a\delta^2$, $w = -4ra\delta - 4a^2\delta^2$ for l_a and $s = 1 - \delta^2$, $t = 2(b - 1)\delta - 2\delta^2$, $u = -2r(2 - b) - 4b\delta^2$, $v = 2r - 2r\delta - 4b\delta^2$, $w = -4rb\delta - 4b^2\delta^2$ for l_b .

The sort of the conic represented by (1.5) depends on the sign of $s^2 - \frac{1}{4}t^2$. We are concerned with the case $0 < a, b < 2$, $0 < \delta < 1$. Now $\frac{1}{2}|t| < \delta + \delta^2$, so that $s - \frac{1}{2}|t| > 1 - \delta - 2\delta^2$ which is positive for $\delta < \frac{1}{2}$. It follows that, for $r > \frac{3}{4}$, l_a and l_b are ellipses.

We have $\overline{OA} - r = a\delta < (2b + a)\delta$ and $\overline{OB} - r = b\delta < (2a + b)\delta$ showing, on the one hand, that A and B are lying on l_a and l_b , and on the other hand, that they are contained in the elliptical disks bounded by l_b and l_a , respectively.

We still need some information about the point of intersection C of the tangent of l_a at A and the tangent of l_b at B . The derivative y' of $y = y(x)$ defined by (1.5) is given by the equation

$$2s(x + yy') + ty + txy' + u + vy' = 0.$$

Writing $y' = y'_A$ at the point A and $y' = y'_B$ at the point B , we obtain, by using the respective coefficients,

$$y'_A = -\frac{a + r(2 - a)\delta + a\delta^2}{r + (a - a^2 - r)\delta - a\delta^2}, \quad y'_B = \frac{b + r(2 - b)\delta + b\delta^2}{r + (b - b^2 - r)\delta - b\delta^2}.$$

For $0 \leq a, b \leq 2$ the denominators attain their minimum for $a = 2$ and $b = 2$. Since the numerators are obviously positive, we have $y'_A < 0$ and $y'_B > 0$, provided that $r(1 - \delta) - 2(a + \delta)\delta > 0$. Since this inequality is satisfied for $r \geq 2$ (even for $r > 1.33$), C is in the intersection of the half-plane $y > 0$ and the parallel strip $-b < x < a$.

We continue to show that, for $r \geq 2$, the angles α and β of the triangle ABC at A and B are less than 60° . Since changing the role of a and b changes the role of α and β simultaneously, it is enough to investigate the angle α .

We have

$$-y'_A = -\frac{dy}{dx} = \frac{\sin \alpha}{\sin(\omega + \alpha)} = \frac{\tan \alpha}{\sin \omega + \cos \omega \tan \alpha},$$

whence

$$\tan \alpha = \frac{-y'_A \sin \omega}{1 + y'_A \cos \omega}.$$

Using the inequality $r\delta < \frac{1}{2}$ and writing $\delta_0 = \delta(r_0)$ it is easy to check that, for $r \geq r_0$, $0 \leq a \leq 2$, we have

$$-y'_A < \frac{2 + 2\delta_0^2}{r - 0, 5 - 2(\delta_0 + \delta_0^2)} = m(r).$$

Since $m(r)$ is a decreasing function,

$$\tan \alpha < \frac{m(r_0)}{1 - m(r_0)\delta_0}.$$

This yields, for $r_0 = 2.6$, $\alpha < 58.4^\circ$. For $2 \leq r < 2.6$ numerical computations show that, for a given r , α is an increasing function of a and, for $a = 2$, α is a decreasing function of r . The maximum of α , i.e., its value for $a = r = 2$ is approximately 54.3° .

Now we may assume that neither A nor B are on $\text{bd } d$ since these points would satisfy condition (1.1) imposed on the point P' . Since the diameter of d was supposed to be at least 2 a part of $\text{bd } d$ lies outside the triangle ABC . Therefore $\text{bd } d$ intersects either l_a or l_b or both. Let $\text{bd } d$ intersect l_a . Proceeding from P on $\text{bd } d$ in positive direction, let $P'(x', y')$ be the first point of intersection. We assume that the y -coordinate of any point of the arc PP' is at most y' . Then

$$\Delta r = \overline{OP} - r = (2a - x + y)\delta > \Delta\lambda \cdot \delta.$$

Assuming on the other hand that the arc PP' contains a point whose y -coordinate is greater than y' , then $\text{bd } d$ cannot intersect the side AC of the triangle ABC . Therefore it must intersect the side BC , and consequently l_b . Now proceeding from P on $\text{bd } d$ in negative direction, the first point at which we reach l_b satisfies (1.1).

Now we have proved (1.1) if the respective point P' is on the departure side of d . In the opposite case we move P' on $\text{bd } d$ so as to increase the distance OP' until the line OP' touches d . We claim that during this operation $\Delta r/\Delta\lambda$ increases. Changing our notation we write P_0 and P_1 for the original points P and P' , and P for a variable point on the arrival side of d beyond P_1 . Writing furthermore $\overline{OP} = r$ and σ for the length of the arc P_0P , we have $dr/d\sigma = \cos \varepsilon$, where ε is the angle made by the directed line OP and the instantaneous direction in which P is moving. Since ε is a decreasing function of σ , r is a convex function of σ . Therefore, in accordance with our statement,

$$\frac{r - r_0}{\sigma} > \frac{r_1 - r_0}{\sigma_1} > \delta(r_0).$$

This completes the proof of the lemma. □

Now our strategy of flight from O is as follows. First we travel from O to a point P_1 of distance 2 from O . We proceed on the half-line OP_1 until we get to a point P_2 on the arrival side of a disk (which may coincide with P_1). We continue our way on the boundary of this disk to the point P_3 on its departure side satisfying (1.1), proceed on the half-line OP_3 to the next disk, and so on.

In order to get an upper bound for the total length λ_n of our path from O to

P_n in terms of $r_n = \overline{OP}_n$ we write (1.1) in the form

$$\frac{\Delta\lambda}{\Delta r} \leq \sqrt{1+r^2} + r.$$

Using the inequality $\sqrt{1+r^2} < r + 1/2r$ and writing $f(r) = 2r + 1/2r$, we have $\Delta\lambda < f(r)\Delta r$. For a straight segment of the path $\lambda_{2i} - \lambda_{2i-1} = r_{2i} - r_{2i-1}$, we have, for all positive integers i ,

$$\lambda_{i+1} - \lambda_i < f(r_i)(r_{i+1} - r_i).$$

Observing that, for $r > \frac{1}{2}$, $f(r)$ is an increasing function, we have

$$\lambda_n - \lambda_1 < \sum_{i=1}^{n-1} f(r_i)(r_{i+1} - r_i) < \int_2^{r_n} f(r) dr = r_n^2 + \frac{1}{2} \ln r_n - 4 - \frac{1}{2} \ln 2.$$

This inequality does not hold generally by replacing λ_n and r_n by the respective data λ and r associated with an arbitrary point of the path constructed above because, during evading a disk, r first increases slowly or definitely decreases. Let P_n be on the arrival side of a disk d we want to evade. Let AB be a segment of length 2 satisfying (1.2) with $\overline{P_nA} = a$, $\overline{P_nB} = b$, and such that AB is a support line of d . Let the circle with center O and radius r , such that $r_n < r \leq \overline{OB}$, intersect the segment P_nB at G and bd in a point P on the arrival side of d , such that the line OP intersects the segment P_nB in a point H . Let σ be the length of the arc P_nP . Then $\sigma \leq P_nH + HP \leq P_nH + HG = P_nG$ with equality if P coincides with G . On the other hand, for a given r , $\overline{P_nG} = \sigma_n$ attains its maximum in the limiting case as $a \rightarrow 0$. For $a = 0$ we have

$$r^2 = r_n^2 + \sigma_n^2 - 2r_n\sigma_n\delta_n, \quad \delta_n = \delta(r_n),$$

whence $\sigma_n = r_n\delta_n + \sqrt{r_n^2\delta_n^2 + r^2 - r_n^2}$. It follows that during evading d we have

$$\lambda < g(r) = \lambda_n + r_n\delta_n + \sqrt{r_n^2\delta_n^2 + r^2 - r_n^2}.$$

We write $h(r) = r^2 + \frac{1}{2} \ln r - 4 - \frac{1}{2} \ln 2$ and consider the difference $D(r) = g(r) - h(r)$. Since $D(r)$ is concave and

$$D'(r_n) = \frac{1}{\delta_n} - 2r_n - \frac{1}{2r_n} = \sqrt{1+r_n^2} - r_n - \frac{1}{2r_n} < 0$$

we have, for $r \geq r_n$,

$$D(r) \leq D(r_n) = \lambda_n + 2r_n\delta_n - h(r_n) < \lambda_1 + 2r_n\delta_n < \lambda_1 + 1$$

and consequently $\lambda < h(r) + 1$.

This ends the proof of (1) with the constant $c = \Lambda(2) - 4 - \frac{1}{2} \ln 2 + 1$.

2. Lower Bound

We construct a packing showing that, for $r \geq \cot(\pi/9) \approx 2.75$,

$$\Lambda(r) \geq r^2 - \left(\frac{\pi}{3} + \frac{3}{\pi}\right)r + 3.6.$$

Let $\dots, P_n^{-2}, P_n^{-1}, P_n^0, P_n^1, P_n^2, \dots$ be regular n -gons centered at O , P_n^i being spanned by the side-midpoints of P_n^{i+1} . These n -gons divide the plane into triangles. Let the inradius of P_n^0 be $R_n = \cot(\pi/n)$. Then the diameter of each triangle in the exterior of P_n^0 is greater than or equal to 2.

Let r_n^k be the circular ring bounded by the incircles of P_n^0 and P_{n+k}^0 . A rather “good” packing (a packing impeding the flight from O) is obtained by considering in each ring r_n^1 the triangles generated by the n -gons. However, here the flight from O is facilitated by the fact that at each new ring r_n^1 we can start from a point which is generally near to a vertex of P_n^0 avoiding hereby a road section of length 1 along a half-side of P_n^0 . Therefore it is convenient to pack two or more consecutive rings with similar triangles. We investigate the packing of triangles in r_n^3 generated by the n -gons for $n = 6, 9, \dots$ completed by the packing of P_6^0 by triangles and pentagons exhibited in Fig. 1.

Let P_n^k be the outermost n -gon in r_n^3 , i.e., the n -gon with $k = k(n)$ given by

$$R_n q^k \leq R_{n+3} < R_n q^{k+1}, \quad q = \frac{1}{\cos(\pi/n)}.$$

Let λ be the distance covered by a point P during its flight from O in this packing. Writing $\lambda = \lambda_n^l$ if P is in a vertex of P_n^l ($n = 6, 9, \dots$), we have

$$1 + \lambda_n^k - \lambda_n^0 = 1 + q + q^2 + \dots + q^k = \frac{q^{k+1} - 1}{q - 1} > \frac{(R_{n+3} - R_n) \sin(\pi/n)}{1 - \cos(\pi/n)}.$$

Fig. 1

In view of

$$\cot x = \frac{1}{x} - \frac{x}{3} - \dots, \quad 0 < x < \frac{\pi}{2},$$

and

$$\frac{\sin x}{1 - \cos x} > \frac{x - x^3/6}{x^2/2} = \frac{2}{x} - \frac{x}{3}, \quad 0 < x < \frac{\pi}{2},$$

then

$$1 + \lambda_n^k - \lambda_n^0 > \left(\frac{3}{\pi} + \frac{\pi}{n(n+3)} \right) \left(\frac{2n}{\pi} - \frac{\pi}{3n} \right) = \frac{6n}{\pi^2} + \frac{1}{n(n+3)} \left(n - 3 - \frac{\pi^2}{3n} \right)$$

and, by our stipulation that $n \geq 6$,

$$\lambda_n^k - \lambda_n^0 > \frac{6n}{\pi^2} - 1.$$

This implies, because $R_n < n/\pi$ and

$$\begin{aligned} \lambda_6^0 &= 2 + \sqrt{32 - 18\sqrt{3}} + \sqrt{28 - 16\sqrt{3}} > 3.44, \\ \Lambda(R_{3n}) &\geq \lambda_{3n-3}^1 > \lambda_6^0 + \lambda_6^k - \lambda_6^0 + \lambda_9^k - \lambda_9^0 + \dots + \lambda_{3n-3}^k - \lambda_{3n-3}^0 \\ &> \lambda_6^0 + \frac{6}{\pi^2} (6 + 9 + \dots + 3n - 3) - (n - 2) \\ &= \frac{9}{\pi^2} n^2 \left(\frac{9}{\pi^2} + 1 \right) n + 2 \left(1 - \frac{9}{\pi^2} \right) + \lambda_6^0 > R_n^2 - \left(\frac{\pi}{3} + \frac{3}{\pi} \right) R_n + 3.6 \end{aligned}$$

proving (2) for values of r of the form $r = R_{3n} = \cot(\pi/3n)$, $n = 3, 4, \dots$.

If the fleeing point P is in a vertex of P_n^l , then $\overline{OP} = r = R_n q^l$. Therefore, for $l = 1, 2, \dots, k$,

$$\lambda_n^l = \lambda_n^0 + q \frac{q^l - 1}{q - 1} = \lambda_n^0 + \frac{q}{q - 1} \left(\frac{r}{R_n} - 1 \right)$$

is in a linear relation with the pertaining value of r . Since, on the other hand, in $(\lambda_n^l, \lambda_n^{l+1})$, $\lambda = \lambda_n^l + \sqrt{r^2 - R_n^2 q^{2l+2}}$ is a concave function of r , we have, in the whole interval $(\lambda_n^0, \lambda_n^k)$,

$$\lambda \geq \lambda_n^0 + \frac{q}{q - 1} \left(\frac{r}{R_n} - 1 \right)$$

which is obviously greater than the (convex) lower bound of $\Lambda(r)$ given in (2).

3. Remarks

Let STU and SVU be triangles defined by $\overline{SU} \leq 2$ and $\overline{ST} = \overline{TU} = \overline{UV} = \overline{VS} = 2$. Let O be the center of the quadrangle $STUV$. We consider two further triangles which arise from the previous ones by a rotation about O through 90° followed by a translation through $\overline{OS}$ and $\overline{OU}$, respectively. These triangles form a packing which shows that, for $r \leq \sqrt{2}$, we have $\Lambda(r) \geq \sqrt{2}r$. It seems to be very likely that, in this inequality, equality holds suggesting that a universally best packing, in which $\Lambda(r)$ is attained for all values of r , does not exist.

Suppose that whenever the point P has evaded a disk we remove all disks and throw a new obstacle in its way. In the family of disks under consideration the most efficient obstacle seems to be a “bar” of length 2 put orthogonally to OP with its center at P . This suggests a sharper upper bound for $\Lambda(r)$ than (1): $\Lambda(r) \leq r^2$ for $r^2 = 2, 3, \dots$ and

$$\Lambda(r) \leq r^2 + \frac{1}{4}$$

for any value of $r \geq \sqrt{2}$.

Among the numerous problems of estimating $\Lambda(F, r)$ for various families of disks, special attention is due to disks with prescribed quotient of their width w and diameter d . More generally, we consider the family F_p of disks with $w/d \geq p$. For the triangles in the packing generated by n -gons in Section 2 we have $w/d = \frac{1}{2} \tan(\pi/n)$. In this packing the length of a path leading from a vertex of P_n^0 to O is given by

$$r \sin \frac{\pi}{n} (1 + q + q^2 + \dots) = r \frac{\sin(\pi/n)}{1 - \cos(\pi/n)},$$

where r is the circumradius of P_n^0 . Is it true that, for $p = \frac{1}{2} \tan(\pi/n)$,

$$\Lambda(F_p, r) = \frac{\sin(\pi/n)}{1 - \cos(\pi/n)} r?$$

What can be said for other values of p ? In particular, what can be said for $p = 1$? A construction by Hortobágyi [1] implies that

$$\Lambda(F_1, r) \geq \frac{8\pi/3 - 4(\arcsin \frac{1}{4} + \arcsin \frac{3}{4})}{\sqrt{15} + \sqrt{7} - \sqrt{12}} r > 1.3r.$$

About other results concerning shortest paths in packings of disks we refer to Section 2.3 of the survey paper [2] and to [3]–[5].

I am grateful to Wlodek Kuperberg for the following comment. “The upper bound (1) obtained for the case of planar disks D with $\text{diam } D \geq 2$ applies directly, as a corollary, to the case of d -dimensional convex domains K with $\text{diam } K \geq 4$. Perhaps, with some elaboration on the proof, this change of the diameter

assumption might prove unnecessary. In any case, if the approach to the problem in dimension 2 could be successfully transferred to dimension 3 by means of an analogous lemma, then it would automatically produce the same upper bound for all $d \geq 3$. On the other hand, obtaining a good lower bound for $\Lambda(r)$ in dimensions $d \geq 3$ might require some essential modification of the construction."

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