Discrete Comput Geom 3:307-324 (1988)

Discrete & Computational
Geometry

Valuations and Polarity

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Abstract. Given a collection $\mathscr{A}$ of convex polytopes, let $\tau(\mathscr{A})$ denote the set of all convex transversals of $\mathscr{A}$. If $\mathscr{A}$ and $\mathscr{B}$ are two such collections, of finite cardinality, then there is a simple, arithmetical condition which holds precisely when $\tau(\mathscr{A}) = \tau(\mathscr{B})$. Another such condition, involving what we call the "Sallee-Shephard mapping," characterizes those pairs $\mathscr{A}$ and $\mathscr{B}$ for which $\tau(\tau(\mathscr{A})) = \tau(\mathscr{B})$.

As these results are established, several distributive lattices involving convex sets are introduced, and relationships between their valuation modules are determined. In particular, it is proven that the Sallee-Shephard mapping is an isomorphism of the additive, abelian group of simple functions generated by the characteristic functions of the open, convex sets and that generated by those of the closed, convex sets.

1. Introduction

The writing of this paper began as an attempt to exploit the notion of "polarity" of convex sets in the theory of valuations on convex polyhedra. In the first sections we give results without explicit reference to the connection with polarity; but in many cases a simpler proof (of a possibly weaker result) can be obtained by appealing to later results more directly concerned with polarity. The final section of this paper is largely self-contained and the reader might wish to glance at this section early on.

In Sections 2, 4, and 5 various distributive lattices are described. Among these are two familiar ones: L_o and L_c, the lattices of finite unions of open and closed, respectively, convex sets in $\mathfrak{R}^d$. Three less-familiar ones are: Φ, the lattice of "complete families of convex transversals;" and $\hat{L}_o$ and $\hat{L}_c$, the "polar lattices of open (closed) convex sets."

The valuation modules of L_o, $\hat{L}_o$, L_c, and $\hat{L}_c$ are isomorphic. Indeed, we show that those of L_o and of $\hat{L}_o$ are isomorphic to $\mathbb{Z} \oplus S_o$, where S_o is the group

($\mathbb{Z}$-module) of simple functions generated by the characteristic functions of open, convex sets. Similarly, those of L_c and $\hat{L}_c$ are isomorphic to $\mathbb{Z} \oplus S_c$, where S_c is the group generated by the characteristic functions of the closed, convex sets. In Section 6 it is shown that S_o and S_c are connected by an isomorphism, the "Sallee-Shephard mapping."

In Section 3 an arithmetical condition is given for two finite collections $\mathscr{A}$ and $\mathscr{B}$ of convex sets to possess the same convex transversals. The Sallee-Shephard mapping is introduced and used in Section 6 to shed further light on this topic.

The paper makes use of fundamental facts concerning convex sets and polytopes, which may be found in [7], [15], and [19]. The inspirational papers of Hadwiger [8], Klee [11], and Rota [16] provide bacground on the Euler characteristic in a suitable setting, and on valuations, more generally. The paper of McMullen and Schneider [14] provides a useful survey of the use of valuations in convexity. The papers of Geissinger [2] provide the basic facts concerning the valuation modules and valuation rings of distributive lattices. The papers of Groemer [3]-[6] contain more related material.

2. Complete Families of Transversals

Given a family $\mathscr{K}$ of convex subsets of $\mathscr{R}^d$, let $\tau(\mathscr{K})$ denote the family of convex transversals of $\mathscr{K}$, so that $\tau(\mathscr{K}) = \{T \subseteq \mathscr{R}^d\colon T \cap \mathscr{K} \neq \varnothing$ for each $K \in \mathscr{K}\}$. We will call sets of the form $\tau(\mathscr{K})$ *complete families of transversals*, and we will denote by Φ the collection of all complete families of transversals.

It follows from the formula $\bigcap_{\lambda\in\Lambda} \tau(\mathscr{K}_\lambda) = \tau(\bigcup_{\lambda\in\Lambda} \mathscr{K}_\lambda)$ that the intersection of any collection of complete families of transversals is again in Φ. The family of all convex sets in $\mathscr{R}^d$ is $\tau(\varnothing)$, an element of Φ. It follows that Φ, as a set partially ordered by inclusion, is a complete lattice.

The meet operation on Φ is intersection. The join of a collection $\mathscr{K}_\lambda$ $(\lambda \in \Lambda)$ of elements of Φ is $\varphi(\bigcup_{\lambda\in\Lambda} \mathscr{K}_\lambda)$, where for any collection $\mathscr{K}$ of convex sets,

$$\phi(\mathscr{K}) = \tau(\tau(\mathscr{K})) = \bigcap_{\substack{\mathscr{T}\in\Phi,\\ \mathscr{K}\subseteq\mathscr{T}}} \mathscr{T}.$$

Note that τ, when restricted to Φ, is a dual automorphism of Φ. It is worthwhile to observe that a criterion for inclusion of a family $\mathscr{T}$ in Φ is that for each convex set $K \notin \mathscr{T}$, there is a convex transversal T of $\mathscr{T}$ such that $K \cap T = \varnothing$.

The foregoing paragraphs describe some essential pieces of the structure one obtains by considering the binary relation "K has nonempty intersection with T" on the family of convex sets in $\mathscr{R}^d$ and using this relation to obtain a "polarity," as in pp. 122-125 of [1].

In this section we study the structure of Φ. In particular we show that it is distributive.

We call a set $H \subseteq \mathscr{R}^d$ a *half-space* if both H and its complement, $\mathscr{R}^d \sim H$, are convex. (Note that a half-space may be neither closed nor open. Our half-spaces are called "hemispaces" by Jamison [10].) Let $\mathscr{H}$ denote the collection of all half-spaces in $\mathcal{R}^d$, partially ordered by inclusion. Let $\hat{\mathcal{H}}$ denote the collection of all upper semi-ideals of $\mathcal{H}$; i.e., a subset $\mathcal{J} \subseteq \mathcal{H}$ is an element of $\hat{\mathcal{H}}$ if, for half-spaces H_1 and H_2, $H_1 \in \mathcal{J}$ and $H_2 \supseteq H_1$ imply $H_2 \in \mathcal{J}$. Clearly, $\hat{\mathcal{H}}$ is a completely distributive lattice. Its operations are intersection and union.

For $\mathcal{T} \in \Phi$, let $\gamma(\mathcal{T}) = \mathcal{T} \cap \mathcal{H}$. This is a function $\gamma: \Phi \to \hat{\mathcal{H}}$, since elements $\mathcal{T}$ of Φ are upper simi-ideals in the lattice of all subsets of the set of convex sets in $\mathcal{R}^d$, ordered by inclusion.

Theorem 1. *The function $\gamma: \Phi \to \hat{\mathcal{H}}$ is an isomorphism of partially ordered sets (and, therefore, of complete lattices).*

Proof. We first show that for complete families of transversals $\mathcal{T}_1$ and $\mathcal{T}_2$, $\mathcal{T}_1 \subseteq \mathcal{T}_2$ if and only if $\gamma(\mathcal{T}_1) \subseteq \gamma(\mathcal{T}_2)$. If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, it is immediate that $\gamma(\mathcal{T}_1) = \mathcal{T}_1 \cap \mathcal{H} \subseteq \mathcal{T}_2 \cap \mathcal{H} = \gamma(\mathcal{T}_2)$. Suppose $\mathcal{T}_1 \not\subseteq \mathcal{T}_2$. Let K be an element of $\mathcal{T}_1 \sim \mathcal{T}_2$. Since $K \notin \mathcal{T}_2$, there is a transversal T of $\mathcal{T}_2$ such that $K \cap T = \emptyset$. By a result of Hammer [9] there is a half-space H such that $K \subseteq H$ and $T \subseteq \mathcal{R}^d \sim H$. Then $H \in \mathcal{T}_1 \sim \mathcal{T}_2$, so $\gamma(\mathcal{T}_1) \not\subseteq \gamma(\mathcal{T}_2)$.

It remains to show that γ is surjective. Suppose $\mathcal{J} \in \hat{\mathcal{H}}$. Let $\mathcal{T} = \{K \subseteq \mathcal{R}^d$: K is convex, and for each transversal T of $\mathcal{J}$ it is true that $T \cap K \neq \emptyset\}$; i.e., $\mathcal{T} = \varphi(\mathcal{J})$. Then $\mathcal{T} \in \Phi$. We need only show that $\gamma(\mathcal{T}) = \mathcal{J}$. Clearly, $\mathcal{J} \subseteq \mathcal{T}$, so $\mathcal{J} \subseteq \mathcal{T} \cap \mathcal{H} = \gamma(\mathcal{T})$. We verify the reverse inclusion. Suppose that H is a half-space which is not in $\mathcal{J}$. We must show that H is not in $\mathcal{T}$. Let $T = \mathcal{R}^d \sim H$. Then T is a transversal of $\mathcal{J}$, for if $J \in \mathcal{J}$ then, since $H \notin \mathcal{J}$, $J \not\subseteq H$, so $J \cap T \neq \emptyset$. Since $T \cap H = \emptyset$ it follows that $H \notin \mathcal{T}$. □

Corollary. *The lattice Φ is completely distributive.*

Let Φ_p denote that subset of Φ consisting of all complete families of transversals which are of the form $\tau(\mathcal{K})$, where $\mathcal{K}$ is a finite family of convex polytopes.

Theorem 2. *The subset $\Phi_p \subseteq \Phi$ is a sublattice of Φ (in the finitary sense). The restriction of τ to Φ_p is a dual automorphism of Φ_p.*

Proof. Temporarily denote by Φ' the sublattice of Φ generated by elements of the form $\tau(\{p\})$, where $p \in \mathcal{R}^d$. Note that such elements are fixed by τ, so that τ maps Φ' to itself. It remains only to show that $\Phi' = \Phi_p$.

Suppose $\{v_1, \ldots, v_n\} \subseteq \mathcal{R}^d$ and $P = \operatorname{conv}\{v_1, \ldots, v_n\}$. Then $\tau(\{\{v_1\}, \ldots, \{v_n\}\})$ consists of precisely those convex sets which contain P; clearly, this is $\varphi(\{P\})$. From the equality $\varphi(\{P\}) = \tau(\{\{v_1\}, \ldots, \{v_n\}\})$ follows $\tau(\{P\}) = \tau(\varphi(\{P\})) = \varphi(\{\{v_1\}, \ldots, \{v_n\}\})$. We see that (i) the join $\varphi(\{\{v_1\}, \ldots, \{v_n\}\})$ of the elements $\varphi(\{\{v_i\}\}) = \tau(\{\{v_i\}\})$ (for $i \in [n]$) is $\tau(\{P\})$, and (ii) $\tau(\{P\}) \in \Phi'$ (for any convex polytope P).

If $\mathcal{K} = \{P_1, \ldots, P_m\}$ is a finite collection of convex polytopes, so that $\tau(\mathcal{K}) \in \Phi_p$, then $\tau(\mathcal{K}) = \tau(\{P_1\}) \cap \cdots \cap \tau(\{P_m\})$, which, utilizing (ii) above, is in Φ'. Then $\Phi_p \subseteq \Phi'$. The reverse inclusion follows easily from the fact that, since Φ is distributive, each element of the lattice Φ' can be written (finitarily) as an intersection of joins of elements of the form $\tau(\{p\})$, so it is of the form $\tau(\mathcal{K})$. □

3. A Consequence of the Equality $\tau(\mathscr{K}_1)=\tau(\mathscr{K}_2)$

Our main objective in this section is the proof of Theorem 3, below. In case $\mathscr{K}_1$ and $\mathscr{K}_2$ are finite families of convex sets and $\tau(\mathscr{K}_1)=\tau(\mathscr{K}_2)$, this result describes an equality involving $\mathscr{K}_1$ and $\mathscr{K}_2$ which must hold in the valuation ring of the lattice of finite unions of convex sets. The proof roughly mimics (through polarity) part of an argument used by Groemer in his proof of the existence of an Euler characteristic. (See [3].)

A consequence of Theorem 3 for "clustered families" of convex sets, introduced in [12], is also described.

If K and G are convex sets, let $\delta(K,G)=\{ag+(1-a)x\colon 0\le a<1,\ g\in G$, and $x\in K\}$. Clearly, this is also a convex set, and $K\subseteq\delta(K,G)\subseteq\operatorname{conv}(K\cap G)$.

It is not difficult to establish that, if $\mathscr{K}$ and $\mathscr{G}$ are families of convex sets, then

$$\delta\left(\operatorname{conv}\bigcup_{K\in\mathscr{K}}K,\ \operatorname{conv}\bigcup_{G\in\mathscr{G}}G\right)=\operatorname{conv}\bigcup_{\substack{K\in\mathscr{K}\\ G\in\mathscr{G}}}\delta(K,G).$$

Also, if K, G_1, and G_2 are convex, then $\delta(\delta(K,G_1),G_2)=\delta(K,\operatorname{conv}(G_1\cup G_2))$; if K_1, K_2, and G are convex, then $\delta(K_1,\delta(K_2,G))=\delta(\operatorname{conv}(K_1\cup K_2),G)$.

Lemma 1. *Suppose $\mathscr{K}$ is a family of convex sets, $K\in\varphi(\mathscr{K})$, and G is also a convex set. Then*

$$\delta(K,G)\in\varphi(\{\delta(W,G)\colon W\in\mathscr{K}\}).$$

Proof. Suppose $T\in\tau(\{\delta(W,G)\colon W\in\mathscr{K}\})$. For each element $W\in\mathscr{K}$, let $p_W=a_Wg_W+(1-a_W)x_W$ be an element of $T\cap\delta(W,G)$, where $0\le a_W<1$, $g_W\in G$, and $x_W\in W$. Let $T'=\operatorname{conv}\{x_W\colon W\in\mathscr{K}\}$. Clearly, $T'\in\tau(\mathscr{K})$, so $T'\cap K\neq\varnothing$. Let w be an element of $T'\cap K$. Since $w\in T'$, we may write $w=\sum_{W\in\mathscr{K}}\beta_Wx_W$, where $\beta_W=0$ except for finitely many $W\in\mathscr{K}$, $\beta_W\ge 0$ for $W\in\mathscr{K}$, and $\sum_{W\in\mathscr{K}}\beta_W=1$ (so that the sum is a convex combination of the x_W's). Let $\sigma=\sum_{W\in\mathscr{K}}[\beta_W/(1-a_W)]$. Then $\sigma>1$. Let $\gamma_W=\beta_W/[(1-a_W)\sigma]$ for $W\in\mathscr{K}$. Then

$$\begin{aligned}\sum_{W\in\mathscr{K}}\gamma_Wp_W&=\sum_{W\in\mathscr{K}}\gamma_W(a_Wg_W+(1-a_W)x_W)\\&=\sum_{W\in\mathscr{K}}\frac{\beta_Wa_W}{(1-a_W)\sigma}g_W+\sum_{W\in\mathscr{K}}\frac{\beta_W}{\sigma}x_W\\&=\frac{\sigma-1}{\sigma}\left(\sum_{W\in\mathscr{K}}\frac{\beta_W/(1-a_W)-\beta_W}{\sigma-1}g_W\right)+\frac{1}{\sigma}w.\end{aligned}$$

From the first expression we see that this point is in T. From the last we see that it is in $\delta(K,G)$. Then $T\cap\delta(K,G)\neq\varnothing$. Since this is true of each such transversal T, it follows that $\delta(K,G)\in\varphi(\{\delta(W,G)\colon W\in\mathscr{K}\})$. □

Suppose $\mathscr{K}=\{K_\lambda:\lambda\in\Lambda\}$ is a finite collection of convex sets indexed by Λ, and suppose $p\in\mathscr{R}^d$. Let

$$\beta(\mathscr{K},p)=\left\{A\subseteq\Lambda\colon p\notin\operatorname{conv}\left(\bigcup_{\lambda\in A}K_\lambda\right)\right\}.$$

Lemma 2. *Suppose $\mathcal{K}$ is a nonempty collection of convex sets in $\mathcal{R}^d$ indexed by Λ, a and b are distinct point of $\mathcal{R}^d$, and p is a point of the relative interior of the line segment connecting a and b. Let*

$$\mathcal{K}_1 = \{\delta(K_\lambda, \{a\}): \lambda \in \Lambda\},$$

$$\mathcal{K}_2 = \{\delta(K_\lambda, \{b\}): \lambda \in \Lambda\},$$

and

$$\mathcal{K}_3 = \{\delta(K_\lambda, \operatorname{conv}\{a, b\}): \lambda \in \Lambda\}.$$

Then

$$\beta(\mathcal{K}_1, p) \cup \beta(\mathcal{K}_2, p) = \beta(\mathcal{K}, p)$$

and

$$\beta(\mathcal{K}_1, p) \cap \beta(\mathcal{K}_2, p) = \beta(\mathcal{K}_3, p).$$

Proof. We establish the first equality. Suppose $A \in \beta(\mathcal{K}_1, p) \cup \beta(\mathcal{K}_2, p)$. Then

$$\begin{aligned} p \notin \operatorname{conv}\Big(\bigcup_{\lambda \in A} \delta(K_\lambda, \{a\})\Big) &\cap \operatorname{conv}\Big(\bigcup_{\lambda \in A} \delta(K_\lambda, \{b\})\Big) \\ &= \delta\Big(\operatorname{conv}\Big(\bigcup_{\lambda \in A} K_\lambda\Big), \{a\}\Big) \cap \delta\Big(\operatorname{conv}\Big(\bigcup_{\lambda \in A} K_\lambda\Big), \{b\}\Big) \\ &\supseteq \operatorname{conv}\Big(\bigcup_{\lambda \in A} K_\lambda\Big), \end{aligned}$$

so $A \in \beta(\mathcal{K}, p)$. Suppose $A \notin \beta(\mathcal{K}_1, p) \cup \beta(\mathcal{K}_2, p)$. Then $p \in \delta(\operatorname{conv}(\bigcup_{\lambda \in A} K_\lambda), \{a\}) \cap \delta(\operatorname{conv}(\bigcup_{\lambda \in A} K_\lambda), \{b\})$, so there exist x, $y \in \operatorname{conv}(\bigcup_{\lambda \in A} K_\lambda)$ such that $p \in \delta(\{x\}, \{a\}) \cap \delta(\{y\}, \{b\})$. Since p is strictly between a and b it is clear that x, y, a, b, and p are colinear, and that $p \in \operatorname{conv}(\{x, y\})$. Therefore $p \in \operatorname{conv}(\bigcup_{\lambda \in A} K_\lambda)$, so that $A \notin \beta(\mathcal{K}, p)$.

Now we establish the second equality. Suppose $A \in \beta(\mathcal{K}_3, p)$. Then

$$\begin{aligned} p \notin \operatorname{conv}&\Big(\bigcup_{\lambda \in A} \delta(K_\lambda, \operatorname{conv}\{a, b\})\Big) \\ &\supseteq \operatorname{conv}\Big(\bigcup_{\lambda \in A} \delta(K_\lambda, \{a\})\Big) \cup \operatorname{conv}\Big(\bigcup_{\lambda \in A} \delta(K_\lambda, \{b\})\Big), \end{aligned}$$

so $A \in \beta(\mathcal{K}_1, p) \cap \beta(\mathcal{K}_2, p)$. Suppose $A \notin \beta(\mathcal{K}_3, p)$. Then

$$p \in \operatorname{conv}\Big(\bigcup_{\lambda \in A} \delta(K_\lambda, \operatorname{conv}\{a, b\})\Big) = \delta(\operatorname{conv}\Big(\bigcup_{\lambda \in A} K_\lambda\Big), \operatorname{conv}\{a, b\}\Big),$$

so there is $x\in\operatorname{conv}(\bigcup_{\lambda\in A}K_\lambda)$ such that $p\in\delta(\{x\},\operatorname{conv}\{a,b\})$. Since p is on the line through a and b, it is clear that x must be, as well. Then

$$\begin{aligned}p\in(\operatorname{conv}\{a,b\})\sim\{a,b\}&\subseteq\delta(\{x\},\{a\})\cup\delta(\{x\},\{b\})\\&\subseteq\delta\Big(\operatorname{conv}\Big(\bigcup_{\lambda\in A}K_\lambda\Big),\{a\}\Big)\cup\delta\Big(\operatorname{conv}\Big(\bigcup_{\lambda\in A}K_\lambda\Big),\{b\}\Big)\\&=\operatorname{conv}\Big(\bigcup_{\lambda\in A}\delta(K_\lambda,\{a\})\Big)\cup\operatorname{conv}\Big(\bigcup_{\lambda\in A}\delta(K_\lambda,\{b\})\Big),\end{aligned}$$

so $A\notin\beta(\mathcal{K}_1,p)\cap\beta(\mathcal{K}_2,p)$. □

If $\mathcal{K}$ is finite, indexed by the finite set $\Lambda=[m]$, then $\beta(\mathcal{K},p)$ is a finite simplicial complex. Let χ be the function which assigns to each (finite) simplicial complex its Euler characteristic.

Theorem 3. *Let $\mathcal{K}=\{K_i: i\in[m]\}$ and $\mathcal{K}'=\{K_j': j\in[n]\}$ be nonempty, finite, indexed collections of convex sets, and let p be a point of $\mathcal{R}^d$. Suppose that $\tau(\mathcal{K})=\tau(\mathcal{K}')$ (or, equivalently, that $\varphi(\mathcal{K})=\varphi(\mathcal{K}')$). Then $\chi(\beta(\mathcal{K},p))=\chi(\beta(\mathcal{K}',p))$.*

Proof. Suppose that this is not the case. Choose $\mathcal{K}$, $\mathcal{K}'$, and p so that the hypotheses are satisfied but $\chi(\beta(\mathcal{K},p))\neq\chi(\beta(\mathcal{K}',p))$, in such a way that $|\beta(\mathcal{K},p)|+|\beta(\mathcal{K}',p)|$ is as small as possible.

Clearly, $\{p\}\notin\tau(\mathcal{K}\cup\{\{p\}\})$ $(=\tau(\mathcal{K}'\cup\{\{p\}\}))$, for otherwise $\beta(\mathcal{K},p)=\beta(\mathcal{K}',p)=\{\varnothing\}$, and the Euler characteristics would be 0. Also, it cannot be the case that there are sets $A\subseteq[m]$ and $A'\subseteq[n]$ such that

$$\beta(\mathcal{K},p)=\{S\subseteq[m]: S\cap A=\varnothing\}$$

and

$$\beta(\mathcal{K}',p)=\{T\subseteq[n]: T\cap A'=\varnothing\},$$

for then (since we have seen that neither $A=[m]$ nor $A'=[n]$) the Euler characteristics would both be 1.

It follows that, for at least one of the two complexes, there is a subset B of the index set which (i) is not in the complex, (ii) has the property that each proper subset is in the complex, and (iii) has cardinality at least 2. For definiteness, we assume that this is true of the complex $\beta(\mathcal{K},p)$, so that $B\subseteq[m]$. Let j be an element of B. Let $B_0=B\sim\{j\}$. We may choose $a\in K_j$ and $b\in\operatorname{conv}(\bigcup_{i\in B_0}K_i)$ such that $p\in\operatorname{conv}\{a,b\}$. Notice that, since both $\{j\}$ and B_0 are in $\beta(\mathcal{K},p)$, $p\notin\{a,b\}$. For $\mathcal{K}$, a, b, and p, the situation is that of Lemma 2. Let $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}_3$ be as in Lemma 2. Also it is clear that Lemma 2 applies to $\mathcal{K}'$, a, b, and p. Let $\mathcal{K}_1'$, $\mathcal{K}_2'$, and $\mathcal{K}_3'$ be the corresponding collections in this case.

By Lemma 2 and the modular property of the Euler characteristic,

$$\chi(\beta(\mathcal{K},p))=\chi(\beta(\mathcal{K}_1,p))+\chi(\beta(\mathcal{K}_2,p))-\chi(\beta(\mathcal{K}_3,p))$$

and

$$\chi(\beta(\mathcal{K}_1', p)) + \chi(\beta(\mathcal{K}_2', p)) - \chi(\beta(\mathcal{K}_3', p)) = \chi(\beta(\mathcal{K}', p)).$$

The complexes appearing on the right in the first equation are properly contained in $\beta(\mathcal{K}, p)$.

Since $\varphi(\mathcal{K}) = \varphi(\mathcal{K}')$, Lemma 1 yields $\varphi(\mathcal{K}_1) = \varphi(\mathcal{K}_1')$, $\varphi(\mathcal{K}_2) = \varphi(\mathcal{K}_2')$, and $\varphi(\mathcal{K}_3) = \varphi(\mathcal{K}_3')$. It follows, considering the minimality criterion for our choice of $\mathcal{K}$ and $\mathcal{K}'$, that $\chi(\beta(\mathcal{K}, p) = \chi(\beta(\mathcal{K}', p)$, a contradiction. □

Recall from [12] that a family of convex sets is *clustered* if $\tau(\mathcal{K}) = \tau(\{\bigcap_{K\in\mathcal{K}} K\})$.

Corollary. *If $\mathcal{K}$ is a finite, clustered family of convex sets indexed by $[m]$ then*

$$\chi(\beta(\mathcal{K}, p)) = \begin{cases} 1 & \text{if } p \in \bigcap_{K\in\mathcal{K}} K, \\ 0 & \text{otherwise.} \end{cases}$$

4. Valuations and the Transversal Characteristic

In this section we utilize facts about valuation modules to produce a partial converse to Theorem 3.

If $\mathcal{K}$ is a finite family of convex sets in $\mathcal{R}^d$ then the function $f\colon \mathcal{R}^d \to \mathbb{Z}$ given by the formula $f(p) = 1 - \chi(\beta(\mathcal{K}, p))$ is an element of the additive group of simple functions generated by the characteristic functions of convex sets. Indeed,

$$f(p) = \sum_{\substack{\mathcal{A}\subseteq\mathcal{K},\\ \mathcal{A}\neq\varnothing}} (-1)^{|\mathcal{A}|-1} C\left(\operatorname{conv}\left(\bigcup_{K\in\mathcal{A}} K\right), p\right),$$

where

$$C(W, p) = \begin{cases} 1 & \text{if } p \in W, \\ 0 & \text{otherwise.} \end{cases}$$

(We shall usually designate this function simply by $C(W)$, and write $C(W)(p)$ instead of $C(W, p)$.) We denote the function f by $\omega_{\mathcal{K}}$ or $\omega(\mathcal{K})$, and call it the *transversal characteristic* of $\mathcal{K}$.

Suppose $\mathcal{D}$ is a distributive lattice of subsets of $\mathcal{R}^d$, and $\varnothing \in \mathcal{D}$. Let $S(\mathcal{D})$ denote the group of simple functions generated by the characteristic functions of elements of $\mathcal{D}$. Let $C\colon \mathcal{D} \to S(\mathcal{D})$ be the mapping which takes $D \in \mathcal{D}$ to its characteristic function. Then each valuation v on $\mathcal{D}$, having values in an abelian group A, such that $v(\varnothing) = 0$, induces a homorphism $\bar{v}\colon S(\mathcal{D}) \to A$, the unique homomorphism such that $v(D) = \bar{v}(C(D))$ for $D \in \mathcal{D}$. (See [2, I and III].) Obviously, given any homomorphism $\bar{v}\colon S(\mathcal{D}) \to A$, the composition $\bar{v} \circ C$ is a valuation on $\mathcal{D}$.

Theorem 4. *Suppose $\mathcal{K}$ and $\mathcal{K}'$ are finite, nonempty families of convex sets and that $\tau(\mathcal{K})=\tau(\mathcal{K}')$. Suppose that $\mathcal{D}$ is a distributive lattice of sets which contains* $\operatorname{conv}(\bigcup_{K\in\mathcal{A}} K)$, *for each subset $\mathcal{A}$ of $\mathcal{K}\cup\mathcal{K}'$. If v: $\mathcal{D}\to A$ is a valuation on $\mathcal{D}$, then*

$$\sum_{\substack{\mathcal{A}\subseteq\mathcal{K},\\ \mathcal{A}\neq\varnothing}} (-1)^{|\mathcal{A}|-1} v\Big(\operatorname{conv}\Big(\bigcup_{K\in\mathcal{A}} K\Big)\Big)=\sum_{\substack{\mathcal{A}\subseteq\mathcal{K}'\\ \mathcal{A}\neq\varnothing}} (-1)^{|\mathcal{A}|-1} v\Big(\operatorname{conv}\Big(\bigcup_{K\in\mathcal{A}} K\Big)\Big).$$

Proof. Suppose, first, that $v(\varnothing)=0$. Let $\bar{v}$ be the corresponding group homomorphism, $\bar{v}$: $S(\mathcal{D})\to A$. If $\tau(\mathcal{K})=\tau(\mathcal{K}')$ then, by Theorem 3, $\omega(\mathcal{K})=\omega(\mathcal{K}')$, so that $\bar{v}(\omega(\mathcal{K}))=\bar{v}(\omega(\mathcal{K}'))$. This immediately yields the desired equality.

The result now follows in general by noting that (i) any valuation is the sum of a constant valuation and one which maps $\varnothing$ to zero, and (ii) the result holds for constant valuations. □

Corollary. *Suppose $\mathcal{K}$ is a finite, clustered family of convex sets, and that $\mathcal{D}$ is a distributive lattice of sets such that* $\operatorname{conv}(\bigcup_{K\in\mathcal{A}} K)\in\mathcal{D}$, *for each collection $\mathcal{A}\subseteq\mathcal{K}$. Let v: $\mathcal{D}\to A$ be a valuation on $\mathcal{D}$. Then*

$$\sum_{\substack{\mathcal{A}\subseteq\mathcal{K},\\ \mathcal{A}\neq\varnothing}} (-1)^{|\mathcal{A}|-1} v\Big(\operatorname{conv}\Big(\bigcup_{K\in\mathcal{A}} K\Big)\Big)=v\Big(\bigcap_{K\in\mathcal{K}} K\Big).$$

Proof. This follows at once from Theorem 4 by setting $\mathcal{K}'=\{\bigcap_{K\in\mathcal{K}} K\}$. □

Next we have, essentially, the converse to Theorem 3, when the sets are open.

Theorem 5. *Suppose $\mathcal{K}$ and $\mathcal{K}'$ are finite, nonempty families of open, convex sets in $\mathcal{R}^d$. Then $\tau(\mathcal{K})=\tau(\mathcal{K}')$ if and only if $\omega(\mathcal{K})=\omega(\mathcal{K}')$.*

Proof. If $\tau(\mathcal{K})=\tau(\mathcal{K}')$ then $\omega(\mathcal{K})=\omega(\mathcal{K}')$, by Theorem 3.

Suppose $\omega(\mathcal{K})=\omega(\mathcal{K}')$. Let χ_o be the Euler characteristic for the lattice of finite unions of open, convex sets in $\mathcal{R}^d$. Let T be an element of $\tau(\mathcal{K})$. If $T\notin\tau(\mathcal{K}')$ then there is an element $K\in\mathcal{K}'$ and a closed half-space H such that $T\subseteq H$ and $K\cap H=\varnothing$. Let H' be the interior of H. The function $v(G)=\chi_o(G\cap H')$ is a valuation on the lattice of finite unions of open convex sets. It induces a homomorphism, $\bar{v}$: $S_o\to\mathbb{Z}$, where S_o is the group generated by the characteristic functions of open convex sets.

Since $H\in\tau(\mathcal{K})$, $H=\operatorname{cl} H'$, and all elements of $\mathcal{K}$ are open, it follows that $H'\in\tau(\mathcal{K})$. Then for each nonempty subset $\mathcal{A}$ of $\mathcal{K}$ it is clear that $(\operatorname{conv}(\bigcup_{A\in\mathcal{A}} A)\cap H')$ is a nonempty, open, convex set, so its Euler characteristic is 1. It follows that $\bar{v}(\omega(\mathcal{K}))=1$.

Let $\mathcal{B}=\{U\in\mathcal{K}'\colon U\cap H'=\varnothing\}$. Clearly, $K\in\mathcal{B}$, so $\mathcal{B}\neq\varnothing$. For nonempty $\mathcal{A}\subseteq\mathcal{K}'$, $(\operatorname{conv}(\bigcup_{U\in\mathcal{A}} U)\cap H')$ is an open, convex set which is empty if and only if $\mathcal{A}\subseteq\mathcal{B}$. It follows that $\bar{v}(\omega(\mathcal{K}'))=0$. This is a contradiction, since $\omega(\mathcal{K})=\omega(\mathcal{K}')$; so it must be the case that $T\in\tau(\mathcal{K}')$.

We have shown that $\tau(\mathcal{K})\subseteq\tau(\mathcal{K}')$. The reverse inclusion becomes apparent upon reversing the roles of $\mathcal{K}$ and $\mathcal{K}'$ in the foregoing. □

In Section 6 we consider another converse to Theorem 3, when the sets involved are assumed to be closed.

5. The Open, Polar Lattice

Let $\mathscr{A}$ be an additive, abelian group. Let S be a set endowed with a binary operation, $\wedge$, under which it is a semilattice. Suppose, further, that there is an injection $\varepsilon: S \to \mathscr{A}$. For $T \subseteq S$ of finite cardinality and nonempty, let

$$\mu(T) = \sum_{\substack{U \subseteq T, \\ U \neq \varnothing}} (-1)^{|U|-1} \varepsilon\left(\bigwedge_{u \in U} u\right).$$

We say that the semigroup $(S, \wedge)$ is *conforming* (*relative to* ε) if, given any finite sets T_1, T_1', T_2, and T_2' contained in S such that $\mu(T_1) = \mu(T_1')$ and $\mu(T_2) = \mu(T_2')$, one has $\mu(T_1 \cup T_2) = \mu(T_1' \cup T_2')$.

Suppose $(S, \wedge)$ is conforming. Let L denote the image of μ, so that $L = \{\mu(T): T \subseteq S,\ 0 < |T| < \infty\} \subseteq \mathscr{A}$. We define a binary operation $\vee$ on L by the rule $a \vee b = \mu(T_1 \cup T_2)$, where $a = \mu(T_1)$ and $b = \mu(T_2)$. This is well defined, since $(S, \wedge)$ is conforming. For $a, b \in L$ we define $a \wedge b = a + b - a \vee b$. Then, for $s, t \in S$, one has $\varepsilon(s \wedge t) = \varepsilon(s) \wedge \varepsilon(t)$, since by definition $\varepsilon(s) \vee \varepsilon(t) = \mu(\{s, t\}) = \varepsilon(s) + \varepsilon(t) - \varepsilon(s \wedge t)$. We use this formula to extend this operation to L. Presently, we shall see that if a and b are in L then $a \wedge b$ is, as well. We shall call the operations $\wedge$ and $\vee$ "meet" and "join".

Theorem 6. *If $(S, \wedge)$ is a conforming semilattice, and L, $\wedge$, and $\vee$ are as above, then $(L, \wedge, \vee)$ is a distributive lattice. The inclusion $c: L \to \mathscr{A}$ is a valuation on L.*

Proof. The operation $\vee$ on L is clearly commutative, associative, and idempotent, so $(L, \vee)$ is a semilattice. It is easily verified that

$$\varepsilon(a_1) \vee \cdots \vee \varepsilon(a_k) = \mu(\{a_1, \ldots, a_k\}) \qquad \text{for} \quad a_1, \ldots, a_k \in S.$$

In particular, L is generated under $\vee$ by the image, $\varepsilon(S)$, of S.

Suppose $a, b_1, \ldots, b_k$ are in $\varepsilon(S)$. Then

$$\begin{aligned} a \wedge (b_1 \vee \cdots \vee b_k) &= a + (b_1 \vee \cdots \vee b_k) - (a \vee b_1 \vee \cdots \vee b_k) \\ &= - \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq k} (-1)^r (a \wedge b_{i_1} \wedge \cdots \wedge b_{i_r}) \\ &= (a \wedge b_1) \vee \cdots \vee (a \wedge b_k). \end{aligned}$$

It follows that

$$(a_1 \vee \cdots \vee a_k) \wedge (b_1 \vee \cdots \vee b_1) = \bigvee_{\substack{i \in [k], \\ j \in [l]}} (a_i \wedge b_j), \qquad (*)$$

whenever a_i, $b_j \in \varepsilon(S)$ $(i \in [k], j \in [l])$. Since, as we have seen, L is precisely the set of joins of elements of $\varepsilon(S)$, L is closed under $\wedge$.

It is not difficult to verify by using $(*)$ that $\wedge$ is associative, commutative, and idempotent, and that $\wedge$ distributes over $\vee$.

The identity $a \wedge b = a + b - (a \vee b)$ establishes the final statement of the theorem. □

For example, suppose $\mathscr{A}$ is the abelian group generated, under addition, by the characteristic functions of sets in a family S which is closed under finite intersection. There is the obvious injection $\varepsilon: S \to \mathscr{A}$, sending each element of S to its characteristic function. If $\wedge$ denotes intersection (making S into a semilattice), then S is conforming. The lattice L is isomorphic to the lattice of finite unions of elements of S.

Let S_o denote the abelian group generated by the characteristic functions of open, convex sets in $\mathscr{R}^d$. We utilize the mapping taking each convex, open set to its characteristic function.

Theorem 7. *The semilattice of open, convex sets in $\mathscr{R}^d$ under the operation $U \wedge V = \operatorname{conv}(U \cup V)$ is conforming in S_o.*

Proof. In this case, the function μ described above coincides with the transversal characteristic, ω.

By Theorem 5, if $\mathscr{K}$ and $\mathscr{K}'$ are finite, nonempty families of open, convex sets such that $\omega(\mathscr{K}) = \omega(\mathscr{K}')$, then $\tau(\mathscr{K}) = \tau(\mathscr{K}')$. If $\omega(\mathscr{K}_1) = \omega(\mathscr{K}_1')$ and $\omega(\mathscr{K}_2) = \omega(\mathscr{K}_2')$ then $\tau(\mathscr{K}_1) = \tau(\mathscr{K}_1')$ and $\tau(\mathscr{K}_2) = \tau(\mathscr{K}_2')$. From these equalities follows

$$\begin{aligned}\tau(\mathscr{K}_1 \cup \mathscr{K}_2) &= \tau(\mathscr{K}_1) \cap \tau(\mathscr{K}_2)\\ &= \tau(\mathscr{K}_1') \cap \tau(\mathscr{K}_2')\\ &= \tau(\mathscr{K}_1' \cup \mathscr{K}_2'),\end{aligned}$$

so that $\omega(\mathscr{K}_1 \cup \mathscr{K}_2) = \omega(\mathscr{K}_1' \cup \mathscr{K}_2')$, as required. □

We denote by $\hat{L}_o$ the distributive lattice corresponding to L, of Theorem 7. We call it the *polar lattice generated by the open, convex sets*, or, briefly, the *open, polar lattice*. The extension of the convex hull operation is the meet operation on $\hat{L}_o$. We shall, therefore, denote the meet of elements A and B by $A \operatorname{conv} B$.

Lemma 3. *The zero function is the maximal element of $\hat{L}_o$. It is irreducible.*

Proof. Suppose $Y \in \hat{L}_o$. We must show that $C(\varnothing) \operatorname{conv} Y = Y$. (Here, $C(\varnothing)$ denotes the characteristic function of $\varnothing$—the zero function. This function is certainly an element of $\hat{L}_o$, since $C(\varnothing) = \omega(\{\varnothing\})$.) Since $Y \in \hat{L}_o$, there is a family $\mathscr{K}$ of open, convex sets such that $Y = \omega(\mathscr{K})$. Then $C(\varnothing)$ conv $Y = C(\varnothing) + Y - \omega(\mathscr{K} \cup \{\varnothing\}) = Y$. It follows that $C(\varnothing)$ is the maximal element of $\hat{L}_o$.

Suppose $X_1 = \omega(\mathscr{K}_1)$ and $X_2 = \omega(\mathscr{K}_2)$. Then the join of X_1 and X_2 in $\hat{L}_o$ is $\omega(\mathscr{K}_1 \cup \mathscr{K}_2)$. It remains only to show that if $\mathscr{K}$ is any finite, nonempty collection of open, nonempty convex sets then $\omega(\mathscr{K})$ is not identically 0. Let $\bar{\chi}_o$ denote the homomorphism, $\bar{\chi}_o$: $S_o \to \mathbb{Z}$, induced by the Euler characteristic. Then we have $\bar{\chi}_o(\omega(\mathscr{K})) = 1$, under the stated conditions on $\mathscr{K}$. It follows that $\omega(\mathscr{K})$ is not the zero function. □

We denote by $\hat{L}'_o$ the lattice $\hat{L}_o$ with the irreducible, maximal element, $\hat{C}(\varnothing)$, removed. It follows from Theorems 6 and 7 that the natural injection $\hat{L}_o \to S_o$, as well as its restriction to $\hat{L}'_o$, is a valuation. It follows that it extends to a homomorphism, η: $V(\hat{L}'_o) \to S_o$, where $V(\hat{L}'_o)$ denotes the valuation ring of $\hat{L}'_o$. Our next objective is to show that this mapping is an isomorphism. First, we develop a useful criterion for determining if an element $X \in S_o$ is the zero function.

By an elementary valuation on a distributive lattice L, we mean a valuation e: $L \to \mathbb{Z}$, whose image is $\{0, 1\}$, such that if $a, b \in L$ and $a \le b$, then $e(a) \le e(b)$. Given a valuation v: $L \to \mathbb{Z}$, let $\tilde{v}$ denote its unique extension to a homomorphism, $\tilde{v}$: $V(L) \to \mathbb{Z}$.

Lemma 4. *Let L be a distributive lattice. Let W be a set of elementary valuations on L which distinguish points of L. Let e_0 be the valuation which is identically 1 on L. Finally, let $\tilde{W} = \{\tilde{e}$: $e \in W\}$. Then the elements of $\tilde{W} \cup \{\tilde{e}_o\}$ distinguish points of $V(L)$. If L has no least element, then the elements of $\tilde{W}$ distinguish points of $V(L)$.*

Proof. For $x \in L$, let $\delta(x) = \{e \in W \cup \{e_0\}$: $e(x) = 1\}$. Then δ is a lattice monomorphism of L into the lattice of subsets of $W \cup \{e_0\}$. Let $\delta(L)$ denote its image, so that $L \simeq \delta(L)$. Notice that $e_0 \in \delta(x)$, for each x in L. Then $\delta(L)$ is a lattice of nonempty sets. (Notice that, if L has no least element, then $\delta(x) \sim \{e_0\}$ is nonempty.) Let S denote the group of simple functions on W generated (under addition) by the characteristic functions of sets $\delta(x)$. According to [2, III], the unique extension to $V(\delta(L))$ of the valuation on $\delta(L)$ taking $\delta(x)$ to its characteristic function is an isomorphism. Composing, we have a valuation ψ: $L \to S$ whose extension $\tilde{\psi}$: $V(L) \to S$ is an isomorphism. Suppose v_1 and v_2 are distinct elements of $V(L)$. Then $\tilde{\psi}(v_1 - v_2) \ne 0$, and we may choose $e \in W$ on which it is nonzero; but its value on e is, clearly, $\tilde{e}(v_1) - \tilde{e}(v_2)$, so $\tilde{e}$ distinguishes v_1 and v_2.

The final statement in the lemma is clear, considering the parenthetical comment above. □

Let H be an open half-space. Then the function $K \mapsto \chi_o(K \cap H)$ is a valuation on L_o, the lattice of finite unions of open, convex sets. Its value on the empty set is 0, so it extends uniquely to a homomorphism $\bar{v}_H$: $S_o \to \mathbb{Z}$. Also, let $\bar{\chi}_o$: $S_o \to \mathbb{Z}$ be the extension of χ_o to S_o.

Theorem 8. *The function η: $V(\hat{L}'_o) \to S_o$ is an isomorphism.*

Proof. If H is an open half-space, let $v'_H(F) = 1 - \bar{v}_H(F)$, for $F \in \hat{L}_o$. We show, first, that the restriction of v'_H to $\hat{L}_o$ is an elementary valuation. Clearly, it is a valuation. Suppose $F=\omega(\mathscr{K})$, where $\mathscr{K}$ is a finite, nonempty collection of open, convex sets. Then

$$v'_H(F)=\begin{cases}1 & \text{if } H\notin\tau(\mathscr{K}),\\ 0 & \text{if } H\in\tau(\mathscr{K}).\end{cases}$$

It is clear from this that if $\mathscr{K}_1\subseteq\mathscr{K}_2$ then $v'_H(\omega(\mathscr{K}_1))\leq v'_H(\omega(\mathscr{K}_2))$, which implies that v'_H is monotone on $\hat{L}_o$.

Next, we establish that elements of $\hat{L}_o$ are distinguished by valuations of the form v'_H. Suppose that $F_1=\omega(\mathscr{K}_1)$ and $F_2=\omega(\mathscr{K}_2)$ are elements of $\hat{L}_o$, and that $F_1\neq F_2$. Then $\tau(\mathscr{K}_1)\neq\tau(\mathscr{K}_2)$. Suppose, for instance, that $\tau(\mathscr{K}_1)\not\subseteq\tau(\mathscr{K}_2)$. Let T be an element of $\tau(\mathscr{K}_1)\sim\tau(\mathscr{K}_2)$. Let K be an element of $\mathscr{K}_2$ such that $T\cap K=\emptyset$. There is a closed half-space H' such that $T\subseteq H'$ and $K\cap H'=\emptyset$. Let $H=\operatorname{int} H'$. Since $T\subseteq H'$, $H'\in\tau(\mathscr{K}_1)$. Since $H'=\operatorname{cl} H$ and all the elements of $\mathscr{K}_1$ are open, $H\in\tau(\mathscr{K}_1)$. However, $H\notin\tau(\mathscr{K}_2)$, since $K\cap H=\emptyset$. It follows that $v'_H(F_1)=0\neq 1=v'_H(F_2)$.

The lemma now yields that the extensions $\tilde{v}'_H$ to $V(\hat{L}_o)$ of the functions v'_H, together with that of the function identically 1 on $\hat{L}_o$, distinguish elements of $V(\hat{L}_o)$. Let the extension of the function identically 1 on $\hat{L}_o$ be $\tilde{v}_o: V(\hat{L}_o)\to\mathbb{Z}$.

Consider the homomorphism $\eta: V(\hat{L}'_o)\to S_o$. Clearly, it is surjective. We need only establish that its kernel is $\{0\}$. Suppose $x\in V(\hat{L}'_o)$ and $x\neq 0$. If there is an open half-space H such that $\tilde{v}'_H(x)\neq 0$ then we have that $0\neq\tilde{v}'_H(x)=\bar{\chi}_o(\eta(x))-\bar{v}_H(\eta(x))$. Clearly, we must have $\eta(x)\neq 0$. Likewise, if $\tilde{v}_o(x)\neq 0$, then we have $0\neq\tilde{v}_o(x)=\bar{\chi}_o(\eta(x))$. (This equality holds in $V(\hat{L}'_o)$ since it holds for elements of $\hat{L}'_o$.) Again it follows that $\eta(x)\neq 0$. □

We observe the following useful by-product of the proof of Theorem 8.

Theorem 9. *Let F be an element of S_o. If $F\neq 0$ then there is an open half-space, H, such that $\bar{v}_H(F)\neq 0$.*

Proof. It is clear from the proof of Theorem 8 that the functions $\bar{v}_H$ are elementary valuations on the dual of $\hat{L}'_o$. This lattice has no least element, so, according to the lemma, they distinguish points of S_o. □

Theorem 10. *The binary operation* conv *on $\hat{L}'_o$ has a unique extension to a binary operation on S_o which distributes over addition. With this multiplication, S_o becomes a ring.*

Proof. Since S_o is isomorphic to the valuation module of $\hat{L}'_o$ and conv is the meet operation of $\hat{L}'_o$, the multiplication of the valuation ring yields such an extension. Uniqueness is not hard to verify. □

Reference [4] contains a result analogous to Theorem 10, but with conv replaced by Minkowski addition. See also [13].

6. The Sallee–Shephard Mapping and the Closed, Polar Lattice

In this section we will describe a lattice. $\hat{L}_c$, related to the closed, convex sets in roughly the same way that $\hat{L}_o$ is related to the open, convex sets. We discuss its valuation module and show, in fact, that $V(\hat{L}_c)$ is isomorphic to $V(\hat{L}_o)$. The isomorphism is what we call the "Sallee-Shephard mapping." Sallee and Shephard studied a function which is essentially an adjoint of this mapping. (See [17] and [18], and, in particular, Theorem 4.2 of [17].)

The section closes with another application of the Sallee-Shephard mapping. We use it to characterize those finite families $\mathscr{P}$ and $\mathscr{Q}$ of convex polytopes which are dually related in the lattice Φ_p of Section 1; i.e., for which $\tau(\mathscr{P}) = \varphi(\mathscr{Q})$.

To begin, we describe a homomorphism cl: $S_o \to S_c$, where S_c is the group of simple functions generated by the closed, convex sets. Given the characteristic function $F = C(U)$ of an open convex set U, let cl F denote the characteristic function of its closure. (If W is a set, we also denote its closure by cl W. Hopefully, this will not cause confusion.)

Lemma 5. *The function* cl *extends uniquely to a homomorphism* cl: $S_o \to S_c$.

Proof. Recall that $\bar{\chi}_o: S_o \to \mathbb{Z}$ is the homomorphism induced by the Euler characteristic $\chi_o: L_o \to \mathbb{Z}$. The desired homomorphism maps $F \in S_o$ to the function whose value at x is $\bar{\chi}_o(F \cap C(B))$, where B is the characteristic function of a sufficiently small, open ball centered at x. (Here, $F \cap C(B)$ denotes pointwise multiplication of the functions. The reader will recognize this as the multiplication in an isomorphic copy of the augmentation ideal of the valuation ring of the lattice $\hat{L}_o$.)

To be more precise, we let $F = \sum_{i \in [n]} \alpha_i F_i$, where the functions F_i are characteristic functions of open, convex sets, U_i. If B is the characteristic function of any open ball centered at x so small that it does not meet any of the sets U_i for which $x \notin \text{cl } U_i$, then cl F has the value

$$\bar{\chi}_o(F \cap B) = \sum_{\substack{i \in [n], \\ x \in \text{cl } U_i}} \alpha_i.$$

This function is the characteristic function of cl U, if F is the characteristic function of U. It is well defined: it is clear from the left-hand side of the above equation that the number does not depend on the decomposition of the function F, and it is clear from the right-hand side that it does not depend on which small ball is chosen.

Uniqueness is clear. □

Next, we show that cl is, in fact, an isomorphism. We shall make use of χ_c, the Euler characteristic on the lattice L_c, or, rather, of the induced homomorphism $\bar{\chi}_c: S_c \to \mathbb{Z}$.

Theorem 11. *The homomorphism* cl: $S_o \to S_c$ *is an isomorphism.*

Proof. First we verify that cl is monomorphic. Suppose $F \in S_o$ and $F \neq 0$. By Theorem 9 there is an open half-space, H, such that $\bar{\chi}_o(C(H) \cap F) \neq 0$. We may suppose $F=\sum_{i\in[n]}\alpha_i F_i$, where F_i is the characteristic function of the open, convex set U_i (for $i\in[n]$). Let H' be a closed half-space contained in H such that $U_i\cap H'\neq\varnothing$ for each i for which $U_i\cap H\neq\varnothing$. Then

$$0\neq\bar{\chi}_o(C(H)\cap F)=\sum_{i\in[n]}\alpha_i\bar{\chi}_o(C(H)\cap F_i)$$
$$=\sum_{i\in[n]}\alpha_i\bar{\chi}_c(C(H')\cap \mathrm{cl}(F_i))=\bar{\chi}_c(\mathrm{cl}\ F).$$

Therefore $\mathrm{cl}\ F\neq 0$.

It remains to show that cl is surjective. For this it is clear that it suffices to show that the characteristic functions of closed, convex sets are in the image. Let K be such a set. If K is d-dimensional then $C(K)=\mathrm{cl}\ F$, where F is the characteristic function of its interior. We may proceed by induction on the codimension of K. Suppose K is lower-dimensional, and that the characteristic functions of all closed sets of dimension one more are in the image. It is clear that we may write K as the intersection, $K=K_1\cap K_2$, of two closed, convex sets of dimension one more than that of K, and such that $K_1\cup K_2$ is also convex. Since $C(K)=C(K_1)+C(K_2)-C(K_1\cup K_2)$, it follows that $C(K)$ is also in the image. □

Let $\mathrm{cl}^{-1}\colon S_c\to S_o$ denote the inverse of cl. We call the function $\sigma=(-1)^d\ \mathrm{cl}^{-1}$ the *Sallee-Shephard mapping*.

For elements F and G of S_c we define F cl conv $G=\mathrm{cl}(\mathrm{cl}^{-1}(F)\ \mathrm{conv}\ \mathrm{cl}^{-1}(G))$, so that cl conv is a multiplication on S_c. Note that if F and G are d-dimensional, closed, convex sets then this is the closure of the convex hull of F and G. We define $\hat{L}_c$ to be the image of $\hat{L}_o$ under cl, a lattice with operations induced from $\hat{L}_o$. We call it the *polar lattice generated by the closed, convex sets.* If we let $\hat{L}_c'=\hat{L}_c\sim\{C(\varnothing)\}$, then it is clear that $V(\hat{L}_c')\simeq S_c$.

An easy argument (used, already, in the proof of Theorem 11, and to be used again in that of Theorem 13) shows that $\hat{L}_c$ contains the characteristic functions of all closed, convex sets, and that, if K is such a set, then

$$\mathrm{cl}^{-1}(C(K))=(-1)^{d-\dim K}C(\mathrm{relint}\ K).$$

We include another useful theorem.

Theorem 12. *Let F be a nonzero element of S_c. Then there is a closed half-space H such that $\bar{\chi}_c(C(H)\cap F)\neq 0$.*

Proof. Since $F\neq 0$, $G=\mathrm{cl}^{-1}(F)\neq 0$. Then, by Theorem 9, there is an open half-space, H', such that $\bar{\chi}_o(C(H')\cap G)\neq 0$. The result follows, with $H=\mathrm{cl}\ H'$. □

Next we see that F cl conv G is what it should be, whenever F and G are characteristic functions of closed, convex sets.

Theorem 13. *If K_1 and K_2 are closed, convex sets then $C(K_1)$ cl conv $C(K_2) = C(\text{cl conv}(K_1 \cup K_2))$.*

Proof. Suppose not. Choose K_1 and K_2 for which the identity fails, in such a way that $\dim K_1 + \dim K_2$ is as large as possible, and so that $\dim K_1 \le \dim K_2$. Then $\dim K_1 \le d$; we have already observed that the identity holds when both K_1 and K_2 are of dimension d, so $\dim K_1 \le d-1$. We may write $K_1 = W_1 \cap W_2$, where W_1, W_2, and $W_1 \cup W_2$ are closed, convex, and of dimension one more than that of K_1.

Since $C(K_1)$ cl conv $C(K_2) \ne C(\text{cl conv}(K_1 \cup K_2))$ there is $x \in \mathscr{R}^d$ on which these functions differ. Suppose $x \in \text{cl conv}(K_1 \cup K_2)$, so that the right-hand side has value 1. The left-hand side is

$$\begin{aligned} &C(W_1) \text{ cl conv } C(K_2) + C(W_2) \text{ cl conv } C(K_2) \\ &\quad - C(W_1) \text{ cl conv } C(W_2) \text{ cl conv } C(K_2) \\ &= C(\text{cl conv}(W_1 \cup K_2)) + C(\text{cl conv}(W_2 \cup K_2)) \\ &\quad - C(\text{cl conv}(W_1 \cup W_2 \cup K_2)). \end{aligned}$$

Since W_1 and W_2 contain K_1, this expression has value 1 on x.

We may now suppose that $x \notin \text{cl conv}(K_1 \cup K_2)$, so that, at x, the right-hand side has value 0. Then there is a closed half-space H such that $x \in H$ and $H \cap \text{cl conv}(K_1 \cup K_2) = \varnothing$. Since $H \cap W_1 \cap W_2 = \varnothing$ and $\{W_1, W_2\}$ is clustered, it follows that one of these, say W_1, also fails to intersect H nontrivially. It is clear that the left-hand side must also have value 0, a contradiction. □

If $\mathscr{K}$ is a finite, nonempty collection of closed, convex sets, the *closed, transversal characteristic of* $\mathscr{K}$ is the function:

$$\omega_c(\mathscr{K}) = \sum_{\substack{\mathscr{A} \subseteq \mathscr{K}, \\ \mathscr{A} \ne \varnothing}} (-1)^{|\mathscr{A}|-1} C\left(\text{cl conv} \bigcup_{K \in \mathscr{A}} K\right).$$

Clearly, $\omega_c(\mathscr{K}) \in S_c$.

Theorem 14. *Suppose $\mathscr{K}_1$ and $\mathscr{K}_2$ are finite, nonempty collections of closed, convex sets. Then $\tau(\mathscr{K}_1) = \tau(\mathscr{K}_2)$ if and only if $\omega_c(\mathscr{K}_1) = \omega_c(\mathscr{K}_2)$.*

Proof. We observe that if H is a closed half-space, then

$$\bar{\chi}_c(C(H) \cap \omega_c(\mathscr{K})) = \begin{cases} 0 & \text{if } H \notin \tau(\mathscr{K}), \\ 1 & \text{if } H \in \tau(\mathscr{K}). \end{cases}$$

A simple argument shows that $\tau(\mathscr{K}_1)$ and $\tau(\mathscr{K}_2)$ are distinct if and only if there is a closed half-space in one of these sets which is not in the other. □

This theorem enables one to construct an injection of $\hat{L}_c$ into the lattice of complete families of transversals.

We consider next the sublattice, Φ_p, of the lattice of complete families of transversals. If $\mathcal{T} = \tau(\{P_1, \ldots, P_m\}) \in \Phi_p$, where $P_1, \ldots,$ and P_m are convex polytopes, let $\beta(\mathcal{T}) = \omega_c(\{P_1, \ldots, P_m\})$. This is a well-defined injection from Φ_p into $\hat{L}_c$, by Theorem 14. It is clear that the image of β is a sublattice $\hat{L}_p$ of $\hat{L}_c$, and $\hat{L}_p$ is generated by the characteristic functions of the closed, convex polytopes. In fact, $\beta: \Phi_p \to \hat{L}_p$ is a (lattice) dual isomorphism.

Recall that, according to Theorem 2, the restriction of τ to Φ_p is a dual automorphism. For $F \in \hat{L}_p$, let $\bar{\tau}(F) = \beta\tau\beta^{-1}(F)$. This is a dual automorphism of $\hat{L}_p$.

Let S_p denote the group generated by the characteristic functions of the convex polytopes, so that S_p is a subgroup of S_c. Since S_o contains the characteristic function of the relative interior of each convex polytope, and since each nonempty polytope is the disjoint union of the relative interiors of its nonempty faces, it is clear that $S_p \subseteq S_o$, as well.

We use Euler's relation to obtain the connection between $\bar{\chi}_o$ and $\bar{\chi}_c$. If F is the characteristic function of a convex, k-dimensional polytope then $F = \sum_K C(K)$, where the summation extends over all relative interiors of faces of F. Applying $\bar{\chi}_o$ to both sides yields

$$\bar{\chi}_o(F) = \sum_K \bar{\chi}_o(C(K)) = \sum_K (-1)^{d-\dim K}$$
$$= (-1)^d = (-1)^d \bar{\chi}_c(F).$$

Since such functions generate S_p, we have that $\bar{\chi}_o | S_p = (-1)^d \bar{\chi}_c | S_p$.

Lemma 6. *Let* $\mathcal{K} = \{\{v_i\}: 1 \le i \le n\}$. *Then* $\omega(\mathcal{K}) = (-1)^{\dim K} C(\text{relint } K)$, *where* $K = \text{conv}\{v_i : i \in [n]\}$.

Proof. Let H be a closed half-space. Then

$$\bar{\chi}_c(C(H) \cap \omega(\mathcal{K})) = \sum_{\substack{\Lambda \subseteq [n], \\ \Lambda \ne \varnothing}} (-1)^{|\Lambda|-1} \chi_c(C(H) \cap \text{conv}\{v_i : i \in \Lambda\})$$
$$= \sum_{\Lambda} (-1)^{|\Lambda|-1},$$

where this last summation extends over nonempty sets $\Lambda \subseteq [n]$ such that $\Lambda \cap \{i: v_i \in H\} \ne \varnothing$. This is 1, if $\{i: v_i \in H\} = [n]$, and 0, otherwise.

Also we have

$$\bar{\chi}_c(C(H) \cap (-1)^{\dim K} C(\text{relint } K))$$
$$= (-1)^{\dim K} \bar{\chi}_c(C(H \cap \text{relint } K))$$
$$= (-1)^{\dim K} \bar{\chi}_c(C(\text{int } H \cap \text{relint } K)) + \bar{\chi}_c(C(\text{bd } H \cap \text{relint } K)),$$

which is 1, if relint $K \subseteq H$, and 0, otherwise. By Theorem 12, the result follows. □

Theorem 15. *The function* $\bar{\tau}$ *is the restriction of the Sallee-Shephard mapping,* σ, *to* $\hat{L}_p$.

Proof. If $K = \text{conv}\{v_i : i \in [n]\}$, then $\bar{\tau}(C(K)) = (\tau(\{K\})) = \omega(\{\{v_i\}: i \in [n]\}) = \sigma(C(K))$. Since $\bar{\tau}$ is a lattice dual isomorphism, $\bar{\tau}: \hat{L}_p \to \hat{L}_p$, we may view it as a valuation $\bar{\tau}: \hat{L}_p \to S_p$; it induces a group homomorphism, $\bar{\tau}': S_p \to S_p$. We have seen that $\bar{\tau}'$ and σ agree on the characteristic functions of convex polytopes, which generate S_p as a group. □

Corollary. *If* $\mathcal{P}$ *and* $\mathcal{Q}$ *are finite, nonempty collections of convex polytopes, then* $\tau(\mathcal{P}) = \tau(\mathcal{Q})$ *if an only if* $\omega(\mathcal{P}) = \omega(\mathcal{Q})$.

7. Polarity as a Homomorphism

In this section we make the easy observation that the normal cone mapping induces an automorphism of the group S_c generated by characteristic functions of closed cones. Analogous results hold for other polarities, e.g., the polar reciprocal mapping.

For $K \subseteq \mathcal{R}^d$, let

$$\text{norm}(K) = \{y \in \mathcal{R}^d : \langle x, y\rangle \leq 0 \text{ for each } x \in K\}.$$

Theorem 16. *The mapping* $\rho_0: C(K) \to C(\text{norm}(K))$, *for closed, convex sets* K, *has an extension* $\rho: S_c \to S_c$, *which is a homomorphism.*

Proof. For $f = \sum a_i C(K_i) \in S_c$ and $x \in \mathcal{R}^d$, define $\rho(f)(x)$ as follows. Choose $\varepsilon > 0$ sufficiently small that if there is $y \in K_i$ such that $\langle x, y\rangle > 0$, then $K_i \cap H_x \neq \varnothing$, where $H_x = \{y \in \mathcal{R}^d : \langle x, y\rangle \leq -\varepsilon\}$. Then the function defined by

$$\rho(f)(x) = \bar{\chi}(f) - \bar{\chi}(H_x \cap f) = \sum_{K_i \cap H_x = \varnothing} a_i$$

is clearly a well-defined homomorphism, $\rho: S_c \to S_c$; and, since $K_i \cap H_x = \varnothing$ if and only if $x \in \text{norm}(K_i)$ (by our choice of ε), it is indeed the case that $\rho(C(K)) = \rho_0(C(K))$ for closed, convex sets K. □

Consider the subgroup M of S_c generated by the characteristic functions of closed cones. This is isomorphic to the valuation module of the lattice of finite unions of such cones; intersection induces a multiplication such that the resulting ring is isomorphic to the valuation ring of this lattice.

The restriction of ρ to this group is an isomorphism of the ring with intersection as multiplication and the ring with cl conv as multiplication. The image of $\hat{L}_c \cap M$ under this mapping is the set of characteristic functions of finite unions of closed cones; we find that $\hat{L}_c \cap M$ is dually isomorphic, as a lattice, to the lattice of finite unions of closed cones. For a related result, see Theorem 1(a) of [12].

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Received June 12, 1985, and in revised form February 23, 1987.