Finite Coverings by Translates of Centrally Symmetric Convex Domains

G. Fejes Tóth

Mathematical Institute of the Hungarian Academy of Sciences, Reáltonada 13-15,
 1053 Budapest, Hungary

Abstract. Bambah and Rogers proved that the area of a convex domain in the plane which can be covered by n translates of a given centrally symmetric convex domain C is at most $(n-1)h(C) + a(C)$, where $h(C)$ denotes the area of the largest hexagon contained in C and $a(C)$ stands for the area of C . An improvement with a term of magnitude $\sqrt{n}$ is given here. Our estimate implies that if C is not a parallelogram, then any covering of any convex domain by at least 26 translates of C is less economic than the thinnest covering of the whole plane by translates of C .

1. Introduction

By a domain we mean a bounded closed set in the plane with interior points. How large can the area $a(K)$ of a convex domain K , which can be covered by n translates of a given centrally symmetric convex domain C be? Let $h(C)$ denote the maximum area of a hexagon contained in C . We recall a theorem of Bambah and Rogers [1] which claims that

$$a(K) \leq (n-1)h(C) + a(C).$$

In the case when C is a circle and $n \geq 2$ a stronger inequality holds [4]:

$$a(K) < nh(C).$$

These estimates give the asymptotically right order of magnitude of the maximum area of a convex domain which can be covered by n translates of C . However, we shall show that they can be improved by a remainder term of magnitude $\sqrt{n}$.

Let $q(C)$ denote the maximum area of a quadrangle contained in C . Then the following theorem holds:

Theorem. *If a convex domain K is covered by $n \geq 2$ translates of a centrally symmetric convex domain C then*

$$\begin{aligned} a(K) \leq & (n-1)h(C) + a(C) \\ & - \frac{h(C) - q(C)}{2(a(C) + q(C))} \\ & \times \{ \sqrt{(a(C) + h(C))^2 + (a(C) + q(C))[4(n-1)h(C) + 3a(C) - q(C)]} \\ & - a(C) - h(C) \}. \end{aligned} \quad (1)$$

It is known [5], [7] that the least density $\theta(C)$ of a covering of the plane by translates of C is equal to $a(C)/h(C)$. The *finite covering density*

$$\theta_n(C) = \inf \left\{ \frac{na(C)}{a(K)}, K \text{ can be covered by } n \text{ translates of } C \right\}$$

was introduced in [6]. In the case when C is a parallelogram we obviously have $\theta_n(C) = \theta(C) = 1$ for all n . In the following we consider the case when C is not a parallelogram. Now our theorem implies that $\theta_n(C) > \theta(C)$ if n is sufficiently large. Moreover, an easy calculation shows that the right-hand side of (1) is less than $nh(C)$, hence $\theta_n(C) > \theta(C)$, if

$$\begin{aligned} n > & \frac{(a(C) + q(C))(a(C) - h(C))^2}{h(C)(h(C) - q(C))^2} + \frac{(a(C) - h(C))(a(C) + h(C))}{h(C)(h(C) - q(C))} \\ & - \frac{3a(C) - q(C)}{4h(C)} + 1. \end{aligned} \quad (2)$$

According to a theorem of Dowker [3] there is a parallelogram P of area $q(C)$ inscribed into C , the centre of symmetry of which coincides with that of C . We observe that to any side xy of P which is not an arc of ∂C there is a triangle xyz contained in the segment of C cut off by xy with an area at least half of the respective segment. It immediately follows that

$$a(C) - h(C) < \frac{3}{4}(a(C) - q(C)).$$

Therefore

$$\frac{a(C) - h(C)}{h(C) - q(C)} < 3, \quad (3)$$

$$\frac{3a(C) - q(C)}{h(C)} > \frac{10a(C)}{3h(C)} - \frac{4}{3} \geq 2, \quad (4)$$

and

$$\frac{a(C) + q(C)}{h(C)} < 4 \frac{a(C) + q(C)}{a(C) + 3q(C)}.$$

A theorem of Sas [8] states that

$$\frac{q(C)}{a(C)} \geq \frac{2}{\pi} \quad \text{and} \quad \frac{h(C)}{a(C)} \geq \frac{\sqrt{27}}{2\pi}.$$

From this, and the fact that the function $(1+x)/(1+3x)$ is decreasing, it follows that

$$\frac{a(C)+q(C)}{h(C)} \leq 4 \frac{\pi+2}{\pi+6} < 2.25 \quad (5)$$

and

$$\frac{a(C)+h(C)}{h(C)} \leq 1 + \frac{2\pi}{\sqrt{27}} < 2.21 \quad (6)$$

The inequalities (3), (4), (5), and (6) imply that the right-hand side of (2) is less than 26 for all centrally symmetric convex domains C other than a parallelogram. Thus we have the following:

Corollary. *We have for all centrally symmetric convex domains C other than a parallelogram and all integers $n \geq 26$*

$$\theta_n(C) > \theta(C).$$

It may be conjectured that this inequality holds for $n \geq 2$. The result in [3] quoted above confirms this conjecture in the case when C is a circle.

Our theorem implies that the area of a convex domain which is covered by n unit circles is at most

$$\frac{3\sqrt{3}}{2}n - 0.425\sqrt{n} + 0.878.$$

This may be compared with a result of Gritzmann and Wills [6] according to which there is a convex domain of area

$$\frac{3\sqrt{3}}{2}n - 2.755\sqrt{n}$$

which can be covered by n unit circles.

The proof of the Theorem is given in Section 4. The main tool to the proof is a special formula for the area of a convex domain covered by translates of C discussed in Section 3. This, in turn, is based on the construction of generalized Dirichlet cells described in Section 2.

2. Generalized Dirichlet Cells

We use an obvious vector notation, so that a point of the plane and the vector pointing from the origin o to the respective point will be identified. For a real number λ , a point p and a point set A , $\lambda A + p$ will denote the set of points $\lambda x + p$ with x in A . The boundary of a set A will be denoted by $\text{bd } A$. Throughout the paper we assume that

(i) C is a strictly convex domain with centre of symmetry at the origin o .

We consider the *distance function* $f(x)$ of C defined by

$$f(x) = \inf_{\substack{\lambda > 0 \\ x \in \lambda C}} \lambda.$$

The function $f(x)$ is continuous for all x and it satisfies

$$f(x+y) \leq f(x) + f(y) \quad (7)$$

for all points x and y , with strict inequality unless $x = \lambda y$ for some $\lambda \geq 0$ or $y = o$. Further, we have

$$f(\lambda x) = |\lambda| f(x)$$

for all points x and all real numbers λ (see e.g. [9]). The sets $\lambda C + a$ and $\text{bd}(\lambda C + a)$ consist of the points x with $f(x-a) \leq |\lambda|$ and $f(x-a) = |\lambda|$, respectively.

We reformulate Lemma 1 from [2] as:

Proposition 1. *For any two distinct points p and q and any real numbers λ and μ $\text{bd}(\lambda C + p)$ and $\text{bd}(\mu C + q)$ have at most two points in common.*

Hence it immediately follows:

Proposition 2. *If x , y , and z are distinct points then there is at most one point p for which $f(x-p) = f(y-p) = f(z-p)$.*

For, if we had $f(x-p) = f(y-p) = f(z-p) = \lambda$ and $f(x-q) = f(y-q) = f(z-q) = \mu$ for distinct points p and q , then x , y , and z would be three points belonging to $(\text{bd}(\lambda C + p)) \cap (\text{bd}(\mu C + q))$, in contradiction to Proposition 1.

Let p and q be distinct points and let x move on a half-ray with endpoint p . It follows by (7) that the quantity $f(x-p) - f(x-q)$ is then a strictly increasing function of the distance of x to p . Together with the continuity of $f(x)$ this implies:

Proposition 3. *For any two distinct points p and q the set of points x with $f(x-p) = f(x-q)$ is a continuous curve. It separates the sets of points x with $f(x-p) < f(x-q)$ and $f(x-p) > f(x-q)$, which are star shaped with centres p and q , respectively.*

Suppose now that p and q ($p \neq q$) are contained in the convex domain K . Then the midpoint m of the segment pq is contained in K , and we have $f(m-p) = f(m-q)$. We recall Lemma 6 from [2]. It states that the set of points x with $f(x-p) = f(x-q)$ has points on both sides of the line pq arbitrarily far from this line. It follows that there is a point of intersection of $\text{bd } K$ with the curve consisting of the points x with $f(x-p) = f(x-q)$ on both sides of the line pq . Suppose that there are two points on $\text{bd } K$, say x and y , on the same side of the line pq such that $f(x-p) = f(x-q)$ and $f(y-p) = f(y-q)$. Then either the open segments px and qy or the open segments py and qx intersect in a point, say z . Proposition 3 implies in any case that for the point z we have $f(z-p) < f(z-q)$ and simultaneously $f(z-p) > f(z-q)$. Since this is impossible, we obtain:

Proposition 4. *If p and q are distinct points in the convex domain K , then there are exactly two points x and y on $\text{bd } K$ such that $f(x-p) = f(x-q)$ and $f(y-p) = f(y-q)$.*

As above, let K be a convex domain and $a_1, \dots, a_n$ distinct points such that

(ii) $a_i \in K$ for $i = 1, \dots, n$.

Let D_i consist of those points x of K for which $f(x-a_i) \leq f(x-a_j)$ for all $j = 1, \dots, n$. D_i will be called a *generalized Dirichlet cell* associated with a_i . It follows by Proposition 3 and assumption (ii) that the sets D_i , $i = 1, \dots, n$, are closed star-shaped sets which cover K , up to common boundary points exactly once. Let D_0 be the closure of the complement of K . The sets D_i , $i = 0, \dots, n$, constitute the *faces* of a cell complex $\mathcal{C}$. A point of the plane is said to be a *vertex* of $\mathcal{C}$ if it belongs to the boundary of at least three faces of $\mathcal{C}$. An *edge* of $\mathcal{C}$ is a connected component of the intersection of the boundaries of two faces of $\mathcal{C}$ containing more than one point.

Proposition 5. *Suppose that*

(iii) *for any point x in the plane and any real number λ , $\text{bd}(\lambda C + x)$ contains at most three of the points a_i , $i = 1, \dots, n$, and, moreover, if $x \in \text{bd } K$, then $\text{bd}(\lambda C + x)$ contains at most two of these points.*

Then each vertex of the cell complex $\mathcal{C}$ is trihedral and $\mathcal{C}$ has exactly $2n - 2$ vertices.

The first statement of Proposition 5 is obvious. Using the fact that the cell complex $\mathcal{C}$ is trihedral, the second statement follows easily by Euler's formula.

3. Two Lemmas Concerning Covering Convex Domains by Translates of C

In this section we use the notations of the previous one. We assume that the sets C , K and $\{a_i\}_{i=1}^n$ satisfy conditions (i)–(iii). We write $C_i = C + a_i$ and assume that

(iv) $K \subset \bigcup_{i=1}^n C_i$, but no proper subset of $\{C_i\}_{i=1}^n$ covers K .

We consider the cell complex $\mathcal{C}$ introduced in Section 2, and observe that now we have, for $i = 1, \dots, n$, $D_i \subset C_i$. For, if we had a point x which is in D_i but not in C_i , then we would have $f(x - a_j) > 1$ for all $j = 1, \dots, n$. But then x would not be covered by the sets C_i , $i = 1, \dots, n$.

Let E and V denote the set of edges and vertices of $\mathcal{C}$, respectively. With each edge $e \in E$ we associate a set $L(e)$ as follows: if e is an edge common to the cells D_i and D_j , and $i, j \neq 0$, then let $L(e) = C_i \cap C_j$. If, on the other hand, e is an edge common to D_i and D_0 , then let $L(e)$ be the connected component of $C_i \cap D_0$, the boundary of which contains e . For a vertex $v \in V$, in which the edges e_1, e_2 , and e_3 meet, let $I(v)$ be the connected component of $L(e_1) \cap L(e_2) \cap L(e_3)$ which contains v . We write

$$t(v) = a(L(e_1)) + a(L(e_2)) + a(L(e_3)) - 2a(I(v)).$$

Now we are in the position to give the formula for $a(K)$ announced in the Introduction:

Lemma 1. *We have*

$$a(K) = na(C) - \frac{1}{2} \sum_{v \in V} t(v).$$

This formula has been proved in [4] for the case when C is a circle. However, the proof given in [4] is too concise and needs a minor correction. In the following we give a detailed proof in the general case.

We consider a set $C_i = C + a_i$ and the generalized Dirichlet cell D_i associated with a_i . The edges of $\mathcal{C}$ bounding D_i will be called the *sides* of D_i . Let $v_1, \dots, v_r$ be the vertices of $\mathcal{C}$ on $\text{bd } D_i$ enumerated in their natural cyclic order when going on $\text{bd } D_i$ in the clockwise direction. We write $v_{r+1} = v_1$. The side of D_i joining v_m to v_{m+1} ($m = 1, \dots, r$) will be denoted by s_m . We associate with s_m an arc s_m^* , called the *chord* of C_i , determined by s_m as follows: if s_m is an arc of $\text{bd } K$ then let s_m^* be the connected component of $C_i \cap \text{bd } K$ containing s_m . If, on the other hand, D_i abuts along s_m on a cell D_h of $\mathcal{C}$ other than D_0 , then s_m^* is defined as the set of points x in C_i for which $f(x - a_i) = f(x - a_h)$. The fact that the set s_m^* is a Jordan arc is obvious in the first case, and it follows immediately also in the second case by Propositions 3 and 4. Moreover, $D_i \subset C_i$ implies that $s_m \subset s_m^*$. We observe that in the case when s_m is an edge of $\mathcal{C}$ common to D_i and $D_h \neq D_0$, s_m^* is contained in $C_i \cap C_h$ and its endpoints are just the points of intersection of $\text{bd } C_i$ and $\text{bd } C_h$. The chord s_m^* cuts the "segment" S_m off from C_i . S_m consists of the points x of C_i for which $f(x - a_i) \geq f(x - a_h)$, if, along s_m , the cell $D_h \neq D_0$ joins to D_i , and $S_m = L(s_m)$ otherwise.

Let p_m and q_m be the endpoints of s_m^* , and suppose that the notations are chosen so that traveling on s_m^* from p_m to q_m we first meet v_m and then v_{m+1} . Let s_m^+ and s_m^- be the open arcs of s_m^* from v_m through v_{m+1} to q_m and from p_m through v_m to v_{m+1} , respectively. Then we have:

Proposition 6. *For every pair j, k ($1 \leq j \neq k \leq n$), the arcs s_j^- and s_k^+ , as well as the arcs s_j^+ and s_k^- , intersect at, at most, one point. Furthermore, we have*

$$s_j^- \cap s_k^- = \emptyset \quad \text{and} \quad s_j^+ \cap s_k^+ = \emptyset. \quad (8)$$

Consider first the case when s_j^* and s_k^* intersect at a single point, say x . Then we have to verify only the relations (8). Now the cyclic order of the points p_j , p_k , q_j , and q_k on $\text{bd } C_i$ is of the type p, p, q, q . Without loss of generality we assume that we meet these points in the order p_j , p_k , q_j , and q_k when going on $\text{bd } C_i$ in the clockwise direction. Then v_j and v_{j+1} are on the arc of s_j^* between p_j and x , while v_k and v_{k+1} are on the arc of s_k^* between x and q_k . This implies (8).

Now suppose that s_j^* and s_k^* have more than one point in common. By Proposition 2 then at least one of s_j and s_k , say s_j , is an arc of $\text{bd } K$. If s_k abuts on a cell D_h other than D_0 , then s_j and s_k intersect at exactly two points by Proposition 4. Moreover, one of the points of intersection, say x , lies on the closed arc of s_k^* from p_k to v_k , and the other one, say y , lies between v_{k+1} and q_k . If v_j and v_{j+1} are situated on s_j^* between x and y , then the order of points on s_j^* is p_j, y, v_j, v_{j+1}, x , and q_j , and it is easy to check that the statements of Proposition 6 are true. Otherwise the subset of those points z of K for which $f(z - a_i) \geq f(z - a_h)$ is enclosed by the arcs of s_j^* and s_k^* between x and y . Therefore this subset of K is contained in S_k , and consequently also in C_i . On the other hand, D_h is contained in this subset of K , so that we have $D_h \subset C_i$. But this means that C_h can be omitted from the sets C_i , $i = 1, \dots, n$, and we still obtain a covering of K , which is impossible by assumption (iv).

It remains to consider the case when, as well as s_j , s_k is also an arc of $\text{bd } K$. Then the supposition that $s_j^* \cap s_k^*$ contains more than one point implies that s_j^* and s_k^* are identical. Since, however, $s_j \neq s_k$, there is a side of D_i between s_j and s_k , say s_l , such that D_i joins along s_l to a cell of $\mathcal{C}$ other than D_0 . Moreover, s_l^* intersects s_j^* in two points, say x and y , so that v_j and v_{j+1} are not contained in the arc of s_j^* between x and y . However, this is just the situation from which we have seen above that it contradicts assumption (iv). This completes the proof of Proposition 6.

We have

$$a(D_i) = a(C) - a\left(\bigcup_{m=1}^r S_m\right). \quad (9)$$

We write $R = \{1, \dots, r\}$. Then a well-known formula states that

$$a\left(\bigcup_{m=1}^r S_m\right) = \sum_{m=1}^r (-1)^{m-1} \sum_{\substack{M \subset R \\ |M|=m}} a\left(\bigcap_{j \in M} S_j\right). \quad (10)$$

Here $|M|$ denotes the cardinality of M .

Let us suppose that for some j and k , $S_j \cap S_k \neq \emptyset$. Then Proposition 6 implies that $S_j \cap S_k$ is either connected or it is composed of two connected components, one containing the point $s_j^+ \cap s_k^-$ and the other containing $s_j^- \cap s_k^+$. Let S be a connected component of $S_j \cap S_k$. Without loss of generality we suppose that $x = s_j^+ \cap s_k^- \in S$. Let s_l be a side of D_i which lies between s_j and s_k when we go around $\text{bd } D_i$ in the clockwise direction. Again, Proposition 6 implies that p_l lies between p_j and p_k , and q_l lies between q_j and q_k when going on $\text{bd } C_i$ in the

clockwise direction. Furthermore, when we go from p_l to q_l on s_l^* we meet s_j^* in a point between p_j and x , and then we meet s_k^* in a point between x and q_k . This implies that

$$S \subset S_l.$$

If there are exactly t ($0 \leq t \leq r-2$) sides of D_i between s_j and s_k , then there are $\binom{t}{m-2}$ possibilities of obtaining S as a connected component of the intersection of m segments of C_i . This yields a contribution to the sum on the right-hand side of (10) equal to

$$a(S) \sum_{m=1}^{t+2} (-1)^{m-1} \binom{t}{m-2}.$$

This sum is equal to $-a(S)$ if $t=0$, i.e., if $\overline{s_j}$ and s_k are adjacent sides of D_i , and it is equal to 0 otherwise. Thus, writing $S_0 = S_r$ and denoting by J_m the connected component of $S_{m-1} \cap S_m$ containing the vertex v_m ($m = 1, \dots, r$), we have

$$a\left(\bigcup_{m=1}^r S_m\right) = \sum_{m=1}^r [a(S_m) - a(J_m)].$$

Hence we obtain by (9)

$$a(D_i) = a(C) - \sum_{m=1}^r [a(S_m) - a(J_m)].$$

Summing up this equality for $i = 1, \dots, n$ we obtain

$$a(K) = \sum_{i=1}^n a(D_i) = na(C) - \sum_{e \in E} a(L(e)) + \sum_{v \in V} a(I(v)).$$

Now Lemma 1 follows by observing that

$$\sum_{e \in E} a(L(e)) - \sum_{v \in V} a(I(v)) = \frac{1}{2} \sum_{v \in V} t(V).$$

Lemma 2. *Let b denote the number of vertices of $\mathcal{C}$ which are situated on $\text{bd } K$. Then we have*

$$a(K) \leq (n-1)h(C) + a(C) - \frac{b}{2}(h(C) - q(C)). \quad (11)$$

Fig. 1

Let v be a vertex of $\mathcal{C}$ common to the cells D_i , D_j , and D_k , and assume that $i, j, k \neq 0$. Now $t(v)$ is nothing else but the area of the part of the plane which is covered at least twice by C_i , C_j , and C_k . Looking for a lower bound for $t(v)$, we may suppose that the intersection $C_i \cap C_j \cap C_k$ consists of the single point v . Then $\text{bd } C_i$ and $\text{bd } C_j$, $\text{bd } C_j$ and $\text{bd } C_k$, and $\text{bd } C_k$ and $\text{bd } C_i$ intersect at the points $x = a_i + a_j - v$, $y = a_j + a_k - v$, and $z = a_k + a_i - v$, respectively (Fig. 1). Let x^* , z^* , and v^* be the points obtained from x , z , and v by reflecting in a_i . Then we have $x^* = a_i - a_j + v$, $z^* = a_i - a_k + v$, and $v^* = 2a_i - v$. Now v , x , z^* , v^* , x^* , and z are the vertices of a centrally symmetric convex hexagon H inscribed into C_i . The sides xz^* and x^*z of H are parallel to the segment vy . This implies that $t(v) = a(C) - a(H)$. Hence we obtain that

$$t(v) \geq a(C) - h(C) \quad \text{if } v \notin \text{bd } K. \quad (12)$$

Now let v be a vertex of $\mathcal{C}$ common to D_0 , D_i , and D_j . Similarly as above, we assume that v is the only point common to D_0 , D_i , and D_j . Let S be the closure of the complement of a supporting half-plane of K such that $v \in \text{bd } S$ (Fig. 2). Let $t^*(v)$ be the area of the part of the plane which is covered at least twice by the sets C_i , C_j , and S . Then we have $t(v) \geq t^*(v)$. Let the point of intersection of $\text{bd } S$ with $\text{bd } C_i$ other than v be p , and that with $\text{bd } C_j$ be q . The area $t^*(v)$ will be minimal in the case when the segments pv and vq are of equal length. Otherwise we can decrease $t^*(v)$ by a small rotation of S about v in a suitable direction. Thus, let us suppose that $pv = vq$. Let the second point of intersection of $\text{bd } C_i$ and $\text{bd } C_j$ be w . Let the line through w parallel to $\text{bd } S$ intersect $\text{bd } C_i$

Fig. 2

in another point z . Then $P = vwzp$ is a parallelogram inscribed into C_i , and we have $t^*(v) = a(C) - a(P)$. It follows that

$$t(v) \geq a(C) - q(C) \quad \text{if } v \in \text{bd } K. \quad (13)$$

Lemma 2 is now an immediate consequence of Proposition 5, Lemma 1, and the relations (12) and (13).

4. Proof of the Theorem

First we show that it suffices to prove the theorem in the case when C and its translates $C_i = C + a_i$, $i = 1, \dots, n$, covering K satisfy conditions (i)–(iv).

Having proved the theorem in the case when C is strictly convex, the general case can be settled by approximating C by strictly convex domains containing it. It is also clear that the assumption in (iv), that no proper subset of $\{C_i\}_{i=1}^n$ covers K , does not affect the generality.

Suppose now that assumptions (i) and (iv) are satisfied, and for some i , $1 \leq i \leq n$, $a_i \notin K$. Then $K \cap C_i \neq \emptyset$. A standard continuity argument yields that there is a supporting line l of K such that a_i and K are separated by l and the midpoint m of the line segment $l \cap C_i$ is on $\text{bd } K$. Denoting the endpoints of $l \cap C_i$ by p and q , we have $2a_i - p \in \text{bd } K$ and $2a_i - q \in \text{bd } K$. The set $C_i \cap K$ is then contained in the parallel strip bounded by the lines through p and $2a_i - q$ and q and $2a_i - p$, respectively. It follows that $C_i \cap K$ is contained in the set $C'_i = C + m$. Thus K is covered by the sets $C_1, \dots, C'_i, \dots, C_n$. This justifies assumption (ii).

Finally, if (iii) is not satisfied then for any positive number ε we can replace the set $\{a_i\}_{i=1}^n$ by another one satisfying (iii) so that $K \subset \bigcup_{i=1}^n ((1 + \varepsilon)C + a_i)$. This shows that (iii) can be assumed without loss of generality.

We observe that if $n = 2$, then the cell complex $\mathcal{C}$ has two vertices both of them lying on $\text{bd } K$. Thus Lemma 2 implies that if K is covered by two translates of C , then

$$a(K) \leq a(C) + q(C). \quad (14)$$

An easy calculation shows that if for some positive numbers λ and μ and some points p and q the intersection $(\lambda C + p) \cap (\mu C + q)$ is not empty, then

$$(\lambda C + p) \cup (\mu C + q) \subset (\lambda + \mu)C + \frac{\lambda p + \mu q}{\lambda + \mu}.$$

Let $e_1, \dots, e_b$ be the edges of $\mathcal{C}$ which are arcs of $\text{bd } K$ enumerated in their natural cyclic order. Let i_j be the index for which $e_j \subset D_0 \cap D_{i_j}$ ($j = 1, \dots, b$). Then we have

$$\text{bd } K \subset \bigcup_{j=1}^b C_{i_j} \quad \text{and} \quad C_{i_j} \cap C_{i_{j+1}} \neq \emptyset \quad \text{for } j = 1, \dots, b-1.$$

Writing $r = [(b+1)/2]$ and $s = [(b+2)/2]$, we obtain by induction that

$$\text{bd } K \subset \left(rC + \frac{1}{r} \sum_{j=1}^r a_{i_j} \right) \cup \left(rC + \frac{1}{r} \sum_{j=s}^b a_{i_j} \right).$$

We note that if two convex domains cover $\text{bd } K$, then they cover K as well. Thus, K can be covered by two translates of $((b+1)/2)C$. Hence we obtain by (14) that

$$a(K) \leq \left(\frac{b+1}{2} \right)^2 (a(C) + q(C)). \quad (15)$$

It is easy to check that the right hand-sides of (11) and (15) are equal if

$$b = b_0 = \frac{1}{a(C) + q(C)} \times \{ \sqrt{(a(C) + h(C))^2 + (a(C) + q(C))[4(n-1)h(C) + 3a(C) - q(C)]} - a(C) - h(C) \}.$$

For $b < b_0$ (15), for $b > b_0$ (11), is stronger than (1). This completes the proof of the theorem.

References

1. R. P. Bambah and C. A. Rogers, Covering the plane with convex sets, *J. London Math. Soc.* **27** (1952), 304-314.
2. R. P. Bambah, C. A. Rogers, and H. Zassenhaus, On coverings with convex domains, *Acta Arith.* **9** (1964), 191-207.
3. C. H. Dowker, On minimum circumscribed polygons, *Bull. Amer. Math. Soc.* **50** (1944), 120-122.
4. L. Fejes Tóth, Über die dichteste Kreislagerung und dünnste Kreisüberdeckung, *Comment. Math. Helv.* **23** (1949), 342-349.
5. L. Fejes Tóth, Some packing and covering theorems, *Acta Sci. Math. (Szeged)* **12/A** (1950), 62-67.
6. P. Gritzmann and J. M. Wills, On two finite covering problems of Bambah, Rogers, Woods, and Zassenhaus, *Monatsh. Math.* **99** (1985), 279-296.
7. C. A. Rogers, The closest packing of convex two-dimensional domains, *Acta Math.* **86** (1951), 309-321.
8. E. Sas, Über eine Extremaleigenschaft der Ellipsen, *Compositio Math.* **6** (1939), 468-470.
9. F. A. Valentine, *Convex Sets*, McGraw-Hill, New York, 1964.

Received September 20, 1985, and in revised form March 19, 1986.