σ -Morphic Sets of Prototiles

Peter Schmitt

Institut für Mathematik, Universität Wien, A-1090 Wien, Austria

Abstract. Using a general construction method, sets of prototiles (with more than one element) are obtained which admit a countable infinity of distinct tilings. In contrast to the tilings described in a previous paper, in this case almost all the tilings are periodic. In particular, sets of two, three, four, and five prototiles are described.

0. Introduction

A closed subset of the plane that is homeomorphic to a circular disk is called a *tile*. A *tiling* is a family of tiles that cover the plane and have pairwise disjoint interiors. A set of *prototiles* is a finite set of tiles. It *admits* a tiling if there exists a tiling such that every tile is congruent to one of the prototiles. If it admits precisely r distinct tilings (that is, tilings that are not congruent) then it is called *r-morphic*. A set of prototiles that admits a countable infinity of tilings is called *σ -morphic*. The subject of *r-morphic* prototiles was first discussed by Grünbaum and Shephard [1], and in [2] they asked whether a *single* *σ -morphic* prototile exists (Problem 5, p. 180). The answer to this question is still unknown, but the existence of *σ -morphic pairs* of prototiles was established in [3].

A tiling is called *periodic* if there exist two nonparallel translations that are symmetries of the tiling. Of the tilings admitted by the *σ -morphic pairs* constructed in [3] only *one* is periodic. It is the purpose of this note to show how *σ -morphic* sets of prototiles can be constructed such that *almost all* (that is, all but a finite number) of the tilings admitted are periodic.

The construction is based on the following idea. If a prototile T admits a periodic tiling of the plane then every prototile T' which is *similar* to T obviously admits a periodic tiling which is similar but not congruent to the tiling admitted by T (see Fig. 1). If the prototile T is cut into several smaller prototiles $T_1, \dots, T_n$ such that copies of $T_1, \dots, T_n$ can be reassembled to form arbitrarily large patches which are similar to T but do not contain patches congruent to T , then these patches can be used to obtain a countable infinity of periodic tilings which are

Fig. 1. (a) Similar tiles of different sizes, (b) and (c) Two similar, but not congruent, tilings.

distinct and admitted by $T_1, \dots, T_n$. If, furthermore, the dissection of T can be done in such a way that $T_1, \dots, T_n$ admit at most finitely many additional tilings of the plane then $T_1, \dots, T_n$ form a σ -morphic set with the desired property.

It will be shown in the sequel that this indeed is possible.

1. A Special Case

Before dealing with the general case we shall describe a special set of five prototiles, as shown in Fig. 2. From it the main points of the construction can

Fig. 2. A σ -morphic set of five prototiles.

be seen more easily. The five prototiles are built up from congruent squares to which small projections (*keys*) are added and from which corresponding indentations (*keyholes*) are deleted. These keys and keyholes restrict the ways in which the tiles fit together. Their exact shape is not important as long as they possess no symmetries. Five keys and five keyholes are used. Their distribution is shown in Fig. 2. A typical example of a tiling admitted by these prototiles is shown in Fig. 3. Figure 3(a) indicates how the keys and keyholes are replaced by the letters d, h, v, s, t (keys) and $\bar{d}, \bar{h}, \bar{v}, \bar{s}, \bar{t}$ (keyholes), respectively. In any tiling d must fit against $\bar{d}$, and so on. In the tiling shown in Fig. 3(b) a patch shaped (almost) like a rectangle is emphasized: four copies of the cross-shaped tile C at the corners, the two horizontal edges built up from copies of the square-tile H , the vertical edges built up from copies of V , a diagonal built up from copies of the rectangle D , and the remaining area filled by copies of the square S . It is obvious that this patch can be extended to a periodic tiling of the plane which we label $\mathcal{F}(4)$ since in each horizontal edge four copies of H occur. Furthermore, it is easily verified that, for each positive integer n , a similar tiling $\mathcal{F}(n)$ exists.

In Fig. 4 $\mathcal{F}(1)$, $\mathcal{F}(2)$, $\mathcal{F}(3)$, and $\mathcal{F}(7)$, and five additional tilings—distinct from the $\mathcal{F}(n)$ —are shown: the tiling $\mathcal{F}(C)$ which uses only one cross C , the tilings $\mathcal{F}(D)$, $\mathcal{F}(H)$, and $\mathcal{F}(V)$ containing a single diagonal, or horizontal line, or vertical line, respectively, and finally $\mathcal{F}(S)$ consisting of copies of S only.

We claim that no further tiling is admitted by C, H, V, D , and S , and therefore these five prototiles together form a σ -morphic set. Note that $\mathcal{F}(n)$ and $\mathcal{F}(S)$ are periodic whereas $\mathcal{F}(D)$, $\mathcal{F}(H)$, $\mathcal{F}(V)$, and $\mathcal{F}(C)$ are not.

To justify our claim we consider an arbitrary tiling of the plane. If it contains a cross C then to the right of its right arm only another copy of C (in precisely one position) or a copy of H (in precisely one position) fits. In the latter case, to the right of the copy of H , again only a copy of C or a copy of H fits, and so on. A similar argument holds for the left arm, and—with copies of V instead of H —for the upper and the lower arm, and—with copies of D —for the upper part of the right arm and for the lower part of the left arm, as well. Therefore, if no further cross occurs, six straight lines extend from C , as indicated in Fig. 5(a). Since there is only one way in which the area between these lines can be filled (using copies of S), in this case the tiling is $\mathcal{F}(C)$.

If, on the other hand, a second copy C' of the cross occurs on one of these lines, as shown in Fig. 5(b) and (c) then, naturally, from this cross six lines extend, as well. Where such a line crosses another line only a cross fits. Therefore, C and C' force the position of two further crosses which, together with C and C' , form either a parallelogram (as shown in Fig. 5(b)) or a square (as shown in Fig. 5(c)). By these crosses the position of further crosses is determined, and so on. We see that by C and C' the position of crosses on a square lattice, together with the connecting edges and diagonals, is forced. The gaps can be filled using copies of S in a unique way. If a copy of C, D, H , or V occurs outside the lattice then a straight line would extend from it which necessarily cuts a line of the lattice thereby forcing the position of a cross. An argument similar to the preceding one would show that the position of crosses on a finer lattice is forced. Therefore, the tiling is congruent to $\mathcal{F}(n)$ for some n .

Fig. 3. A typical tiling admitted by the prototiles of Fig. 2: (a) the distribution of keys and keyholes is indicated by letters; (b) these letters are used to show how the tiles fit together; and (c) the same tiling—but the proportions of the prototiles have been altered (keys are omitted).

Fig. 4. The tilings admitted by the prototiles of Fig. 2: (a) $\mathcal{T}(S)$, (b) $\mathcal{T}(D)$, (c) $\mathcal{T}(H)$, (d) $\mathcal{T}(V)$, (e) $\mathcal{T}(C)$, and three of the tilings $\mathcal{T}(n)$: (f) $\mathcal{T}(1)$, (g) $\mathcal{T}(2)$, (h) $\mathcal{T}(3)$, and (i) $\mathcal{T}(7)$. Keys are ignored, edges are emphasized.

Fig. 5. A single cross determines six straight lines (a), two crosses (C and C') determine the position of two other crosses (C_1 and C_2) ((b) and (c)).

If the tiling contains no copy of the cross C and is not congruent to $\mathcal{J}(S)$, then it necessarily contains a copy of D , H , or V . From this tile a straight line of copies of the same tile extends in two directions. The half-plane on both sides of this line can be tiled in a unique way, using only copies of S . No further copy of D , H , or V can occur: it would determine another line which—if it met the first line—would either force the occurrence of a cross or lead to a contradiction (if the lines met under an angle not allowed by the cross), or which—if it were parallel to the first line—would force the occurrence of a third, nonparallel line which separates them or, again, lead to a contradiction (see Fig. 6). We conclude that the tiling contains a single line built up from copies of either D , H , or V , and is therefore congruent to $\mathcal{J}(D)$, $\mathcal{J}(H)$, or $\mathcal{J}(V)$. This completes the proof of our assertion.

2. The Construction in the General Case

The basis of the construction is the observation that any triangle can be tiled using (at most) six prototiles in such a way that—using the same prototiles—similar triangular patches of arbitrary size can be tiled. The examples given in Fig. 7 show how this can be achieved. The tiles are modified by small projections and indentations as indicated in the figure. These keys and keyholes are necessary to prevent unwanted tilings. Note that any desired number of distinct keys can be provided, for instance, by using different distributions of small triangles, as shown in Fig. 8(a). Keys possessing a symmetry (as shown in Fig. 8(b)) or keys which are the mirror image of some other key should be avoided because it might be difficult to check whether some unexpected tiling using the tiles and their mirror-images were possible.

In each of the examples shown in Fig. 7 three tiles are used for the sides of the triangle: one (say B) for the base, one each (say S_1 and S_2) for the left and for the right side, respectively. Two tiles (say C_1 and C_2) represent the two bottom corners. The sixth tile (say A) is used to fill the interior of the triangle. If a copy of S_1 is fitted against a copy of S_2 (this can be done in precisely one way) then

Fig. 6. Two parallel lines cannot occur: (a) only direct copies of the tiles (either H or V) are used—keys would have to fit against keys which is impossible, therefore the lines have to be separated by a diagonal; (b) mirror images of the tiles are used in one line—again the patches cannot fit together; and (c) two parallel diagonals are shown—again this is not possible—keyholes would have to fit against keyholes in this case.

Fig. 7. Dissections of a triangle which can be used in the construction. Keys are indicated by small strokes. The dissection shown in (g) is not compatible with the other dissections.

this patch represents the top corner of the triangle—we shall therefore denote it by T . Of course, instead of T a seventh prototile could have been used.

Forced by the matching conditions represented by the keys and keyholes, two straight lines of copies of S_1 , S_2 , or B extend from each corner until they are terminated by another corner. Likewise, from each copy of S_1 , S_2 , or B a straight line extends in two directions until terminated by corners. An analysis similar to that in the preceding section shows that only the following tilings are possible

Fig. 8. Some examples of keys which possess no symmetry (a) or are symmetric (b).

(we assume that the sides of the tiles forming the boundary of the triangle—indicated by the dotted lines in Fig. 7—cannot be fitted against any side of any tile): a countable infinity of triangular patches using all the prototiles; three tilings of angular sectors, determined by one copy of C_1 , C_2 , or T , respectively, in which one side-tile and two of the corners do not occur; three tilings of a half-plane consisting of one side-tile B , S_1 , or S_2 , from which the boundary is built up, and copies of A ; and finally a periodic tiling of the plane by copies of the single prototile A .

We now consider an edge-to-edge tiling of the plane by triangles. Each triangle can be dissected as described. We further assume that this dissection is done in such a way that the dissections are compatible with each other in the sense that, if a copy of the corresponding side-tile is added to each side, a tiling similar to the original one is obtained. The dissections are compatible if, on each side, the same number of edge-tiles is used, and if the ratio of the length of the side-tile and the length of the side is the same in each case. For example, from the seven dissections shown in Fig. 7 six of them (Fig. 7(a)–(f)) are compatible with each other, but the last (Fig. 7(g)) is not compatible with them. We further assume that along the sides of the triangles (the dotted lines in Fig. 7) the tiles are furnished with keys and keyholes which prevent them from being fitted against any other tile, and which therefore guarantee that any triangular patch can be extended to a tiling of the plane in a unique way: along each side the neighbor and its size is determined by the fact that the two corners can only be fitted against the corresponding corners. Further triangles are uniquely determined by these three triangles and so on.

Identical dissections can be used for triangles of the underlying tiling which belong to the same transitivity class. Therefore, if there are k transitivity classes then at most $6k$ prototiles are necessary. In many cases the number of prototiles can be further reduced by fusing together contiguous corners or side-tiles. The tilings admitted by these prototiles can all be obtained by combining patches corresponding to the k distinct dissections: a countable infinity of tilings similar to the original one; a finite number of tilings determined by a corner and obtained

Fig. 9. The dissection of a quadrangle into thirteen distinct prototiles is shown (b). Again keys are ignored. Only one corner-tile is necessary, since in the corresponding tiling (a) the position of tiles is the same at each corner.

by combining angular sectors; a finite number of tilings containing a single infinite side obtained by combining two half-planes; and k distinct tilings using only one tile.

Now we consider a more special case. We start from a tiling of the plane by an n -sided polygon such that the symmetry group acts transitively on the tiles. We choose a point in the interior of the tile (the *center*) and connect it (by *spokes*) with the n corners. By this we obtain a tiling of the plane by triangles with at most n transitivity classes. When dissecting these triangles the sides of the polygon can be chosen as the bases. An example of this construction is shown in Fig. 9 where a tiling by a quadrangle is used. By fusing together some of the tiles the number of prototiles can—in this case—be reduced from 24 to 13. Figure 10 shows how a single corner, a single center, an edge together with two corners,

Fig. 10. (a) Two corners (C, C') determine the position of a center. (b) A corner (C) and a center (S) determines the position of two corners. (c) and (d) The lines radiating from a single corner or center, respectively.

Fig. 11. Four prototiles and a typical tiling. Letters are used to indicate keys and keyholes, and how they fit together. Key *d* has to be chosen centrally symmetric.

Fig. 12. A corner (*C*) and a center (*S*) determine a hexagonal net (a). A single center, respectively a single corner (b) and (c). Two spokes meet in a center (d), and two parallel lines force the occurrence of a third which cuts both lines (e). (f) The hexagonal net of the tilings in Fig. 13—arrows indicate the orientation of the sides.

or a spoke together with a corner and a center determine the tiling. If additional symmetries exist the number of prototiles can be further reduced. An example is shown in Fig. 11: the tiling is based on the tiling by a regular hexagon and, since identical dissections have been chosen for the six congruent triangles, four prototiles suffice. Figure 12 shows how the triangular net arises in various cases.

3. Examples with Three and Two Prototiles

In the last example four prototiles were used: a tile A to fill the triangles, a corner-tile C , and two tiles B and S for the sides. If one takes advantage of the additional symmetries of the equilateral triangle one may use only one tile S for the sides. But then it is necessary to use the same corner-tile C in each of the three corners. Therefore the tile used to fill the triangle must possess the same symmetry properties as the triangle. If this were not the case the copies of A could be oriented in three distinct directions independently in each triangle, and the number of distinct tilings would become uncountable. Therefore the tile A must be a regular hexagon. Two examples of prototiles based on this idea are shown in Figs. 13 and 14: the three prototiles used in Fig. 13 are mirror-symmetric and the corresponding tilings use both the dissected triangle and its mirror-image; the prototiles of Fig. 14 possess rotational symmetry and only direct copies of the dissected triangle are used in the tilings.

The proof that the examples given are indeed σ -morphic is analogous to that in the previous cases: if only copies of A are used then only one tiling is possible. If copies of S but no corners occur, then the tiling necessarily contains an infinite line, and the two half-planes are tiled using A in a unique way—no additional copy of S can occur since the line determined by it would meet the other line and force the occurrence of a corner, and no parallel line can occur (see Fig. 15, it is the necessity of this property which seems to prohibit the use of a single triangle A in the dissections). A unique tiling is determined by the occurrence of a single corner, and if two corners occur, then the tiling is congruent to one of the tilings consisting of triangular patches. Figure 12 may be consulted in this case as well. Figure 12(f) indicates how sides and corners are oriented in the tiling of Fig. 14.

A σ -morphic pair of prototiles is shown in Fig. 16. The number of prototiles could be reduced from three to two by replacing the corner-tile by a patch of tiles built up from the side-tile S and the hexagon A . The tilings admitted by this pair are shown in Fig. 17. We shall prove that no other tiling is possible, and therefore the pair is indeed σ -morphic.

We begin with the remark that if no copy of S is used only one tiling (shown in Fig. 17(a)) is possible. We next investigate how two copies of S can be fitted together: the prototile S has eight sides. The single key is added to four of these sides and the corresponding keyhole is deleted from the other four sides. Therefore there are $4 \cdot 4 = 16$ possibilities of how two copies can touch. These positions are shown in Fig. 18. Seven of them (Fig. 18(b), (c), (e), (k), (n), (o), and (p)) cannot occur in any tiling since the tiles either overlap or leave a gap (an acute angle) which cannot be filled using S or A . Four more (Fig. 18(a), (f), (h), and (i)) leave a right angle which can only be filled using S in precisely one way, as shown in Fig. 19, and a fifth (Fig. 18(l)) possesses two sides against which S and A can be fitted in only four positions, as shown in Fig. 20—in both cases the tiles necessarily overlap or leave a gap which cannot be filled, so that these positions, too, cannot occur in any tiling. Therefore only three positions (Fig. 18(d), (g) = (m), and (j)) turn out to be possible.

Fig. 13. σ -Morphic sets of three prototiles, and the corresponding tilings. The prototiles are shown on top of the figure. Keys are indicated by + and -, respectively.

Fig. 14. σ -Morphic sets of three prototiles, and the corresponding tilings. The prototiles are shown on top of the figure. Keys are indicated by + and -, respectively.

Fig. 15. If two lines occur in a tiling then the patches they belong to fit together as in (a)–(d) and (g), but do not fit as in (e), (f), and (h) since keys would have to fit against keys. Only some keys are indicated.

Fig. 16. A σ -morphic pair of prototiles.

Fig. 17. The tilings admitted by the σ -morphic pair in Fig. 16.

Fig. 18. The 16 possibilities how two sides of S can fit together. Only three ((d), (j), (g) = (m)) can occur in a tiling. Keys and keyholes are indicated by $+$ and $-$, respectively. Their distribution is shown on top of the figure.

Fig. 19. The positions 18(a), (f), (h), and (i) cannot occur in a tiling.

Fig. 20. The position 18(l) cannot occur in a tiling.

Fig. 21. Three pairs of contiguous tiles which determine the same patch of four tiles.

A hexagon can be placed against a side-tile in seven distinct positions. The position of A_1 in Fig. 21(a), or equivalently that of A_2 , excludes that of A_3 , and therefore forces the position of a copy of S , as shown in Fig. 21(d). Similarly, the position of A_4 , or that of A_5 , prohibits that of A_6 or A_7 , respectively, as shown in Fig. 21(b). Again, the position of a copy of S is forced. Finally, if against two copies of S a further copy of S is placed, as S_1 or S_2 in Fig. 21(c), then one of the positions arises which cannot occur in any tiling, as seen in Fig. 19. We conclude that in the patch shown in Fig. 21(d) any two contiguous tiles determine the position of the other two.

The position of A_3 excludes various positions of copies of A and S , as can be seen from Fig. 22: against the top side, or similarly, against the bottom side, either another copy of A (as in Fig. 22(a)) or one of four sides of a copy of S can be fitted. Three positions, shown in Fig. 22(b), (c), and (d), cannot occur in a tiling. The fourth is shown in Fig. 22(h)—in this patch the two copies of S are in the position shown in Fig. 18(d) or (j). They, in turn, force the position of the hexagon, since no copy of S can be used instead, as can be seen from Fig. 22(e) and (f), or Fig. 22(e) and (g), respectively (only copies of S in the position shown in Fig. 18(m) need to be checked since the other positions have

Fig. 22. Three pairs of contiguous tiles which determine the same patch.

Fig. 23. Three pairs of contiguous tiles which determine the same patch.

been recognized as impossible). Therefore, the position of two contiguous tiles of the patch shown in Fig. 22(h) determines the position of the other two. Quite similarly we see from Fig. 23 that a hexagon in the position of A_6 or A_7 , as well as two copies of S in the position shown in Fig. 18(d) or (j), determine the patch shown in Fig. 23(h): Fig. 23(a)–(d) shows the four possibilities of how a side of a copy of S can be fitted against the bottom side of a hexagon (as A_6), and against the right angle on top only a copy of S , as in Fig. 23(h), can be fitted; two copies of S in the position shown in Fig. 18(d) force the presence of a hexagon, as can be seen from Fig. 23(e) and (f). The case of a hexagon in position A_7 is treated analogously.

The two patches shown in Figs. 22(h) and 23(h) contain congruent configurations. Therefore even larger patches are uniquely determined, as indicated in Fig. 24(a) and (b), until the patch shown in Fig. 24(c) is completed: any two

Fig. 24. The patches of Figs. 22 and 23 both determine the corner-patch shown in (c).

Fig. 25. Three copies of S , as shown in (a) or (b), determine the same tiling (see Fig. 17(d)), since they both determine the patch shown in (e).

contiguous tiles of this patch (with one exception: two hexagons) determine the position of the others. It is this patch that replaces the corner-tile.

If in a tiling two copies of S in the position shown in Fig. 18(d) or (j) occur then, as we have seen, they uniquely determine a corner-patch, but on the other side of the pair two possibilities exist: if a third copy of S is fitted against them as shown in Fig. 25(a) or (b), respectively, then two more corner-patches are determined (see Fig. 25(c) and (d)). Both these patches are uniquely extended to the patch shown in Fig. 25(e) which, in turn, uniquely determines the tiling in Fig. 17(d) which consists of overlapping corner-patches. If instead of S a copy of the hexagon occurs (as shown in Fig. 26(a) or (b)) then the patch of Fig. 21(d) is determined (see Fig. 26(c) or (d), respectively). Now the same two possibilities arise: a copy of S (as in Fig. 26(e)) determines a second corner-patch and leads to the tiling in Fig. 17(e), a copy of A determines another patch of Fig. 21(d).

Fig. 26. The patch in Fig. 21(d) can be used to build arbitrarily long lines. It is terminated if a patch as in (a) or (b) occurs which determines a corner-patch.

Now a copy of S (as in Fig. 26(f)) determines the tiling in Fig. 17(f), and so on. Either a copy of S determines a second corner-patch and therefore a tiling consisting of triangular patches, or only one corner-patch exists, and the tiling is that of Fig. 17(c).

If no pair as in Fig. 18(d) or (j) occurs, then no hexagon in the position of A_3 , A_6 , or A_7 in Fig. 21 can occur because it would determine a corner-patch. Therefore only the position shown in Fig. 18(g) = (m) is possible: from each copy of S a straight line extends in two directions. As in the previous cases, no further copy of S can occur (see Fig. 27): the positions of S_1 , S_4 , and S_7 are not possible since they would determine a corner-patch; the positions of the other copies of S would determine a second line meeting the first forcing the occurrence of a corner-patch (S_3 or S_6) or leading to a contradiction (S_2 or S_5). The tiling is therefore that of Fig. 17(b). This completes the proof.

Fig. 27. Copies of the hexagon A extend from a line of copies of S , covering part of the plane. A copy of S can be added to such a patch in seven positions, as indicated. But only two of them— S_3 and S_6 —can occur in a tiling.

Fig. 28. From the five prototiles of Fig. 2 a set of prototiles can be obtained such that in any tiling copies of all the prototiles occur. (a) Thin tiles are cut from the tiles such that congruent squares remain. (b) These thin tiles can be “stowed” inside the square.

4. Final Remarks

As already mentioned in the introduction, it is not known if a σ -morphic prototile exists. But there are many questions open, too, where sets of prototiles are concerned, from which we shall formulate only a few. Is there a σ -morphic set which admits periodic tilings only? Conversely, is there an aperiodic σ -morphic set (that is a set which admits no periodic tiling)?

All the sets of prototiles described up to this point admit some tilings in which one or more of them do not occur. This need not be the case, however, as the following example shows: if, in the tiling of Section 1, the keys and keyholes are chosen to be very small then the five prototiles can be cut into pieces in such a way that the interiors are made up from copies of a single square tile and 35 thin tiles, as indicated in Fig. 28(a). Furthermore, by using suitable keys and keyholes, this can be done in such a way that these pieces can only be fitted together in such a way as to reconstitute the original five pieces. If the edge-tiles are thin enough (much thinner than shown in the figure) they can be packed into the interior of the square tile, as indicated in Fig. 28(b). By cutting the resulting hollow square into two parts, as indicated, we obtain a σ -morphic set of 37 prototiles such that copies of all the prototiles occur in every tiling. The σ -morphic pair described in [3] has this property, too, but almost all tilings are not periodic. What is the minimum number of prototiles in a set by which a countable infinity of periodic tilings is admitted, but no subset of which admits a tiling?

Finally, can all σ -morphic sets, or at least all σ -morphic sets with few prototiles, be described in some way?

References

1. B. Grünbaum and G. C. Shephard, Patch-determined tilings, *Math. Gaz.* **61** (1977), 296–299.
2. B. Grünbaum and G. C. Shephard, Some problems on plane tilings, in *The Mathematical Gardner* (D. A. Klarner, ed.), 167–196, Prindle, Weber & Schmidt, Boston Wadsworth International, Belmont, 1981.
3. P. Schmitt, Pairs of tiles which admit finitely or countably infinitely many tilings, *Geom. Dedicata*, **20** (1986), 133–142.

Received July 22, 1985.