Steiner Minimal Trees for Regular Polygons

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Abstract. Fifty years ago Jarnik and Kössler showed that a Steiner minimal tree for the vertices of a regular n -gon contains Steiner points for $3 \leq n \leq 5$ and contains no Steiner point for $n = 6$ and $n \geq 13$. We complete the story by showing that the case for $7 \leq n \leq 12$ is the same as $n \geq 13$. We also show that the set of n equally spaced points yields the longest Steiner minimal tree among all sets of n cocircular points on a given circle.

1. Introduction

A Steiner minimal tree (SMT) for a set of points P in the plane is a shortest network interconnecting P . The construction of an SMT for a general set P is known [7] to be an NP -complete problem. Recently, SMTs have been constructed for special point sets P such as ladders [1], splitting trees [9], zigzag lines [5], cocircular points [6], and bar waves [4]. However, a special class of sets for which the study of SMTs was started a half century back has remained an unsolved problem. Let A_n denote the set of vertices of a regular n -gon. The SMT problem for A_n was first studied by Jarnik and Kössler [10] in 1934. They obtained SMTs for $n \leq 6$ and also proved a beautiful theorem which says that for $n \geq 13$ an SMT can be obtained by deleting an edge from the perimeter of the regular n -gon. Since an SMT can also be obtained in this manner for $n = 6$, an obvious conjecture is that an SMT can be so obtained for all $n \geq 6$. Kotzig [11] discussed some properties of the angles of an SMT for $n \leq 8$. In this article we will prove this conjecture in its entirety as our Theorem 1.

Theorem 1. *The perimeter of a regular n -gon minus any side is an SMT for A_n for $n \geq 6$.*

We also prove

Theorem 2. *For any n cocircular points on a given circle, the set of n equally spaced points yields a longest SMT.*

2. The Case $n \geq 11$

In this section we show that some recent results on the Steiner ratio (to be defined shortly) can be used to dispose of the conjecture for all $n \geq 11$.

A minimal spanning tree (MST) for a set of points P is a shortest tree interconnecting P such that the vertex-set of the tree is P . The *Steiner ratio* ρ is defined as

$$\inf_P \frac{\text{length of an SMT for } P}{\text{length of an MST for } P}.$$

Gilbert and Pollak [8] conjectured that $\rho = \sqrt{3}/2$ while Du and Hwang [3] proved that $\rho \geq 0.8$. Recently, Chung and Graham [2] announced a proof that $\rho \geq 0.8241$. The Steiner ratio was surprisingly used in [6] to prove a result about SMTs for cocircular points, via the following lemma:

Lemma 0. *Suppose that an n -gon circumscribed in a unit circle has at most one side longer than m with*

$$m = \min\{[\alpha\beta + \sqrt{\alpha^2 + (1 - \beta^2)/4}]/(\alpha^2 + \frac{1}{4}), \gamma\},$$

where

$$\alpha = \sqrt{3} + 1 - 1/(2\bar{\rho}),$$

$$\beta = 1 - (1 - \bar{\rho})\pi/\bar{\rho}$$

($\bar{\rho}$ is a lower bound for ρ) and

$$\gamma = 2(\sqrt{3} + 1)/[(\sqrt{3} + 1)^2 + \frac{1}{4}] = 0.708 \dots$$

Then its MST (which is the perimeter of the n -gon minus the longest side) is also its SMT.

Set $\bar{\rho} = 0.824$. We obtain $m > 0.6034$. On the other hand, the length of a side of the regular n -gon

$$l_n = \sqrt{2\left(1 - \cos \frac{2\pi}{n}\right)} = 2 \sin \frac{\pi}{n}$$

is monotone decreasing in n for $n \geq 3$. Furthermore,

$$l_n \leq l_{11} < 0.5635 < 0.6034 < m \quad \text{for } n \leq 11.$$

By Lemma 0 we obtain

Theorem 1. *The MST of a regular n -gon is also its SMT for $n \geq 11$.*

3. Some Facts About SMTs

Consider any tree T interconnecting a set of points $P = \{p_1, \dots, p_n\}$. We will refer to the p_i 's as the *regular points* and any other points in T as *Steiner points*. T is called a *Steiner tree* if all subtending angles are at least 120° and each Steiner point has three incident edges (this implies that the subtending angles are exactly 120° for a Steiner point). It is well known [8] that a Steiner tree for n points has at most $n - 2$ Steiner points and is called a *full Steiner tree* if it has $n - 2$ points. It is also well known [8] that an SMT must be a Steiner tree and can always be decomposed into subtrees which are full Steiner trees. Finally, it is well known [8] that an SMT always lies within the convex hull of P .

A *topology* of a Steiner tree T is a specification of all edges in T . A Steiner tree for a given topology either exists uniquely or does not exist. When a full Steiner tree with a given topology exists, Melzak [12] gave a recursive construction for it which also yields a line segment, which we call the *axis*, whose length equals that of the Steiner tree.

Let C denote a unit circle with center o . Let R_n denote a regular n -gon inscribed in C with vertex set $A_n = \{a_1, \dots, a_n\}$. Throughout the paper we denote the line segment between two points x and y by $[x, y]$ and its length by (xy) .

Lemma 1. *Let T be an SMT for R_n . Then we may assume that no Steiner point s of T can have an incident edge as long as l_n .*

Proof. Suppose to the contrary that l is such an edge. Delete l and decompose T into two subtrees. Then there must exist a j such that a_j and a_{j+1} are not in the same subtree. Connect a_j , a_{j+1} and we obtain an interconnecting tree not longer than T . $\square$

Lemma 2. *Let C be a unit circle with center o . Let p, q be two points such that $(po) \geq 1 \geq (qo)$ and $\angle oqp \leq 60^\circ$ (see Fig. 1). Then $(pq) \geq (po)$.*

Proof. In $\triangle opq$, $\angle qpo \leq \angle oqp \leq 60^\circ$ since $(po) \geq (qo)$. Hence $\angle poq \geq 60^\circ \geq \angle oqp$. It follows $(pq) \geq (po)$. $\square$

Fig. 1. $(pq) > (po)$.

The following Lemma is not directly related to SMTs but often facilitates an argument that a certain topology does not exist.

Lemma 3. *Let $CA_1 \dots A_mD$ be a polygon lying within another polygon $CB_1 \dots B_nD$. Then*

$$\sum_{i=1}^m \angle A_i - \sum_{i=1}^n \angle B_i \geq (m-n)180^\circ.$$

Proof. Using the fact that an n -gon has total inner degrees $(n-2)180^\circ$ and the fact that $\angle A_1CD \leq \angle B_1CD$ and $\angle CDA_m \leq \angle CDB_n$. $\square$

A path $a_i s_1 \dots s_m a_j$ in an SMT T is called a *Steiner path* if $s_1, \dots, s_m$ are all Steiner points and $\angle s_i = 120^\circ$ for $i = 1, \dots, m$ in the $(m+2)$ -gon $a_i s_1 \dots s_m a_j$.

Lemma 4. *Suppose that T is an SMT for R_n . Let $P = a_i s_1 s_2 \dots s_m a_j$ be a Steiner path.*

- (i) $m \leq 3$. *There are no regular points between a_i and a_j .*
- (ii) $m = 4$. *There is at most one regular point between a_i and a_j but none if $n \leq 9$. No such P can exist for $n \leq 6$.*
- (iii) $m = 5$. *No such P can exist for $n \leq 11$.*
- (iv) $m \geq 6$. *No such P can exist.*

Proof. It is easily verified that $(a_i a_j) < 2l_n$ for all m . Thus at most one regular point can exist between a_i and a_j .

$m = 1$ or 2 . Suppose to the contrary that a_i and a_j are not adjacent. Then $n \geq 4$. Let a_k be the regular point between a_i and a_j . For $m = 1$ consider the quadrilateral $a_i s_1 a_j a_k$. We have

$$\angle a_i + \angle a_j = 360^\circ - \angle s_1 - \angle a_k \leq 360^\circ - 120^\circ - 90^\circ = 150^\circ.$$

For $m = 2$ consider the pentagon $a_i s_1 s_2 a_j a_k$. We have

$$\angle a_i + \angle a_j = 540^\circ - \angle s_1 - \angle s_2 - \angle a_k \leq 540^\circ - 120^\circ - 120^\circ - 90^\circ = 210^\circ.$$

Hence either $\angle a_i$ or $\angle a_j$ is less than 120° . Suppose that $\angle a_i < 120^\circ$. Then a_k is connected to a_j in T . Let T' be obtained by substituting $[a_k, a_i]$ for $[a_k, a_j]$ in T . Then T' and T have the same length. Yet T' cannot be optimal since $\angle a_k a_i s_1 < 120^\circ$, a contradiction to the optimality of T .

$m = 3$. We have $n \geq m + 2 = 5$. If $n = 5$, then there is not other Steiner point and each of s_1, s_2 , and s_3 must connect a distinct regular point outside of the pentagon $a_i s_1 s_2 s_3 a_j$. Thus there is no more regular point to fill between a_i and a_j . Therefore we may assume that $n \geq 6$.

Suppose to the contrary that a_k exists between a_i and a_j . Consider the hexagon $a_i s_1 s_2 s_3 a_j a_k$. Note that all angles except $\angle a_i$ and $\angle a_j$ are at least 120° . Hence either the hexagon is regular, or at least one of $\angle a_i$ and $\angle a_j < 120^\circ$. The former case is impossible since we do not allow $(s_1 s_2) = (a_k a_i) = l_n$. The latter case is also impossible by analogous argument as used in the case $m = 1$ or 2 .

$m = 4$. $n \geq m + 2 = 6$. If $n = 6$, the hexagon $a_i s_1 s_2 s_3 s_4 a_j$ has internal degree less than 720° , an absurdity. For $n \geq 7$ suppose that a_k exists between a_i and a_j . For $n \leq 9$, the heptagon $a_i s_1 s_2 s_3 s_4 a_j a_k$ has internal degrees less than 900° , an absurdity.

$m = 5$. Consider the heptagon $a_i s_1 s_2 s_3 s_4 s_5 a_j$,

$$\angle a_i + \angle a_j = 900^\circ - 5 \cdot 120^\circ = 300^\circ > \angle a_{i-1} a_i a_{i+1} + \angle a_{j-1} a_j a_{j+1} \quad \text{for } n \leq 11,$$

an absurdity.

$m \geq 6$. Consider the $(m+2)$ -gon $a_i s_1 \dots s_m a_j$,

$$\angle a_i + \angle a_j = m \cdot 180^\circ - m \cdot 120^\circ = m \cdot 60^\circ \geq 360^\circ \quad \text{for } m \geq 6, \text{ an absurdity. } \square$$

Lemma 5. *Let T be an SMT for A_n , $n \leq 10$, with a Steiner point s . Then T must be full.*

Proof. Let $T' \subseteq T$ be a full Steiner tree containing s . Then T' partitions the unit circle into convex regions each bounded by a Steiner path and an arc. By Lemma 4 such an arc can contain at most one additional regular point. In fact, the only case in which an additional regular point may exist is when $n = 10$ and the Steiner path bounding the region has $m = 4$. We now show that even for this case no additional regular point can exist on the arc, i.e., T is a full SMT.

Suppose to the contrary that a regular point a_k exists on the arc $a_i a_j$ (see Fig. 2). In the heptagon $a_i s_1 s_2 s_3 s_4 a_j a_k$

$$\angle a_i + \angle a_j = 900^\circ - 4 \times 120^\circ - 144^\circ = 276^\circ.$$

Therefore

$$\angle s_1 a_i a_{i+1} + \angle a_{j-1} a_j s_4 = 2 \times 144^\circ - 276^\circ = 12^\circ$$

and

$$\min\{(\angle s_1 a_{i+1}), (\angle s_4 a_{j-1})\} \geq l_n \sin 48^\circ / \sin 120^\circ > 0.858 l_n$$

Fig. 2. A Steiner path with $m = 4$ and a regular point.

(it is easily verified that $(s_1 a_{i+1})$ will be greater if s_1 connects a_{i+1} through another Steiner point). Now it is a simple matter to show that $(s_1 s_4) > 2l_n$ which implies that one of $(s_1 s_2)$, $(s_2 s_3)$, and $(s_3 s_4) > l_n$, a contradiction to Lemma 1. $\square$

Define D to be the diameter (of the unit circle c) $\perp [a_1, a_n]$.

Lemma 6. *Suppose that a topology is symmetric with respect to an edge e . Then the Steiner tree it yields is symmetric with respect to D with e overlapping D for odd n and $e \perp D$ for even n . Suppose that a topology is symmetric with respect to a point p . Then the Steiner tree it yields is symmetric with respect to the center o with p being o .*

Proof. Clear from Melzak's construction for SMT. $\square$

4. Proof of Theorem 1 for $8 \leq n \leq 10$

Suppose that T is an SMT for A_n with a Steiner point. By Lemma 5 we may assume that T is full. Let d be a point of T closest to the center o and let d lie on the edge e . Let q be an endpoint of e . Partition T into two trees T_1 and T_2 at q and without loss of generality assume that T_1 contains e and the k regular points $\{a_1, a_2, \dots, a_k\}$. By Melzak's construction of the full Steiner tree, there exists a line segment $[p, q]$ which is the axis of T_1 and overlaps e . Our goal is to show that for certain n T_1 cannot exist by proving $(pq) > kl_n$, so T_1 can be replaced by the path $a_1 a_2 \dots a_k$ and some suitable $[a_j, a_{j+1}]$ to obtain a shorter connecting tree. However, since (po) is much easier to compute than (pq) , we will prove $po > kl_n$ instead and use Lemma 2 to justify the replacement. One condition of Lemma 2 is that $\angle pqo \leq 60^\circ$. The following lemma will take care of that condition.

Lemma 7. $\angle d q o > 60^\circ$.

Proof. Let e' be a second edge of T at q such that o lies in the 120° angle enclosed by e and e' (possibly their extensions). Let (od') be the distance from o to e' . Since $(od) \leq (od')$, $\angle d q o \leq \angle d' q o$. But $\angle d q o + \angle d' q o = 120^\circ$. Hence $\angle d q o \leq 60^\circ$. $\square$

Lemma 8. $k \neq 1$ for $n > 6$.

Proof. From Lemmas 7 $\angle a_1 q o \leq 60^\circ$. From Lemma 2

$$(a_1 q) \geq (a_1 o) = 1 > l_n \quad \text{for } n > 6. \quad \square$$

Lemma 9. $k \neq 2$ for $8 \leq n \leq 10$.

Proof. Suppose to the contrary that $k = 2$. Let $pa_1 a_2$ be a regular triangle with p outside of the unit circle. Then by Melzak's construction $[p, q]$ is the axis of

T_1 . We now prove that $(po) > 2l_n$, or $((po)/l_n)^2 > 4$:

$$\begin{aligned}
 (po)^2 &= (a_1o)^2 + (a_1p)^2 - 2(a_1o)(a_1p) \cos \angle oa_1p \\
 &= 1 + l_n^2 - 2l_n \cos(90^\circ - 180^\circ/n + 60^\circ) \\
 &= 1 + l_n^2 + \sqrt{3}l_n \cos(180^\circ/n) - l_n \sin(180^\circ/n) \\
 &= 1 + l_n^2 + \sqrt{3}l_n \sqrt{1 - l_n^2/4} - l_n^2/2 \\
 &= 1 + l_n^2/2 + \sqrt{3}l_n \sqrt{1 - l_n^2/4}.
 \end{aligned}$$

Clearly, $(po)^2/l_n^2$ is monotone decreasing in l_n^2 . For $8 \leq n \leq 10$ l_n^2 is largest for $n = 8$ and $l_8^2 = 2(1 - \cos 45^\circ) = 2 - \sqrt{2}$. Now

$$\left(\frac{(po)}{l_8}\right)^2 = \frac{1}{2 - \sqrt{2}} + \frac{1}{2} + \sqrt{3} - \frac{1}{4} \geq 4.297 > 4. \quad \square$$

Lemma 10. $k \neq 3$ for $8 \leq n \leq 10$.

Proof. Suppose to the contrary that $k = 3$. Let $a_3a_2p_1$ and $p_1a_1p_2$ be regular triangles such that p_1 and o are on different sides of $[a_2, a_3]$, and p_2 and o are on different sides of $[p_1a_1]$ (see Fig. 3). We now prove that $(p_2o) > 3l_n$ or $(p_2o)/l_n > 9$. Note that

$$(p_2o)^2 = (p_1o)^2 + (p_1p_2)^2 - 2(p_1o)(p_1p_2) \cos \angle op_1p_2.$$

Now

$$\begin{aligned}
 (p_1p_2) &= (p_1a_1) = [(a_1a_2)^2 + (p_1a_2)^2 - 2(a_1a_2)(p_1a_2) \cos \angle a_1a_2p_1]^{1/2} \\
 &= 2l_n[(1 - \cos \angle a_1a_2p_1)/2]^{1/2} \\
 &= 2l_n \sin(\angle a_1a_2p_1/2) \\
 &= 2l_n \sin(60^\circ + 180^\circ/n).
 \end{aligned}$$

Hence

$$(p_1p_2)/l_n = 2 \sin(60^\circ + 180^\circ/n) \geq 2 \sin(60^\circ + 18^\circ) > 1.956$$

and $(p_1p_2)^2/l_n^2 > 3.827$ for $n \leq 10$. Furthermore,

$$\begin{aligned}
 \angle op_1p_2 &= \angle op_1a_2 + \angle a_2p_1a_1 + \angle a_1p_1p_2 \\
 &= 30^\circ + 30^\circ - 180^\circ/n + 60^\circ = 120^\circ - 180^\circ/n
 \end{aligned}$$

Fig. 3. $(p_2o) > 3l_n$ for $k = 3$.

and

$$-2 \cos \angle op_1 p_2 = 2 \cos(60^\circ + 180^\circ/n),$$

which is clearly monotone increasing in n for $n \geq 6$. Therefore, for $8 \leq n \leq 10$

$$-2 \cos \angle op_1 p_2 \geq 2 \cos 82.5^\circ = 0.261.$$

Therefore, for $8 \leq n \leq 10$ we have

$$\left(\frac{p_2 o}{l_n}\right)^2 \geq 4.297 + 3.827 + (0.261)\sqrt{(4.297)(3.827)} = 9.182 > 9. \quad \square$$

Lemma 11. $k \neq 4$ for $8 \leq n \leq 10$.

Proof. Suppose to the contrary that $k = 4$. There are three nonisomorphic topologies for T_1 which we will call topologies 4, 5, and 6 and their Melzak's constructions are shown in Figs. 4–6, respectively. We show that the axis of T_1 is too long for all three topologies.

Topology 4 (Fig. 4)

As shown in the proof of Lemma 10,

$$\begin{aligned} (a_2 p_2) &= 2l_n \sin\left(60^\circ + \frac{180^\circ}{n}\right), \\ \angle a_3 a_2 p_1 &= 90^\circ - \left(60^\circ + \frac{180^\circ}{n}\right) \\ &= 30^\circ - \frac{180^\circ}{n}. \end{aligned}$$

Fig. 4. Melzak's construction for topology 4.

Therefore,

$$\begin{aligned}\angle a_1 a_2 p_2 &= 90^\circ + \frac{540^\circ}{n}, \\ (a_1 p_2)^2 &= (a_1 a_2)^2 + (a_2 p_2)^2 - 2(a_1 a_2)(a_2 p_2) \cos \angle a_1 a_2 p_2 \\ &= l_n^2 \left[1 + 4 \sin^2 \left(60^\circ + \frac{180^\circ}{n} \right) + 4 \sin \left(60^\circ + \frac{180^\circ}{n} \right) \sin \frac{540^\circ}{n} \right].\end{aligned}$$

Furthermore,

$$\sin \angle a_2 p_2 a_1 = \frac{l_n}{(a_1 p_2)} \sin \angle a_1 a_2 p_2 = \frac{l_n}{(a_1 p_2)} \cos \frac{540^\circ}{n}.$$

Hence

$$\begin{aligned}\angle o a_1 p_3 &= \left(90^\circ - \frac{180^\circ}{n} \right) + 60^\circ + \angle a_2 a_1 p_2 \\ &= 150^\circ - \frac{180^\circ}{n} + \left(180^\circ - \left(90^\circ + \frac{500^\circ}{n} \right) - \angle a_2 p_2 a_1 \right) \\ &= 240^\circ - \frac{720^\circ}{n} - \angle a_2 p_2 a_1.\end{aligned}$$

It follows

$$\left(\frac{(p_3 o)}{l_n} \right)^2 = \left(\frac{1}{l_n} \right)^2 + \frac{(a_1 p_2)^2}{l_n^2} + 2 \cdot \frac{1}{l_n} \cdot \frac{(a_1 p_2)}{l_n} \cos \left(\frac{720^\circ}{n} + \angle a_2 p_2 a_1 - 60^\circ \right).$$

We compute $((p_3 o)/l_n)^2$ for $n = 8, 9, 10$.

n	$(a_1 p_2)^2/l_n^2$	$(a_1 p_2)/l_n$	$\sin \angle a_2 p_2 a_1$	$1/l_n$	$(p_3 o)^2/l_n^2$
8	> 8.595	> 2.937	< 0.137	> 1.306	> 16.3
9	> 8.290	> 2.879	< 0.175	> 1.461	> 17.7
10	> 7.990	> 2.826	< 0.210	> 1.618	> 18.9

Topology 5 (Fig. 5)

Note that $\triangle p_1 p_2 a_3 = \triangle p_1 a_2 a_4$. Hence $(p_2 a_3) = (a_3 a_1)$. Also note that $(p_1 p_2) = (p_1 a_4) = (p_1 a_1)$. Hence $\triangle p_1 a_3 p_2 = \triangle a_1 a_3 p_1$. It follows

$$\angle p_1 a_3 p_2 = \angle a_1 a_3 p_1 = 60^\circ + 180^\circ/n$$

and

$$\angle a_1 a_3 p_2 = 120^\circ + 360^\circ/n.$$

Fig. 5. Melzak's construction for topology 5.

Furthermore,

$$\begin{aligned}(p_2 a_3) &= (a_2 a_4) = (a_1 a_3) = 2l_n \left[\frac{1 - \cos(180^\circ(n-2)/n)}{2} \right]^{1/2} \\ &= 2l_n \sin\left(180^\circ - \frac{180^\circ}{n}\right) = 2l_n \cos \frac{180^\circ}{n}.\end{aligned}$$

Therefore,

$$\begin{aligned}(p_3 a_1) &= (a_1 p_2) = [(a_1 a_3)^2 + (a_3 p_2)^2 - 2(a_1 a_3)(a_3 p_2) \cos \angle a_1 a_3 p_2]^{1/2} \\ &= 4l_n \cos \frac{180^\circ}{n} \left[\frac{1 - \cos(120^\circ + 360^\circ/n)}{2} \right]^{1/2} \\ &= 4l_n \cos \frac{180^\circ}{n} \sin\left(60^\circ + \frac{180^\circ}{n}\right) \\ &= 2l_n \left[\sin\left(60^\circ + \frac{360^\circ}{n}\right) + \sin 60^\circ \right] \quad \text{using } 2 \sin A \cos B \\ &= \sin(A+B) + \sin(A-B) \\ &\geq 2l_n \left[\sin\left(60^\circ + \frac{360^\circ}{8}\right) + \sin 60^\circ \right] > 3.663l_n \quad \text{for } n \geq 8.\end{aligned}$$

Finally

$$\begin{aligned}\angle oa_1 p_3 &= 60^\circ + \left(90^\circ - \frac{360^\circ}{n}\right) - 90^\circ - \left(60^\circ + \frac{160^\circ}{n}\right) = 180^\circ - \frac{540^\circ}{n}, \\ \frac{(p_3 o)^2}{l_n^2} &= \frac{(a_1 o)^2}{l_n^2} + \frac{(p_3 a_1)^2}{l_n^2} + \frac{(p_3 a_1)^2}{l_n^2} - 2 \frac{(a_1 o)}{l_n} \frac{(p_3 a_1)}{l_n} \cos \angle oa_1 p_3 \\ &> \frac{1}{l_8^2} + (3.663)^2 + 2 \frac{1}{l_8} (3.663) \cos \frac{540^\circ}{8} \\ &= (1.306)^2 + (3.063)^2 + 2(1.306)(3.663)(0.382) \\ &= 18.778 > 16.\end{aligned}$$

Fig. 7. Two topologies for $m_1 = m_2 = 4$.

and each edge in T is shorter than l_{10} , there must be at least four edges connecting a_1 and a_5 . Therefore $m_1 = m_2 = 4$.

There exist two nonisomorphic topologies for $m_1 = m_2 = 4$ as shown in Fig. 7. If T has topology 7(a), then by Lemma 6 T must be asymmetric with respect to the center which is on e . Therefore we can turn the left half of the tree upside down and obtain a tree of the same length but having 7(b) as its topology (Fig. 8). Namely, it suffices to prove that T cannot have 7(b) as its topology.

In $\triangle s_4 a_4 a_5$, $\angle s_4 a_4 a_5 = \frac{2}{10} \cdot 180^\circ = 36^\circ$:

$$(a_5 a_4) = \frac{(a_4 a_5) \sin 36^\circ}{\sin 120^\circ} = \frac{(0.618)(0.588)}{0.866} = 0.42.$$

Extend $[a_5, s_4]$ and $[a_1, s_1]$ to meet at b . Then $bs_1 s_3 s_4$ is a parallelogram. Hence $(s_4 b) = (s_3 s_1)$ and $(s_1 b) = (s_3 s_4)$. In $\triangle ba_5 a_1$, $\angle ba_5 a_1 = \angle a_5 a_1 b = 30^\circ$ and $\angle a_1 b a_5 = 120^\circ$. Furthermore,

$$(a_1 b) = (a_5 b) = (a_5 s_4) + (s_3 s_1) \leq 0.4204 + 0.618 = 1.038.$$

Therefore

$$(a_1 a_5) = \sqrt{3} a_1 b \leq (1.732)(1.038) = 1.798.$$

Fig. 8. A tree for topology 7(b).

Fig. 9. A unique topology for $n=6$.

But from $\triangle a_1 a_5 o$

$$(a_1 a_5) = \sqrt{2(1 - \cos 144^\circ)} > 1.9, \quad \text{a contradiction.}$$

5. Proof of Theorem 1 for $n=6, 7$

For $n=6$, Lemma 4 reduces the nonisomorphic topologies to the unique one shown in Fig. 9. Since this topology is symmetric with respect to s_2 , s_2 must be the center o and T must be symmetric with respect to o . Therefore the length of T is $3\sqrt{3} > 5$ which is the length of an MST.

For $n=7$ Lemma 4 reduces the nonisomorphic topologies to the three shown in Fig. 10. Topology 10(a) can be quickly disposed of by comparing the angles of the polygonal path $a_1 a_2 a_3 a_4 a_5 a_7$ and those of the Steiner path $a_1 s_1 s_2 s_3 s_4 a_7$, using Lemma 3. The length of the tree yielded by topology 10(c) has been computed in [13] to be $5.6676 > 5.2068$ which is the length of an MST. We now show that the tree yielded by topology 10(b) is not an SMT as $(a_4 s_3) > l_7$.

Since the topology 10(b) is symmetric with respect to $[a_4, s_3]$, T must be symmetric to D and $[a_4, s_3]$ must overlap with D (see Fig. 11) Extend $[a_3, s_2]$ to b such that $[a_1, b] \parallel [s_1, s_2]$. Extend $[a_4, s_3]$ to c such that $[a_1, c] \parallel [s_1, s_3]$. Then

$$(a_4 s_3) = (a_4 c) - (a_3 b) + (a_3 s_2).$$

Now

$$\angle a_4 a_1 c = \angle a_3 a_2 s_2 = \angle a_3 a_2 a_6 - \angle s_2 a_2 a_6 = \frac{3}{7} \cdot 180^\circ - 30^\circ = \frac{330^\circ}{7}.$$

Fig. 10. Three topologies for $n=7$.

Lemma 12. For $3 \leq n \leq 5$, if one of the angles of P_n is 120° or larger, than an SMT for C_n is shorter than that for A_n .

Proof. We show that an MST for C_n is shorter than the SMT for A_n . □

Without loss of generality, assume $\angle a_1 a_2 a_3 \geq 120^\circ$. Then

$$\angle a_1 o a_2 + \angle a_2 o a_3 \leq 120^\circ.$$

By standard minimization techniques it is easily seen that the longest MST for n cocircular points satisfies the angle conditions

$$\angle a_1 o a_2 = \angle a_2 o a_3 = 60^\circ,$$

and

$$\angle a_3 o a_4 = \cdots = \angle a_n o a_1 = 240^\circ / (n - 2).$$

The length of such an MST is

$$\begin{aligned} & 2\sqrt{2(1 - \cos 60^\circ)} + (n - 3)\sqrt{2\left[1 - \cos\left(\frac{240^\circ}{n - 2}\right)\right]} \\ &= \begin{cases} 2 < 3 & \text{for } n = 3, \\ 2 + \sqrt{3} < \sqrt{2} + \sqrt{6} & \text{for } n = 4, \\ 2 + 2.572 < 4.574 & \text{for } n = 5, \end{cases} \end{aligned}$$

where the right side of the inequality is the length of an SMT for A_n .

Corollary. If an SMT for C_n is not full, then its length is shorter than that of A_n .

We now prove Theorem 2 for $n = 3$. Consider C_3 such that all angles of P_3 are less than 120° . Construct a regular $\triangle BCD$ such that A and D are on different sides of $[B, C]$. Then (AD) is the length of the SMT for C_3 (see Fig. 12). Let $\angle oBD = \theta$:

$$\begin{aligned} (AD) &\leq (Ao) + (oD) \\ &= 1 + \frac{(oB) \sin \theta}{\sin 30} \leq 3. \end{aligned}$$

Fig. 12. A Steiner tree for $n = 3$.

Fig. 14. A Steiner tree for $n = 5$.

Let M_1, M_2, M_3, M_4 , and M_5 denote the lengths of these five trees, respectively. We prove that

$$\sum_{i=1}^5 M_i < 5M,$$

where M is the length of an SMT for A_5 . Therefore the SMT for C_5 , which is the shortest one among the five trees, must be shorter than M .

Construct equilateral triangles $\triangle ABD'$, $\triangle BCE'$, $\triangle CDA'$, $\triangle DEB'$, $\triangle EAC'$, $\triangle A'C'B''$, $\triangle B'D'C''$, $\triangle C'E'D''$, $\triangle D'A'E''$, and $\triangle E'B'A''$ (see Fig. 15). Then $[A, A'']$, $[B, B'']$, $[C, C'']$, $[D, D'']$, and $[E, E'']$ are the five axes. Since $(x'', x) \leq (x'', o) + (o, x) = (x'', o) + 1$ for $x = A, B, C, D, E$, it suffices to prove

$$S \equiv (A''o) + (B''o) + (C''o) + (D''o) + (E''o) \leq 5(M - 1).$$

Construct a circle through the three points A'', B' , and E' and meet $[A'', o]$ (or its extension) at G . Then

$$(A''o) = (GB') + (GE') + (Go) \quad (\text{or } -(Go)).$$

Fig. 15. Axes for the five full Steiner trees.

Define

$$\begin{aligned}\angle A''oB' &= \alpha_1, \quad \angle B''oC' = \alpha_2, \quad \dots, \quad \angle E''oA' = \alpha_5, \\ \angle E'oA'' &= \beta_1, \quad \angle A'oB'' = \beta_2, \quad \dots, \quad \angle D'oE'' = \beta_5, \\ \angle CoD &= 2\theta_1, \quad \angle DoE = 2\theta_2, \quad \dots, \quad \angle BoC = 2\theta_5.\end{aligned}$$

Since

$$\angle A''GB' = \frac{1}{2}\angle E'GB' = \frac{1}{2}(180^\circ - \angle B'A''E') = 60^\circ,$$

we have

$$(A''o) = \frac{\sin \alpha_1}{\sin 60^\circ} (oB') + \frac{\sin \beta_1}{\sin 60^\circ} (oE') + \frac{\sin(60^\circ - \alpha_1)}{\sin 60^\circ} (oB').$$

Note that

$$(oB') = 2 \sin(30^\circ + \theta_2),$$

$$(oE') = 2 \sin(30^\circ + \theta_5),$$

$$\frac{\sin(60^\circ - \alpha_1)}{\sin 60^\circ} (oB') = \frac{\sin(60^\circ - \beta_1)}{\sin 60^\circ} (oE').$$

we have

$$\begin{aligned}(A''o) &= \frac{1}{\sqrt{3}} \{ [(oB') \sin \alpha_1 + (oE') \sin \beta_1] \\ &\quad + (oB') [\sin \alpha_1 + \sin(60^\circ - \alpha_1)] + (oE') [\sin \beta_1 + \sin(60^\circ - \beta_1)] \} \\ &= \frac{1}{\sqrt{3}} \{ [(oB') \sin \alpha_1 + (oE') \sin \beta_1] \\ &\quad \times (oB') \cos(30^\circ - \alpha_1) + (oE') \cos(30^\circ - \beta_1) \} \\ &= \frac{1}{\sqrt{3}} \{ (oB') [\sin \alpha_1 + \cos(30^\circ - \alpha_1)] + (oE') [\sin \beta_1 + \cos(30^\circ - \beta_1)] \} \\ &= \frac{1}{\sqrt{3}} \{ (oB') [\sin \alpha_1 + \sin(60^\circ + \alpha_1)] + (oE') [\sin \beta_1 + \sin(60^\circ + \beta_1)] \} \\ &= \frac{1}{\sqrt{3}} \{ (oB') \sin(30^\circ + \alpha_1) \cos 30^\circ + (oE') \sin(30^\circ + \beta_1) \cos 30^\circ \} \\ &= \sin(30^\circ + \alpha_1) \sin(30^\circ + \theta_2) + \sin(30^\circ + \beta_1) \sin(30^\circ + \theta_5) \\ &= \cos(\alpha_1 - \theta_2) - \cos(60 + \alpha_1 + \theta_2) + \cos(\beta_1 - \theta_5) - \cos(60 + \beta_1 + \theta_5) \\ &= \cos(\alpha_1 - \theta_2) + \sin(\alpha_1 + \theta_2 - 30^\circ) + \cos(\beta_1 - \theta_5) + \sin(\beta_1 + \theta_5 - 30^\circ).\end{aligned}$$

Therefore we can write

$$S = S' + S'',$$

where

$$\begin{aligned} S' &= \cos(\alpha_1 - \theta_2) + \cos(\beta_1 - \theta_5) + \cdots + \cos(\alpha_5 - \theta_1) + \cos(\beta_5 - \theta_4), \\ S'' &= \sin(\alpha_1 + \theta_2 - 30^\circ) + \sin(\beta_1 + \theta_5 - 30^\circ) + \cdots + \sin(\alpha_5 + \theta_1 - 30^\circ) \\ &\quad + \cos(\beta_5 + \theta_4 - 30^\circ). \end{aligned}$$

To bound S' and S'' we need the following lemma.

Lemma 13. *Let $\angle XYZ = y$ where $60^\circ < y < 180^\circ$. Construct equilateral triangle $\triangle XZW$ and define $\angle WYZ = w$ (see Fig. 16). Then*

$$\min\{60^\circ, y - 60^\circ\} \leq w \leq \max\{60^\circ, y - 60^\circ\}.$$

Proof. Construct a circle circumscribing the three points X , Y , and Z . Then W lies outside of the circle if $y < 120^\circ$, on the circle if $y = 120^\circ$, and inside the circle if $y > 120^\circ$. Consider the first case. When Y moves from Z to X along the arc ZX , clearly, w increases from $y - 60^\circ$ to 60° since the angle of the arc it faces also increases. An analogous argument proves Lemma 13 for the other two cases. $\square$

We may assume without loss of generality that $\theta_i \leq 41.25^\circ$ for $1 \leq i \leq 5$ since otherwise the MST for C_5 is already shorter than M .

Define $\theta_6 = \theta_1$ and $\theta_0 = \theta_5$. By Lemma 13

$$\alpha_i \geq \min\{60^\circ, \theta_{i-1} + 2\theta_i + \theta_{i+1} - 60^\circ\} > \theta_{i+1}.$$

Furthermore, $\alpha_i - \theta_{i+1} \leq \max\{60^\circ, \theta_{i-1} + 2\theta_i + \theta_{i+1} - 60^\circ\} - \theta_{i+1} \leq 63.75^\circ$. Similarly, we can show $\theta < \beta_i - \theta_{i-1} \leq 63.75^\circ$. Since $\cos x$ is concave for $0^\circ \leq x \leq 90^\circ$ and

$$\sum_{i=1}^4 (\alpha_i - \theta_{i+1}) + \alpha_5 - \theta_1 + \beta_1 - \theta_5 + \sum_{i=2}^5 (\beta_i - \theta_{i-1}) = 360^\circ,$$

S' achieves its maximum when

$$\alpha_1 - \theta_2 = \alpha_2 - \theta_3 = \cdots = \alpha_5 - \theta_1 = \beta_1 - \theta_5 = \cdots = \beta_5 - \theta_1 = 360^\circ/10 = 36^\circ.$$

Fig. 16. The range of angle w .

Next note that $\theta_1 + \theta_2 = 180^\circ - \angle CDE > 60^\circ$ and $\theta_1 \leq 41.25^\circ$ implies $\theta_2 > 15^\circ$. Hence

$$\begin{aligned} 0^\circ \leq \alpha_i + \theta_{i+1} - 30^\circ &\leq \max\{30^\circ + \theta_{i+1}, \theta_{i-1} + 2\theta_i + 2\theta_{i+1} - 90^\circ\} \\ &\leq \max\{71.25^\circ, 113.75^\circ\}. \end{aligned}$$

Since $\sin x$ is concave for $0^\circ \leq x \leq 180^\circ$ and

$$\alpha_1 + \theta_2 - 30^\circ + \beta_1 + \theta_5 - 30^\circ + \cdots + \alpha_5 + \theta_1 - 30^\circ + \beta_5 + \theta_4 - 30^\circ = 780^\circ,$$

S'' achieves its maximum when

$$\alpha_1 + \theta_2 - 30^\circ = \beta_1 + \theta_5 - 30^\circ = \cdots = \alpha_5 + \theta_1 - 30^\circ = \beta_5 + \theta_4 - 30^\circ = 780^\circ/10 = 78^\circ.$$

It is easily verified that when $C_5 = A_5$ the conditions on α_i , β_i , and θ_i to maximize S' and S'' are exactly fulfilled and $S = 5(M-1)$.

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