A Criterion for the Affine Equivalence of Cell Complexes in R^d and Convex Polyhedra in R^{d+1} *

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Abstract. A criterion is given that decides, for a convex tiling C of R^d , whether C is the projection of the faces in the boundary of some convex polyhedron P in R^{d+1} . Its applicability is restricted neither by the properties nor by the dimension of C . It turns out to be conceptually simpler than other criteria and allows the easy examination of various classes of cell complexes. In addition, the criterion is constructive, that is, it can be used to construct P provided it exists.

1. Introduction

A j -polyhedron is a convex subset of dimension j of the d -dimensional Euclidean space R^d ($0 \leq j \leq d$) that can be expressed as the intersection of a finite number of closed halfspaces of R^d . The boundary $\text{bd } P$ of a j -polyhedron P consists of finitely many i -polyhedra, for $i < j$, called i -faces of P . A (convex) tiling C of R^d is a finite collection of j -polyhedra (that are accordingly called j -faces or just faces of C) such that

- (i) the faces of C cover R^d , and
- (ii) the relative interiors of the faces of C pairwise do not intersect.

Thus the relative interiors of the faces of a tiling define a convex partition of R^d . We will use interchangeably the terms *cell*, *facet*, *edge*, and *vertex* for d -, $(d-1)$ -, 1 -, and 0 -face, respectively. A tiling C is called a *cell complex in R^d* (or, as also common in the literature, a *face-to-face tiling of R^d*) if the following condition holds additionally for C .

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- (iii) Let f and f' be any two faces of C . If $g = f \cap f' \neq \emptyset$ then g is a face of f , of f' , and of C .

We now identify R^d with some hyperplane h of R^{d+1} and call a tiling C of h *polytopical*, or *affinely equivalent* to a $(d+1)$ -polyhedron P , iff C is the central or parallel projection onto h of the faces of P . Obviously, affine equivalence between C and P implies *combinatorial equivalence*, i.e., the lattices induced by the faces of respective C and P by set inclusion are isomorphic. Since condition (iii) is evident for the faces of P , we deduce that any polytopical tiling is a cell complex. Not any cell complex, however, needs to be polytopical. By a theorem of Steinitz [16], there exists a combinatorially equivalent 3-polyhedron for every cell complex C in R^2 , but (as is easily shown by example) there need not be an affinely equivalent one.

The problem of interest here is a characterization of polytopical tilings of R^d , that is, of polytopical cell complexes in R^d . This issue has been frequently investigated in the mathematical literature. We start with a brief review of the results known to the author.

A criterion that is applicable if a cell complex C in R^d is simple is derived in [10]. C is *simple* if each of C 's vertices is a vertex of exactly $d+1$ cells of C (the minimum possible).

Several authors treat the equivalent problem for complexes on spheres. Let C be a cell complex in the hyperplane h of R^{d+1} , let s denote some sphere in R^{d+1} with $s \cap h = \emptyset$, and let m be the center of s . For each face f of C , define $f' = s \cap \text{conv}(f \cup \{m\})$. ($\text{conv } X$ stands for the convex hull of $X \subseteq R^{d+1}$.) Then $C' = \{f' | f \text{ in } C\}$ is a *spherical complex* on (a hemisphere of) s , and our question now translates to: Is C' the radial projection of the faces of some $(d+1)$ -polyhedron? An answer is given in [17] for $d=2$ and in [15] and [14] for arbitrary $d \geq 1$.

As shown in [14] and in [2], a tiling C of R^d is polytopical iff C can be interpreted as a power diagram. The *power diagram* (synonyma: *Dirichlet cell complex*, *Laguerre-Voronoi diagram*) of a finite set S of spheres in R^d , abbreviated $\text{PD}(S)$, associates each sphere $s \in S$, with center m and radius r say, with its cell

$$\text{cell}(s) = \{x \in R^d | \text{pow}(x, s) \leq \text{pow}(x, t), \forall t \in S\},$$

where $\text{pow}(x, s) = \langle x - m, x - m \rangle - r^2$ denotes the *power function* with respect to s , and $\langle \cdot, \cdot \rangle$ stands for the scalar product in R^d .

If the spheres in S are pairwise congruent then $\text{PD}(S)$ is the well-known *Voronoi diagram* of the set M of centers of the spheres in S . It realizes, for each $m \in M$, the cell containing all points of R^d that are at least as close to m as to any other point in M . A criterion that decides whether a given tiling of R^d is a Voronoi diagram is described in [1] (compare also [4]).

The primary concern of this paper is to develop another criterion for identifying a tiling C of R^d as polytopical. (We recently learned that, for the particular case $d=2$, the sufficiency of the criterion has been shown by Maxwell [13] as early as 1864. More than a century later, its necessity has been proved in [9].) Employing the geometric concepts introduced in Section 2, the general result for $d \geq 1$ is

established in Section 3. We demonstrate that the criterion in fact indicates whether or not there exists a set S of (generalized) spheres in R^d such that $C = \text{PD}(S)$. (For C in R^2 this has been observed independently and in different terms in [1].) Section 4 shows that the criterion can be used in an easy manner to classify several types of common cell complexes. Finally, Section 5 reviews the results obtained and mentions some of their applications.

2. Imaginary Spheres

The fact that the class of polytopical tilings coincides with the class of power diagrams relies on a close relationship between spheres in R^d and halfspaces in R^{d+1} that can be realized by various geometric transforms, see, e.g., [2]. These transforms, however, map all spheres in R^d onto a proper subclass of all halfspaces in R^{d+1} only. To remedy this deficiency, we introduce a more general concept of a sphere.

Recall that a sphere s in R^d with center m and radius r is described by the equality $\langle x - m, x - m \rangle = r^2$. If we allow imaginary values of r , too, then the vectors x satisfying the equality have complex coordinates in general. Nevertheless, the power function $\text{pow}(x, s)$ is real-valued for any $x \in R^d$. (In fact, $\text{pow}(x, s) > 0$ if $r^2 < 0$.) Since we will use a sphere only through its power function, we may as well call the above set of vectors x , for $r^2 < 0$ or $r^2 \geq 0$, a *generalized* or *imaginary sphere*, or *i-sphere* for short. For our purposes it suffices to view an i-sphere s , for $r^2 < 0$, as the sphere s' in R^d with the same center m , with radius $r' = -ir$, and with $\text{pow}(x, s')$ changed to $\langle x - m, x - m \rangle + r'^2$. Imaginary-spheres have already been considered by the famous mathematician Laguerre (see, e.g. [5]). We state some basic properties of i-spheres without proofs.

Lemma 1. *Let s and s' be two i-spheres with distinct centers m and m' . Define $\text{hsp}(s, s') = \{x \in R^d \mid \text{pow}(x, s) \leq \text{pow}(x, s')\}$ and $\text{chor}(s, s') = \text{bd } \text{hsp}(s, s')$.*

- (i) *$\text{chor}(s, s')$ is a hyperplane in R^d perpendicular to the line joining m and m' .*
- (ii) *For $x \in \text{chor}(s, s')$, real c , and $y = x + c(m - m')$, we have $y \in \text{hsp}(s, s')$ iff $c \geq 0$.*

For circles s and s' , $\text{chor}(s, s')$ is known as the *power line*, *radical axis*, or *chordale*. Accordingly, we will term $\text{chor}(s, s')$ in Lemma 1 the *chordale* of the i-spheres s and s' .

Lemma 2. *Let s and h be an i-sphere with center m and a hyperplane in R^d , respectively. For each point $m' \neq m$ on the line perpendicular to h and through m , there is a unique i-sphere s' with center m' and $\text{chor}(s, s') = h$.*

For the sake of completeness of this paper, we generalize the proof of the affine equivalence of power diagrams and polyhedra in [2] to the case of i-spheres. Let h_0 denote the hyperplane $x_{d+1} = 0$ of R^{d+1} . Call a halfspace hsp of R^{d+1} an *upper halfspace* if the point on the x_{d+1} -axis in $+\infty$ is in hsp and $\text{bd } \text{hsp}$ is nonvertical (i.e., not parallel to the x_{d+1} -axis).

Lemma 3. *For every $(d+1)$ -polyhedron P that is the intersection of upper halfspaces there exists some set S of i -spheres in h_0 such that $\text{PD}(S)$ and P are affinely equivalent, and vice versa.*

Proof. For an i -sphere s in h_0 , with center m and radius r say, we define the linear function $g(x, s) = -2\langle x, m \rangle + \langle m, m \rangle - r^2$, and the mapping π that transforms s into the halfspace $\pi(s): x_{d+1} \geq g(x, s)$. Straightforward arguments show that: (i) π is a bijection between i -spheres in h_0 and upper halfspaces of R^{d+1} , and (ii) $g(x, s) \leq g(x, s')$, for any i -sphere s' in h_0 , iff $\text{pow}(x, s) \leq \text{pow}(x, s')$. Let now $P = \bigcap_{i=1}^n \text{hsp}_i$, for upper halfspaces $\text{hsp}_1, \dots, \text{hsp}_n$ of R^{d+1} . Due to (i), $S = \pi^{-1}(\{\text{hsp}_i \mid i = 1, \dots, n\})$ exists, while (ii) implies that the faces of P project vertically to the faces of $\text{PD}(S)$. Since the existence of $P = \bigcap_{s \in S} \pi(s)$ is guaranteed by (i) for any finite set S of i -spheres in h_0 , the proof is complete. $\square$

Lemma 3 immediately implies:

Corollary 1. *The class of polytopical tilings of R^d coincides with the class of power diagrams in R^d .*

3. The Orthogonal Dual

In order to recognize when a tiling C of R^d is the power diagram of some set of i -spheres (or, by Corollary 1, equivalently: when C is polytopical) we now define a structure that is, in a way, dual to C . The *orthogonal dual* $D(C)$ of C is a point-set in R^d that satisfies the following three properties:

- (i) *Duality.* $D(C)$ contains exactly one point p_i for each cell Z_i of C such that distinct points correspond to distinct cells.
- (ii) *Orthogonality.* For distinct cells Z_i and Z_j of C , let $L_{i,j}$ denote the line connecting p_i and p_j . Then $L_{i,j}$ is orthogonal to $Z_i \cap Z_j$.
- (iii) *Orientation.* Any ray parallel to $L_{i,j}$, directed from p_i to p_j , and intersecting both of Z_i and Z_j , first meets Z_i .

Clearly, $D(C)$ is necessarily not unique if it exists since (ii) and (iii) only concern the relative rather than the absolute position of its points. Figure 1 suggests that $D(C)$ may not exist although C is a cell complex. Since $p = L_{i,k} \cap L_{j,k} \cap L_{m,k} = \emptyset$, for any choice of p_i, p_j, p_k valid for Z_i, Z_j, Z_k , we deduce that p_k for cell Z_k does not exist. The reader may convince himself of the following:

Observation 1. p is a point iff the extensions of the three unbounded edges of C do concur.

The concept of the orthogonal dual allows the complete characterization of the class of power diagrams.

Lemma 4. *A tiling C of R^d , for $d \geq 1$, is the power diagram $\text{PD}(S)$ of some set S of i -spheres iff an orthogonal dual $D(C)$ of C exists.*

Fig. 1. Plane cell complex having no orthogonal dual.

Proof. “ $\Rightarrow$ ”: Let S be a set of i -spheres in R^d such that $C = PD(S)$ and $|S|$ is minimal. C has the set M of centers of the i -spheres in S as an orthogonal dual: Duality of M trivially holds by definition of $PD(S)$, and orthogonality and orientation of M are a consequence of Lemma 1(i) and (ii), respectively.

“ $\Leftarrow$ ”: Conversely, assume the existence of an orthogonal dual $D(C) = \{p_1, \dots, p_n\}$ for a tiling C of R^d . We will show that there are i -spheres s_i centered at p_i , for $i = 1, \dots, n$, such that $PD(\{s_1, \dots, s_n\}) = C$. Since the assertion is trivial for $n < 2$, let $n \geq 2$. To aid the presentation, we write $s_i \sim s_j$ iff $Z_i \subseteq \text{hsp}(s_i, s_j)$ and $Z_j \subseteq \text{hsp}(s_j, s_i)$ hold for the cells Z_i and Z_j of C that correspond to p_i and p_j , respectively ($1 \leq i < j \leq n$). Obviously, we have $PD(\{s_1, \dots, s_n\}) = C$ iff $s_i \sim s_j$ for $1 \leq i < j \leq n$. (A set of i -spheres is called *proper* in this case.)

Fact 1. For any radius of s_i , and for any Z_j distinct from Z_i , there exists an i -sphere s_j such that $s_j \sim s_i$.

Proof. By duality and orthogonality of $D(C)$, we deduce from Lemma 2 the existence of s_j (centered at p_j) with $Z_i \cap Z_j \subseteq \text{chor}(s_i, s_j)$. In fact, s_j is unique iff $Z_i \cap Z_j \neq \emptyset$. Moreover, $Z_i \subseteq \text{hsp}(s_i, s_j)$ holds by Lemma 1(ii) and orientation of $D(C)$. This yields $s_j \sim s_i$. $\square$

In proving that Fact 1 suffices for the existence of a proper set of i -spheres, two cases have to be distinguished.

Case 1. $D(C)$ is collinear. Without loss of generality, assume that $p_1, \dots, p_n$ are numbered according to their sorted order on the line containing $D(C)$. Then $Z_{i-1} \cap Z_i$ is a hyperplane of R^d , and Fact 1 implies that the i -sphere s_i with $s_i \sim s_{i-1}$ uniquely exists, for $i = 2, \dots, n$. But $s_i \sim s_j$ trivially holds for $j < i - 1$ (if s_j exists) because of $Z_i \cap Z_j = \emptyset$.

Case 2. $D(C)$ is not collinear such that C contains three cells Z_i, Z_j , and Z_m with $f = Z_i \cap Z_j \cap Z_m \neq \emptyset$ and $1 \leq i < j < m \leq n$. Let $s_j \sim s_i$ and $s_m \sim s_i$. (s_j and s_m

Fig. 2. Tiling with one "dual" point at infinity.

uniquely exist by Fact 1.) Thus $f \subseteq \text{chor}(s_i, s_j) \cap \text{chor}(s_i, s_m)$. Since the chordales defined by three i -spheres in R^d intersect in a $(d-2)$ -dimensional subspace of R^d unless they are parallel, we conclude $f \subseteq \text{chor}(s_j, s_m)$. On the other hand, $f \subseteq Z_j \cap Z_m$ trivially holds. As $\text{chor}(s_j, s_m)$, as well as $Z_j \cap Z_m$, are orthogonal to the line joining p_j and p_m (by Lemma 1(i) and orthogonality of $D(C)$, respectively), we have $Z_j \cap Z_m \subseteq \text{chor}(s_j, s_m)$. Clearly, $Z_j \subseteq \text{hsp}(s_j, s_m)$ is then implied by orientation of $D(C)$, so that $s_j \sim s_m$. In other words, the relation " $\sim$ " is transitive for $f \neq \emptyset$, which is sufficient for the existence of a proper set $\{s_1, \dots, s_n\}$ of i -spheres since $Z_1, \dots, Z_n$ define a tiling C of R^d .

This finally shows that C is a power diagram and thus completes the proof of Lemma 4. $\square$

It is worth mentioning that, if we relax the duality condition so that $D(C)$ may contain multiple points or points at infinity, then there are tilings, e.g., in R^2 that would have an orthogonal dual although they are not even cell complexes, cf. Fig. 2.

Plugging the result of Lemma 4 into Corollary 1 yields a main theorem of this paper.

Theorem 1. *Let C be an arbitrary (convex) tiling of R^d , for $d \geq 1$. C is polytopical iff an orthogonal dual $D(C)$ of C exists.*

4. Polytopical Cell Complexes

Since the property "polytopical" outrules tilings of R^d that are not face-to-face, we shall consider only cell complexes in R^d from now on. Some interesting and frequently investigated types of cell complexes are introduced and, by means of the concept of the orthogonal dual, classified into polytopical or not. Since $D(C)$ trivially exists for any cell complex C in R^1 , $d \geq 2$ is assumed throughout.

Subsections 4.1 and 4.2 deal with two classes of cell complexes that are created by applying a specific construction rule, while in Subsections 4.3 to 4.5 cell complexes are treated that are characterized by local properties of their faces regardless of their generation.

4.1. Order- k Power Diagrams

Let S denote a set of n i -spheres in R^d . In the preceding sections, the concept of the power diagram $PD(S)$ has been introduced. However, $PD(S)$ is not the only cell complex that can be generated by means of S and the power function. The *order- k power diagram* of S , for short k - $PD(S)$, is the cell complex that contains, for each subset S_i of k i -spheres of S , the cell $Z_i = \bigcap \text{hsp}(s, s')$, for all $s \in S_i$ and all $s' \in S - S_i$ ($1 \leq k \leq n-1$). Hence order-1 power diagrams are power diagrams. Corollary 1 implies that 1- $PD(S)$ is polytopical. Interestingly, this property holds for k - $PD(S)$ and $2 \leq k \leq n-1$.

Theorem 2. *Any order- k power diagram in R^d is polytopical.*

Proof. Let k - $PD(S)$ be the order- k power diagram of a set S of n i -spheres in R^d , and let $2 \leq k \leq n-1$ since for $k=1$ there is nothing to prove. Define the symmetric difference $\text{diff}(S_i, S_j)$ of two subsets S_i and S_j of S as $\text{diff}(S_i, S_j) = (S_i - S_j) \cup (S_j - S_i)$. For $\text{diff}(S_i, S_j) = \{s, s'\}$ and s and s' not cocentric, S_i and S_j define in a unique way the hyperplane $h_{i,j} = \text{chor}(s_i, s_j)$ of R^d . On the other hand, each subset S_i of S with $2 \leq |S_i| = k \leq n-1$ corresponds to a cell Z_i of k - $PD(S)$. (By abuse of notation, we shall call Z_i a cell here although Z_i may be *degenerate*, i.e., $Z_i = \emptyset$ or $0 \leq \dim Z_i \leq d-1$; for the dimension $\dim Z_i$ of Z_i .)

Let $S_1, \dots, S_r$ be the $r = \binom{n}{k}$ subsets of S with fixed cardinality k . Since we also take degenerate cells of k - $PD(S)$ into consideration, each face f of a cell Z_i is also a face of some cell Z_j with $|\text{diff}(S_i, S_j)| = 2$. Thus we have $f \subseteq h_{i,j}$. Let now $D'(S)$ denote a point-set in R^d such that: (i) $D'(S)$ contains exactly one point p_i for each S_i , and $p_i \neq p_j$ if $S_i \neq S_j$; (ii) if $h_{i,j}$ is defined then the line $L_{i,j}$ orthogonal to $h_{i,j}$ and through p_j contains p_i ; and (iii) if $\dim Z_i = \dim Z_j = d$ then p_i and p_j satisfy orientation with respect to Z_i and Z_j ($1 \leq i < j \leq r$). Since $Z_i \cap Z_j \subseteq h_{i,j}$, for $1 \leq i < j \leq r$, we observe that the subset $\{p_i \in D'(S) | \dim Z_i = d\}$ of $D'(S)$ is an orthogonal dual of k - $PD(S)$. So, in conjunction with Theorem 1, the existence of $D'(S)$ proves k - $PD(S)$ polytopical.

By induction on n , we show the existence of $D'(S)$ for an arbitrary set $S = \{s_1, \dots, s_n\}$ of i -spheres in R^d and any fixed k , $2 \leq k \leq n-1$. The case $n=3$ is evident. As inductive hypothesis, we assume the existence of $D'(S)$ for some $n \geq 3$ and prove it of $D'(S \cup \{s_{n+1}\})$, for any i -sphere s_{n+1} in R^d . Let $T \subsetneq S$, $|T| = k-1$, and define $S_{n+1} = T \cup \{s_{n+1}\}$. Then any set $S_i = T \cup \{s_i\}$, for $s_i \in S - T$, has the property $|\text{diff}(S_{n+1}, S_i)| = 2$ so that $L_{n+1,i}$ is defined since $h_{n+1,i}$ and p_i are. It is to be shown first that there is a point p_{n+1} for S_{n+1} with $p_{n+1} \in \bigcap_{i \in I} L_{n+1,i}$, for $I = \{i | s_i \in S - T\}$. This clearly is equivalent to showing $p_{n+1} \in g(j, m, t) = \bigcap_{i=j,m,t} L_{n+1,i}$, for any pairwise distinct indices $j, m, t \in I$. But $p_{n+1} \in g(j, m, t)$

holds by straightforwardly generalizing Observation 1 to d dimensions, and because of the hyperplanes $h_{a,b}$ of R^d ($1 \leq a < b \leq n+1$) are chordales so that any three of them concur in a subspace of dimension $d-2$ or are mutually parallel. This proves orthogonality of p_{n+1} with respect to p_j , p_m , and p_t . If there are no distinct $j, m, t \in I$ with $p_{n+1} = g(j, m, t)$ then we deduce $g(j, m, t) = L_{n+1,i}$ for $i = j, m, t$. In this case, p_{n+1} is not unique, but can be chosen on $L_{n+1,i}$ so as to satisfy duality and orientation. Otherwise, p_{n+1} is obviously distinct from p_j , p_m , and p_t . Moreover, if S_{n+1} defines a cell Z_{n+1} of k -PD($S \cup \{s_{n+1}\}$) with $\dim Z_{n+1} = d$ then the convexity of Z_{n+1} ensures the proper orientation of p_{n+1} with respect to p_j , p_m , and p_t . Hence we conclude $D'(S \cup \{s_{n+1}\}) = D'(S) \cup \{p_{n+1}\}$.

Since no restrictions have been drawn on S and k , this proves the existence of an orthogonal dual for any order- k power diagram in R^d , and thus Theorem 2. $\square$

An order-3 power diagram for eight circles in R^2 that realizes 18 cells is depicted in Fig. 3. For any two cells that share a common edge, their points in the corresponding orthogonal dual are connected by broken line segments. The reader is encouraged to check duality, orthogonality, and orientation for this particular example.

Theorem 2 and Corollary 1 imply that any order- k power diagram is an (order-1) power diagram. We pose the question if the converse is also true: Is there a set S_* of i -spheres for every set S of i -spheres in R^d such that k -PD(S_*) = PD(S) for $2 \leq k \leq n-1$? The question is settled affirmatively in [2] for the particular case $k = n-1$. In this context, we mention:

Conjecture 1. Any set S_* of n i -spheres in R^d (the centers of which are not collinear) defines at least one chordale that yields $\min\{k, n-k\}$ facets of k -PD(S_*), for $1 \leq k \leq n-1$.

Fig. 3. Order-3 power diagram and its orthogonal dual.

If Conjecture 1 proves true then the answer to the above question is no, for $2 \leq k \leq n-2$. $PD(S)$ would realize at least two facets that are contained in the same hyperplane of R^d for any choice of S , otherwise.

In Section 1, *Voronoi diagrams* are introduced as particular power diagrams. A direct consequence of Theorem 2 concerning this important class of cell complexes should be mentioned here. If we generalize Voronoi diagrams to order- k in a matter similar to power diagrams, we obtain:

Corollary 2. *Let M be any finite set of points in R^d . The order- k Voronoi diagram of M is polytopical.*

4.2. Arrangements of Hyperplanes

A (*hyperplane*) *arrangement* is the cell complex that represents the dissection $A(H)$ of R^d induced by a finite set H of hyperplanes of R^d . It is well known that arrangements in R^d are combinatorially equivalent to polyhedra that are dual to zonotopes in R^{d+1} ; see [6], [8]. A *zonotope* is a polytope (i.e., a bounded polyhedron) with centrally symmetric faces. Two $(d+1)$ -polyhedra P and P' are *dual* if there is a bijective mapping Δ between the j -faces of P and the $(d-j)$ -faces of P' ($0 \leq j \leq d$) such that $f \subseteq g$, for any two faces f and g of P , iff $\Delta(f) \supseteq \Delta(g)$. We shall prove:

Theorem 3. *Any arrangement in R^d is polytopical.*

Proof. Let $H = \{h_1, \dots, h_n\}$ be a set of hyperplanes in R^d , and let $\langle a_j, x \rangle = c_j$ be the equation describing h_j , for $j = 1, \dots, n$. Without loss of generality, $c_j \neq 0$ is assumed ($j = 1, \dots, n$). For each cell Z_i of $A(H)$, we define the signum function $\sigma_j(Z_i) = -1$ if Z_i and the origin are contained in the same halfspace of R^d bounded by h_j , and $\sigma_j(Z_i) = 1$, otherwise. This allows to associate Z_i in a unique way with the point $p_i = \sum_{j=1}^n \sigma_j(Z_i) a_j$. As a matter of fact, $D = \{p_i | Z_i \text{ is a cell of } A(H)\}$ is an orthogonal dual of $A(H)$: Consider two cells Z_1 and Z_2 such that $Z_1 \cap Z_2 \subseteq h_j$ and $\dim Z_1 \cap Z_2 = d-1$. Obviously, $p_2 = p_1 - 2a_j$ holds. It is an easy analytic exercise now to prove the existence of position vectors $a_1, \dots, a_n$ for $h_1, \dots, h_n$ such that duality and orientation of D is met. Clearly, orthogonality of D is satisfied, too, such that the proof is complete. $\square$

The arrangement induced by four lines in R^2 and its orthogonal dual drawn in Fig. 4 illustrate the construction. It can also be shown directly that $A(H)$ is polytopical by exhibiting a particular $(d+1)$ -polyhedron that is affinely equivalent to $A(H)$.¹ Let the hyperplane $h_0: x_{d+1} = 0$ of R^{d+1} contain $A(H)$. Then the $(d+1)$ -polyhedron $P(H)$ bounded by the surface $x_{d+1} = \sum_{j=1}^n |\langle a_j, x \rangle - c_j|$ projects vertically onto $A(H)$. (The easy proof is left to the interested reader.)

¹ As pointed out by one of the referees.

Fig. 4. Line-arrangement with orthogonal dual.

We thus have seen that order- k power diagrams as well as arrangements in R^d are the projected boundaries of particular convex polyhedra in R^{d+1} . Here is the right point to note that these both classes of cell complexes are directly interrelated via projection: For any set S of n i -spheres in h_0 and for g the linear function defined in the proof of Lemma 3, $1\text{-PD}(S), \dots, (n-1)\text{-PD}(S)$ can be obtained by vertically projecting onto h_0 certain faces of $A(\{g(x, s) | s \in S\})$ in R^{d+1} . (For details, consult [2].)

In the remainder of Section 4, three classes of cell complexes that are defined via certain properties of their faces are examined.

4.3. Simple Cell Complexes

Now we turn our attention to so-called *simple cell complexes*. (Also, the notation *primitive face-to-face tilings* is commonly used.) Note that their definition given in Section 1 is equivalent to that each j -face of the complex is also a face of exactly $d - j + 1$ cells. As a matter of fact, every simple cell complex in R^d is polytopical if $d \geq 3$ [10]. An alternative proof is given below that seems to be preferable because it is conceptually easier.

Theorem 4. *Every simple cell complex in R^d , $d \geq 3$, is polytopical.*

Proof. Let C denote an arbitrary simple cell complex of R^d . We first introduce a certain ordering $Z_1, \dots, Z_n$ of C 's cells and then show the existence of an orthogonal dual $D(C) = \{p_1, \dots, p_n\}$. Without loss of generality, let C contain at least one vertex (which implies $n \geq d + 1$) since the problem can be treated in R^{n-1} , otherwise.

Define $Q_i = \bigcup_{j=1}^i Z_j$, let v be some vertex in the boundary $\text{bd } Q_i$ of Q_i , and denote by $e_1 \dots e_i$ the edges in $\text{bd } Q_i$ that have v as one vertex ($d + 1 \leq i \leq n$).

Q_i is *concave at v* if the convex hull $\text{conv}\{e_1, \dots, e_t\}$ is not contained in Q_i . Since C is simple, there is a unique cell of C that contains all faces f in $\text{bd } Q_i$ with $v \in f$, provided Q_i is concave at v . This cell is said to be *proper with respect to Q_i* (at v). An ordering is now imposed on the cells of C as follows: $Z_1, \dots, Z_d$ are chosen such that they share a common edge. For $i = d+1, \dots, n$, Z_i is chosen proper with respect to Q_{i-1} and so that Q_i is simply connected. (The easy proof that Z_i always exists is omitted.)

We are now prepared for an iterative construction of $D(C)$. For $1 \leq i < j \leq n$, use the notations $f_{i,j}$ for the facet $Z_i \cap Z_j$ (if existent) and $L_{i,j}$ for the line orthogonal to $f_{i,j}$ and through p_j . As easily seen, points $p_1, \dots, p_d$ that satisfy duality, orthogonality, and orientation exist. It remains to deduce the existence of $p_i \in D(C)$ from $\{p_1, \dots, p_{i-1}\} \subseteq D(C)$, for $d+1 \leq i \leq n$. Let F_i denote the polyhedral surface $Q_{i-1} \cap Z_i$, and define an *inner vertex* of F_i as a vertex in $F_i - \text{bd } Q_i$.

Claim 1. For any inner vertex v of F_i , $g = \bigcap_{v \in f_{i,j}} L_{i,j}$ is a point.

Proof. Since C is simple, F_i contains exactly d edges e of Z_i that emanate from v . Each e belongs to exactly $d-1$ cells $Z_t \neq Z_i$ whose points p_t lie in a common hyperplane h_e orthogonal to e . However, $p = \bigcap_{v \in e} h_e$ is a point so that $L_{i,t} \subseteq h_e$ implies $p = g$. $\square$

Claim 2. The graph G consisting of the inner vertices of F_i and the edges joining them is connected.

Proof. Assume that G splits into two connected subgraphs G' and G'' . Then $F_i = F' \cup F''$ holds for the union F' (F'') of all facets of F_i that contain a vertex of G' (G''). Now observe that F_i is a connected set as Q_i is simply connected due to the cell ordering imposed. Hence $g = F' \cap F''$ has to be a $(d-2)$ -face. g has no inner vertex which implies that Q_{i-1} is not concave at any vertex of g . For the facets f' of F' and f'' of F'' with $g = f' \cap f''$, their union $f' \cup f''$ thus is not a subset of Z_i . Z_i would not be convex, otherwise. But this contradicts $f' \cup f'' \subseteq F_i$. $\square$

Based on Claims 1 and 2 we finally demonstrate that $\bigcap_{f_{i,j} \in F_i} L_{i,j}$ is a point p_i . This suffices for the orthogonality of $\{p_1, \dots, p_i\}$. Moreover, duality and orientation is guaranteed by the convexity of Z_i so that $p_i \in D(C)$ follows. By Claim 2 we only have to show that, for any two vertices v and v' that define an edge e of G , the (due to Claim 1 uniquely existing) points $p = \bigcap_{v \in f_{i,j}} L_{i,j}$ and $p' = \bigcap_{v' \in f_{i,t}} L_{i,t}$ coincide. e is contained in exactly $d-1$ facets of F_i , hence $j = t$ occurs for $d-1$ values of j . Now $p = p'$ results from $d \geq 3$. (It should be noted that the last conclusion is wrong for $d = 2$. For $j = a, b$ and $t = b, c$, $L_{i,c}$ is not forced to pass through $L_{i,a} \cap L_{i,b}$ by Claim 1.)

This completes the proof of Theorem 4 $\square$

Fig. 5. Nonpolytopal three-dimensional complex.

Figure 1 reveals that simple cell complexes in R^2 are, in general, not polytopal. To establish the existence of nonpolytopal (and thus, by Theorem 4, nonsimple) cell complexes in R^3 that contain at least one vertex, let us study Fig. 5. We denote by $1, \dots, 7$ the unbounded edges of the complex C shown, by a, b, c the edges of the shaded triangle t , with e' the unbounded edge that is the prolongation of e , for $e = 4, 5, 6$, and by (x, y) the facet of C spanned by the distinct edges x and y . The construction of C starts with triangle t and edges 4, 5, and 6 orthogonal to t . Facets $(3, 4)$, $(5, 7)$, and $(1, 6)$ can be chosen such that (*) the three planes containing them have an empty intersection, and that further $a \subset (3, 7)$ and $b \subset (1, 7)$. Observe now that 3 does not lie in the plane through $(1, c)$ because of (*). But we can select an edge 2 in this plane such that the cell below $(2, 3)$ and $(3, 4)$ is convex. However, only two facets meet in 2 so that the faces integrated so far define no cell complex. To remedy this, facets $(2, 4')$, $(3, 4')$, $(7, 5')$, $(1, 6')$, $(4', 5')$, $(4', 6')$, and $(5', 6')$ (that obviously exist) are added which completes C .

Now let h be the plane behind and parallel to t . Since the facets that separate the four cells intersecting h are all orthogonal to h , the subset D' of $D(C)$ that corresponds to these cells can, without loss of generality, be assumed to lie in h . Consequently, $D(C') = D'$ holds for $C' = C \cap h$. However, D' violates orthogonality since, due to (*), C' is just the cell complex in Fig. 1. This implies that $D(C)$ does not exist. Compare also Fig. 6 that suggests an example of a nonpolytopal cell complex in R^3 which also contains bounded cells.

4.4. Complexes of Unbounded Cells

Let C be a cell complex in R^d such that each cell of C is unbounded. For $d \geq 3$, we learn from Theorem 4 that C is polytopical provided its simplicity, and from Fig. 5 that this is not true any longer if the simplicity condition is dropped. In addition, Fig. 1 and the comments on Fig. 6 suggest that a cell complex in R^2 (whether simple or nonsimple) is not polytopical, in general. Nevertheless, the following weaker assertion holds true.

Theorem 5. *If all cells of a two-dimensional cell complex C are unbounded then C is polytopical.*

Proof. We assume that the (without loss of generality at least three) cells of C are unbounded and provide an ordering of C 's vertices that implies an iterative argument for the existence of $D(C)$. Notice first that the graph G induced by the set $\text{vert } C$ of vertices of C and the bounded edges of C is a *tree*, i.e., G contains no cyclic sequence of edges (since there are no bounded cells). For some vertex v of G , the *preorder of* $\text{vert } G = \text{vert } C$ (at v) is defined as follows: first v and then $\text{vert } G_w$ in preorder at w , for all vertices w adjacent to v (i.e., connected to v by an edge), and for G_w being the subtree of G defined by w . (For graphs and, in particular, trees consult, e.g., [12].)

Now let $v_1, \dots, v_m$ be such that v_i is the i th vertex of G in preorder ($i = 1, \dots, m$). It is obvious that points in $D(C)$ exists for the at least three cells that have v_1 as a vertex. For each v_i , $i = 2, \dots, m$, the following situation occurs: By definition of a preorder, v_i is adjacent to exactly one vertex v_j with $j < i$. All cells containing v_j , two of which also contain v_i , have got their dual points assigned. But is not difficult to verify that dual points for all cells that contain v_i can be constructed if two points are fixed in advance. This proves C polytopical. $\square$

If C contains bounded cells, no ordering of $\text{vert } C$ will avoid that, for at least one vertex v_i , the dual points of all cells containing v_i are fixed before v_i is encountered.

4.5. Simplicial Complexes

Instead of R^d , e.g., some d -polytope Q can also be taken as the underlying space of a cell complex C . In this case, C is called a *complex in* Q . C is said to be *simplicial* if each j -face of C is a j -simplex, that is, the convex hull of $j+1$ affinely independent points ($0 \leq j \leq d$).

Simplicial complexes and simple cell complexes can be related to each other by means of the orthogonal dual. Let C' denote any polytopical cell complex in R^d and let $D(C) = \{p_1, \dots, p_n\}$. C is defined as the complex in $\text{conv } D(C')$ that

contains the vertex $p_i \in D(C)$ for each cell Z_i of C' , and the j -face $\Delta(f) = \text{conv}\{p_1, \dots, p_t\}$ for each $(d-j)$ -face $f = \bigcap_{i=1}^t Z_i$ of C' ($1 \leq j < t \leq d$). Now observe that C is simplicial iff C' is simple, and that $\text{vert } C'$ is an orthogonal dual of C . If the property "polytopical" is restricted to $\text{conv } D(C')$ which means that there exists a $(d+1)$ -polyhedron such that a subset of its faces projects onto C , we obtain:

Observation 2. A simplicial complex C in some d -polytope is polytopical if $\text{vert } C$ is an orthogonal dual of a simple cell complex in R^d .

The existence of nonpolytopical simplicial complexes in R^2 (i.e., *triangulations* of bounded convex polygons) is shown in [7]. Their example can be used to construct a nonpolytopical *tetrahedralization* C of the 3-polytope illustrated in Fig. 6.

Let T be the triangulation of the shaded triangle, let h be the plane that contains T , and let v and w denote the two vertices of C not in h . Each triangle t_i of T defines two tetrahedra $Z_i = \text{conv}\{t_i, v\}$ and $Z'_i = \text{conv}\{t_i, w\}$ of C ($i = 1, \dots, 7$). Assume that $D(C)$ exists and let $p_i \in D(C)$ correspond to Z_i . If $Z_i \cap Z_j$ is a facet then p_i and p_j lie in a plane orthogonal to h ($1 \leq i < j \leq 7$). Thus the vertical projection D' of $\{p_1, \dots, p_7\}$ onto h is an orthogonal dual of T . However, D' can be shown to violate orientation. This contradiction proves that $D(C)$ does not exist.

Fig. 6. Nonpolytopical tetrahedralization.

5. Concluding Remarks

Let us review the contents of this paper. First of all, a conceptually simple criterion that indicates whether or not an arbitrary convex tiling of R^d is the projection of the faces of a convex polyhedron in R^{d+1} is outlined. Although independently discovered, the criterion turned out to be previously known for the case $d = 2$. In this paper, the criterion is derived in a natural way by means of power diagrams for generalized spheres. Another goal of the paper is to demonstrate the usefulness of the criterion for proving that various types of known cell complexes are polytopical (cf. Theorems 2 to 5), as well as for providing examples of certain nonpolytopical complexes (cf. Figs. 5 and 6).

Beside the theoretical interest, the existence of an affinely equivalent polyhedron can be exploited to construct particular polytopical cell complexes that have applications in computer science and other areas (computation of weighted Voronoi diagrams [2] and order- k Voronoi diagrams [3]). Also the inverse problem has practical applications: If the facial structure of a cell complex C in R^d given explicitly, decide whether C is polytopical and, if yes, determine an affinely equivalent $(d + 1)$ -polyhedron. In a companion paper [4], both problems that, e.g., arise in examining plane frameworks [13], [9] and point-location problems [11], respectively, are solved efficiently by means of our criterion.

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