On Lattices with Möbius Function $\pm 1, 0$

Jeff Kahn*

Department of Mathematics and Center for Operations Research, Rutgers University,
 New Brunswick, NJ 08903, USA

Abstract. We provide simple arguments of a geometric nature to explain why the Möbius functions of certain lattices take only the values $-1, 0, 1$.

1. Introduction

With a collection K of nonempty subsets of a finite set A we associate two larger collections,

$$L_1(K) = \left\{ \bigcup_{X \in M} X : M \subseteq K \right\}$$

and

$$L_2(K) = \{\phi\} \cup \{X : \exists Y \in K, Y \subseteq X\}.$$

Each of these, ordered by $\subseteq$, is a lattice (note $\phi \in L_1(K)$). Our interest here is in the Möbius functions of these lattices for various types of K . (For background on Möbius functions see [1] or [13].)

Let us note at the outset that the cross-cut theorem of [13] implies:

- (a) for any K , $\mu(L_1(K)) = \mu(L_2(K))$, which common value we denote simply $\mu(K)$; and
- (b) if K_0 is the set of minimal elements of K then $\mu(K) = \mu(K_0)$.

In view of (b) we may confine our attention to K 's which are *antichains* (i.e., $X, Y \in K$ and $X \neq Y \Rightarrow X \not\subseteq Y$). On the other hand, (a) allows us to deal with either $L_1(K)$ or $L_2(K)$ as the situation requires; we will find both viewpoints useful in what follows.

* Supported in part by NSF Grant MCS83-01867 and by a Sloan Research Fellowship.

A number of recent papers are concerned with showing that in various circumstances these Möbius functions take only the values -1 , 0 , and 1 . (See [6], [7], [9], and [15]–[18] but note that of these only [6] and [9] deal with Möbius functions explicitly.) We begin by describing two such results.

Let $N = (V, A)$ be a network (=digraph, see, e.g., [11]) with a distinguished $s \in V$ and $T = \{t_1, \dots, t_k\} \subseteq V$, and

- (a) for each t_i a threshold $d_i \in \mathbb{N} \cup \{\infty\}$,
- (b) for each $e \in A$ a length $l(e) \in \mathbb{N}$.

An (s, t_i) -path is *short* if the sum of the lengths of its arcs is at most d_i , and *long* otherwise. A set $X \subseteq A$ is *good* if it contains short (s, t_i) -paths for all i , and *bad* otherwise. We take our K to be the collection of minimal good subsets of A . (Consideration of these objects and the associated lattices arises naturally, most often with all thresholds infinite, in the study of network reliability. See, e.g., [15] and [9].) The following theorem generalizes results of [7] and [15]–[18].

Theorem 1 [9]. *With the above notation:*

- (a) $\mu(K) = 0$ if either
 - (i) $A \neq \bigcup_{X \in K} X$, or
 - (ii) A contains a directed circuit, or
 - (iii) A contains a long (s, t_i) -path for some i ,
- (b) $\mu(K) = (-1)^{|V|+|A|}$ otherwise.

The second result, due to Greene [6], is easier to state.

Theorem 2. *If K is any collection of intervals from $\{1, \dots, n\}$ then $\mu(K) \in \{-1, 0, 1\}$.*

It was, indeed, a desire to see whether or to what extent Theorems 1 and 2 were related which stimulated the work discussed here. As it turns out (see Section 3) they do not appear to be, as one might wish, different manifestations of a single phenomenon. What our investigation did produce on the positive side was two quite satisfying and, in different degrees, geometric methods for proving these results (and in the case of Theorem 1 some generalizations), and it is with these methods that we concern ourselves here.

In Section 2 we give a novel approach based on the notion of evasiveness (reviewed below) which can be used to prove Theorems 1(a) and 2. As observed at the end of that section, these results have a geometric interpretation in terms of collapsibility properties of simplicial complexes associated with the lattices in question.

In Section 3 we present our candidate for the “right” proof of Theorem 1(b), showing that here $L_1(K)$ is in fact the face lattice of a polytope, and we indicate how these results may be generalized to $\mathbb{R}$ -representable oriented matroids, which may be their correct level of generality.

In the interests of brevity, and since our purpose is mainly to indicate some connections, details of proofs have been kept to a minimum; but the proofs are in any case fairly easy once begun correctly, and the reader should have little difficulty supplying what we have omitted.

2. Evasiveness

In this section we deal with the lattices $L_2(K)$ for various K . Let us note, to begin, that since each $X \in L' := L_2(K) \setminus \{\phi\}$ satisfies

$$\mu_{L_2(K)}(X, A) = (-1)^{|A \setminus X|},$$

we have

$$\mu(K) = \mu(L_2(K)) = (-1)^{|A|}(k_0 - k_e),$$

where k_0 and k_e are, respectively, the numbers of odd and even subsets in L' . In particular, $\mu(K) = 0$ iff L' is *balanced*, i.e., contains equal numbers of odd and even sets. From this we easily conclude that $\mu(K) = 0$ whenever $A \neq \bigcup_{X \in K} X$, and observation which is in any case immediate from consideration of $L_1(K)$.

We will prove Theorems 1(a) and 2 using the algorithmic notion of evasiveness, for which we now briefly recall the relevant definitions. (Further information may be found, e.g., in Chapter 8 of [3] under the name “elusiveness.”) Suppose G is a family of subsets of a finite set A , and X an unknown subset of A . We are assigned the task of determining whether X belongs to G by asking a sequence of questions (of an oracle, say) of the form “is $x \in X$?” (This is referred to in what follows simply as “asking x .”). If there exists a strategy whereby we can always decide membership in G without asking all $|A|$ questions, then G is said to be *nonevasive*. Otherwise it is *evasive*. (This is often best thought of as an adversary situation: the oracle’s answers are chosen to make our task as difficult as possible.)

For instance if A is the set of unordered pairs from $\{1, \dots, n\}$ with $X \in G$ iff the pairs in X form a connected subgraph of the complete graph K_n , then the reader might try to verify that G is evasive.

For the purposes of the present discussion the key fact is:

Proposition. *If G is nonevasive, then it is balanced.*

This result, which is easily proved inductively, was first observed in [2] and [12].

Now let $N = (V, A)$ be a network as described in Section 1 and let G be the collection of good subsets of A . As noted above, part (i) of Theorem 1(a) is obvious. In view of the preceding remarks we will have parts (ii) and (iii) if we show:

Theorem 3. *If A contains either a directed circuit or for some i a long (s, t_i) -path, then G is nonevasive.*

Proof. Assume first that C is a directed circuit of A . The reader may easily verify that the following questioning strategy avoids at least one question. First ask all arcs not belonging to C . If there is at least one path from s to C all of whose arcs have been answered yes, and if $x \in C$ is the terminal vertex of a shortest such path, then there is no need to ask the arc of C entering x . But if there is no such x then there is no need to ask *any* arc of C .

We may thus assume that A is acyclic and contains at least one long (s, t_i) -path. The strategy is now as follows. Choose a longest such (s, t_i) -path P and ask all arcs not in P . Let Y be the set of arcs which have been answered yes, x the first vertex of P which can be reached in $Y \cup P$ by a path other than $P[s, x]$, and α the arc of P entering x . If there is no such x then $Y \cup P$ is bad and we are done. Otherwise it is not hard to see that α is irrelevant, since in fact for any $Y \subseteq B \subseteq Y \cup (P \setminus \alpha)$,

$$d_B(s, x) = d_{B \cup \alpha}(s, x). \quad \square$$

We turn next to Theorem 2. In this case, atypically, evasiveness gives the result even when $\mu(K) \neq 0$. Recall that here K is a collection of intervals from $\{1, \dots, n\}$ and we must show that $\mu(K) = \mu(L_2(K)) \in \{-1, 0, 1\}$, or equivalently that

$$|k_0 - k_e| \leq 1.$$

Of course we may assume that K is an antichain, and so may speak unambiguously of one member of K being to the left of another.

Again, we are trying to decide whether an unknown X belongs to $L' = L_2(K) \setminus \{\phi\}$ by asking elements of $\{1, \dots, n\}$. We will describe a questioning strategy for which either:

- (α) every sequence of answers allows us to decide membership in L' without asking all questions (so L' is nonevasive), or
- (β) there is a unique sequence of answers to our first $n-1$ questions which forces us to ask all questions; if Y is the set of elements receiving yes answers under this sequence, then Y is a maximal nonelement of L' and $L' \cup \{Y\}$ is nonevasive.

Our questioning proceeds in stages. For the i th stage let $k = k(i)$ be the least as yet unasked element of $\{1, \dots, n\}$. (For example, $k(1) = 1$.)

Let I be the leftmost interval satisfying

- (a) $k \in I \in K$, and
- (b) if $j \in I$ and $j < k$ then (note j has been asked in an earlier stage) the answer to j was yes.

It is easy to see that if no I satisfies (a) and (b) then there is no need to ask k . So for any sequence of answers which will force us to ask all questions, I must exist. Let m be the largest element of I . We first ask $k+1, \dots, m$. (If $m = k$ this step is null.) If any of these is answered no then it is no longer necessary to ask k , as it is no longer possible that X contains any I' with $k \in I' \in K$. We may thus assume that all of these questions are answered yes. If $m < n$ we now ask k and see that only a no answer forces us to ask additional questions (a yes answer would give $I \subseteq X$). If $m = n$ then we have asked $n-1$ questions and the procedure halts.

As we have indicated, there is at most one sequence of answers for which our first $n-1$ questions do not suffice to decide whether $X \in L'$. If there is no such sequence then L' is nonevasive and $\mu(K) = 0$.

Suppose, on the other hand, that there is such a sequence, say σ . Since σ answers a question no only when a yes answer would place the current set of elements which have received yes answers in L' , the set Y of elements to which σ answers yes is a maximal nonmember of L' . Furthermore, if we repeat the same sequence of questions relative to the family $L'' = L' \cup \{Y\}$ then it is easy to see that:

- (i) The first $n-2$ answers of σ are the only ones which can prevent us from deciding whether $X \in L''$ in the first $n-2$ question.
- (ii) If the first $n-2$ questions are answered as in σ , then if the $(n-1)$ st answer in σ was no we already have $Y \subseteq X$; while if the $(n-1)$ st answer in σ was yes then a yes answer to our $(n-1)$ st question implies $Y \subseteq X$ (so $X \in L''$), and a no implies $X \notin L''$.

Thus L'' is nonevasive and it follows that $\mu(K) \in \{-1, 1\}$.

This completes the proof of Theorem 2. □

It would be interesting to know whether a proof of Theorem 1(b) could also be given along these lines.

In order to interpret the results of this section geometrically we recall the fact, first observed in [8] that if a family G of subsets of S is an (abstract) simplicial complex (i.e., $X \subseteq Y \in G \Rightarrow X \in G$), then nonevasiveness implies collapsibility of G (and hence contractibility of its geometric realization).

If we write $\bar{L}'$ for the set of complements of members of L' , then the nonevasiveness of $\bar{L}'$ is easily seen to be equivalent to nonevasiveness of L' , so we have collapsibility of $\bar{L}'$ in Theorem 1(a) and those cases of Theorem 2 in which $\mu(K)=0$. (In case (i) of Theorem 1, which is not covered by the present discussion, $\bar{L}'$ is of course a cone.) For those cases of Theorem 2 in which $\mu \neq 0$, a little further consideration along the lines of [8] (and using the fact that Y is a maximal nonelement of L') shows that $\bar{L}'$ collapses to the boundary of a simplex (with vertex set $\{1, \dots, n\} \setminus Y$).

3. Polytopes

Recall ([14]) that if L is the lattice of faces of a d -dimensional (convex) polytope, then

$$\mu(L) = (-1)^{d-1}.$$

In this section we prove Theorem 1(b) by showing:

Theorem 4. *If N is a network as described in Section 1 with K its associated collection of minimal good sets, and if none of (i)–(iii) of Theorem 1 occurs, then $L_1(K)$ is the face lattice of a polytope of dimension $|A| - |V| + 1$.*

Actually we will be proving something considerably more general. Although we do not wish to get involved in the requisite definitions, for the reader versed in oriented matroids we briefly outline what we have in that direction. (Our notation here follows [10].) Let $M = (E, \mathcal{O})$ be an oriented matroid and Y a distinguished subset of E satisfying:

- (i) every element of E is in a positive circuit (=circuit with all signs positive), and
- (ii) every positive circuit uses exactly one element of Y .

Let K consist of the minimal subsets of E which can be written as $\bigcup_{y \in Y} C(y)$ with $C(y)$ a positive circuit on y . Then our proof of Theorem 4 actually gives:

Theorem 5. *If M is representable over $\mathbb{R}$ then $L_1(K)$ is the face lattice of a d -dimensional polytope, where $d = |E \setminus Y| - \text{rank}(M)$.*

We do not know whether anything along these lines is true for general oriented matroids, though the results of Cordovil *et al.* [19] are certainly suggestive. We also do not know whether there is any connection between this theorem and the work of Cordovil [4] or Edelman [5].

We should also mention that it is possible to give a complete proof of Theorem 1 using the present approach, but the details become more cumbersome and we have chosen to omit them.

Finally, before beginning, we remark that in Theorem 2 it is possible to have $\mu(F) \in \{-1, 1\}$ without $L_1(F)$ being a polytope lattice (e.g., take $n = 5$ and $F = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$), which would seem to say that Theorems 1 and 2 really are different.

Our proof of Theorem 4 is based on:

Lemma. *Let P be a convex polytope and $Y \subseteq P$ such that*

$$(*) \quad \text{for each facet } F \text{ of } P, |Y \setminus F| = 1.$$

Let $\Delta = \{F: F \text{ a face of } P, F \cap Y = \emptyset\}$. Then Δ (ordered by inclusion) is isomorphic to the poset of faces of ∂Q where Q is the projection of P in the direction of $\text{aff}(Y)$. (Note ∂Q denotes the boundary of Q .)

Remark. It is natural to wonder whether the situation described by (*) can occur in any but some quite special situations (the simplest being that Y consists of a single point in the interior of P). That it occurs more often than one might expect will appear below.

Proof of Lemma. Let $A = \text{aff}(Y)$. We may assume $0 \in A$. Say an (affine) hyperplane H is a Q -hyperplane if H is a supporting hyperplane of P parallel to A . Then the poset of faces of ∂Q is isomorphic to

$$\{H \cap P: H \text{ a } Q\text{-hyperplane}\}$$

(ordered by inclusion). Moreover, it is clear that if H is a Q -hyperplane then $H \cap P \in \Delta$, so we only have to show:

(1) If $F \in \Delta$ then $F = H \cap P$ for some Q -hyperplane H .

This in turn is easily seen to follow from:

(2) If $F \in \Delta$ then $\text{aff}(F)$ is skew to A .

(Note we say the affine subspaces A, B are *skew* if $A \cap B = \emptyset$ and $(A - A) \cap (B - B) = 0$.)

Suppose $Y = \{y_1, \dots, y_k\}$ and let $F \in \Delta$. Since F is an intersection of facets of P , for each y_i there is some facet F_i with $y_i \in F_i \supseteq F$. Note that $y_j \in F_i, \forall j \neq i$, so in particular the F_i are all distinct. Let $H_i = \text{aff}(F_i)$ and note $y_i \notin H_i$ (since $H_i \cap P = F_i$). Now for $i = 1, \dots, k$ we have

$$y_i \in \left(A \cap \bigcap_{j=1}^{i-1} H_j \right) - \left(A \cap \bigcap_{j=1}^i H_j \right),$$

whence

$$\dim \left(A \cap \bigcap_{j=1}^k H_j \right) \leq \dim(A) - k \leq -1.$$

From this follow:

- (i) $\text{aff}(F) \cap A = \emptyset$, and
- (ii) there is no direction common to $\text{aff}(F)$ and A (for we know $A \cap \bigcap_{j=1}^{k-1} H_j = \{y_k\}$, which could not happen if there were some direction common to $A, H_1, \dots, H_{k-1}$).

This proves (2) and lemma. $\square$

We are now ready to prove Theorem 4. Let $N = (V, A)$, $s, T = \{t_1, \dots, t_k\}$, and K be as in Section 1, and assume that none of (i)–(iii) of Theorem 1 occurs. Form a new network $N' = (V, A \cup Y)$ by adding new arcs $y_i = (t_i, s)$ for $i = 1, \dots, k$ and setting $Y = \{y_1, \dots, y_k\}$. In the absence of (i)–(iii) (note especially the relevance of (iii)) we have a natural, union-preserving bijection (via $Z \leftrightarrow Z \cup Y$) between $L_1(K)$ and

$$L := \{W \subseteq A \cup Y: W \text{ is a union of directed circuits of } N'; Y \subseteq W\}.$$

(Note this is understood to include the empty set.) Recall (see, e.g., [10]) that if D is a directed graph in which every arc is contained in a directed circuit then it is possible to identify the arc set (say X) of D with a set of points in $\mathbb{R}^d$ ($d = |X| - |V(D)|$) such that, setting $P = \text{conv}(X)$, we have $\dim(P) = d$ and

- (C) $X' \subseteq X$ is a directed circuit of D iff there is a (necessarily unique) facet F of P for which $X' = X \setminus F$.

(The map $X \hookrightarrow \mathbb{R}^d$ is just an affine dependence representation of the dual of the oriented matroid associated with D .)

If this construction is applied with $D = N'$, then it is easy to see that the resulting polytope P and the arc set Y (regarded now as a subset of P) satisfy

condition (*) of the lemma. (Note this uses the exclusion of (ii).) Letting Δ and Q be as in the lemma we see, moreover, that the correspondence (C) induces an order-reversing bijection between $L \setminus \{\phi\}$ and Δ , and so L is isomorphic to the face lattice of the dual polytope Q^* of Q .

To complete the proof of Theorem 4 we need only note that, as $\dim(P) = |A \cup Y| - |V|$ and the elements of Y are affinely independent (immediate from (*)), we have $\dim(Q) = |A| - |V| + 1$.

Acknowledgments

I thank Dean Sturtevant for several stimulating discussions and the referee for his careful reading of the manuscript and many helpful suggestions.

References

1. M. Aigner, *Combinatorial Theory*, Springer-Verlag, New York, 1979.
2. M. R. Best, P. van Emde Boas, and H. W. Lenstra, Jr., A sharpened version of the Aanderaa-Rosenberg conjecture, *Math. Centrum*, Amsterdam, 1974.
3. B. Bollobas, *Extremal Graph Theory*, Academic Press, London, 1978.
4. R. Cordovil, A combinatorial perspective on the non-Radon partitions, *J. Combin. Theory Ser. A* **38** (1985), 38-47.
5. P. H. Edelman, The acyclic sets of an oriented matroid, *J. Combin. Theory Ser. B* **36** (1984), 26-31.
6. C. Greene, A class of lattices with Möbius function $\pm 1, 0$, preprint.
7. J. Hagstrom, Combinatorial properties for directed network reliability with a path length criterion, preprint.
8. J. Kahn, M. Saks, and D. Sturtevant, A topological approach to evasiveness, *Combinatorica* **4** (1984), 297-306.
9. J. Kahn and D. Sturtevant, Reliability and Möbius functions, preprint.
10. M. Las Vergnas, Convexity in oriented matroids, *J. Combin. Theory Ser. B* **29** (1980), 231-243.
11. E. Lawler, *Combinatorial Optimization: Networks and Matroids*, Holt, Rinehart and Winston, New York, 1976.
12. R. L. Rivest and J. Vuillemin, On recognizing graph properties from adjacency matrices, *Theoret. Comput. Sci.* **3** (1976), 371-384.
13. G.-C. Rota, On the foundations of combinatorial theory I: theory of Möbius functions, *Z. Wahrsch. Verw. Gebiete* **2** (1964), 340-368.
14. G.-C. Rota, On the combinatorics of the Euler characteristic, in *Studies in Pure Mathematics* (L. Mirsky, ed.), 221-233, Academic Press, London, 1971.
15. A. Satyanarayana, A unified formula for analysis of some network reliability problems, *IEEE Trans. Reliability* **31** (1982), 23-32.
16. A. Satyanarayana and J. Hagstrom, Combinatorial properties of directed graphs useful in computing network reliability, *Networks* **11** (1981), 357-366.
17. A. Satyanarayana and A. Prabhakar, A new topological formula and rapid algorithm for reliability analysis of complex networks, *IEEE Trans. Reliability* **27** (1978), 82-100.
18. R. Willie, A theorem concerning cyclic directed graphs with applications to network reliability, *Networks* **10** (1980), 71-78.
19. R. Cordovil, M. Las Vergnas, and A. Mandel, Euler's relation, Möbius functions and matroid identities, *Geom. Dedicata* **12** (1982), 147-162.

Received June 2, 1986, and in revised form July 18, 1986.