Shelling Pseudopolyhedra

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Abstract. The notion of shellability originated in the context of polyhedral complexes and combinatorial topology. An abstraction of this concept for graded posets (i.e., graded partially ordered sets) was recently introduced by Björner and Wachs first in the finite case [1] and then with Walker in the infinite case [11]. Many posets arising in combinatorics and in convex geometry were investigated and some proved to be shellable. A key achievement was the proof by Bruggesser and Mani that boundary complexes of convex polytopes are shellable [4].

We extend here the result of Bruggesser and Mani to polyhedral complexes arising as boundary complexes of more general convex sets, called pseudopolyhedra, with suitable asymptotic behavior. This includes a previous result on tilings of a Euclidean space $\mathbb{R}^d$ which are projections of the boundary of a $(d + 1)$ -pseudopolyhedron [7].

1. Definitions and Notations

The scalar product on $\mathbb{R}^d$ is denoted by $\langle \cdot, \cdot \rangle$. Let A be any subset of $\mathbb{R}^d$. Then $\text{aff } A$, $\text{conv } A$ denote the affine and convex hull of A , respectively, cone A is the smallest convex cone with apex 0 containing A . $\dot{A}$ or A° is the interior of A , $\text{relint } A$ is its interior in $\text{aff } A$, and ∂A is its frontier. The *polar* of A is $A^* = \{x \in \mathbb{R}^d \mid \forall a \in A: \langle x, a \rangle \leq 0\}$. The *barrier cone* of A is $\mathbb{B}(A) = \{x \in \mathbb{R}^d \mid \exists M: \forall a \in A: \langle a, x \rangle \leq M\}$, its *asymptote cone* is $A_\infty = \bigcap_{\varepsilon > 0} [0, \varepsilon] \cdot A$. If A is convex, then $A_\infty = \{x \in \mathbb{R}^d \mid \exists y \in \mathbb{R}^d: \forall \lambda \geq 0: y + \lambda x \in A\}$. A convex subset A of $\mathbb{R}^d$ is *parabolic* if, for each $u \in A_\infty \setminus \{0\}$ and each $x \in \mathbb{R}^d$, there is $\lambda \geq 0$ such that $x + \lambda u \in A$. For the first properties of these, see, for instance, [6] and [3]. A *facet* of a convex set A is a convex subset F of A such that any segment $[x: y]$ contained in A and which meets F in some point other than x or y , is contained in F . The set of facets of A is denoted by $\mathcal{F}(A)$, and $\mathcal{B}(A) = \mathcal{F}(A) \setminus \{A\}$. A facet of A is *proper* if it is not A or $\emptyset$.

A *convex polytope* is the convex hull of some finite set. A *polyhedral complex*

$\mathcal{P}$ is a set of convex polytopes of $\mathbb{R}^d$ which contains the facets of its elements and such that if $F, G \in \mathcal{P}$, then $F \cap G$ is a facet of both F and G (possibly empty). The elements of $\mathcal{P}$ are called *faces*, and $\mathcal{P}^{(k)}$ is the set of all faces of dimension k (k -*faces*). The *dimension* m of $\mathcal{P}$ is the least upper bound of the dimensions of its faces; $\mathcal{P}$ is *pure* if all its maximal faces have the same dimension.

Let P be a poset (i.e., a partially ordered set). The dual poset (defined on the same set, with the reverse order) is denoted by $\tilde{P}$. The *closed*, *lower*, and *upper intervals* of P are respectively defined by $[a, b] = \{x \in P \mid a \leq x \leq b\}$, $[a) = \{x \in P \mid x \leq a\}$, and $[a) = \{x \in P \mid a \leq x\}$ ($a, b \in P$). P is *graded* if it has a minimum element $\hat{0}$ and a maximum element $\hat{1}$, and all its maximal totally ordered subsets have the same finite cardinality; this cardinality minus one is called the *length* $l(P)$ of P . An *atom* of a poset with a minimum element $\hat{0}$ is an element $a \neq \hat{0}$ such that $[\hat{0}, a] = \{\hat{0}, a\}$. A *recursive atom ordering* (in short, R.A.O.) of a graded poset P with set of atoms A is defined inductively on $l(P)$, according to [11, Section 7]:

- (i) if $l(P) = 1$, then an R.A.O. of P is a well-ordering of A ;
- (ii) if $l(P) > 1$, then an R.A.O. of P is a well-ordering, $\leq_R$ of A , such that, for each $a \in A$, $[a)$ has an R.A.O. that begins with the minimal elements of the subposet

$$H_a = [a) \cap \bigcap_{b \in A, b <_R a} [b)$$

(which are therefore atoms of $[a)$), and H_a is nonempty unless a is minimum for $\leq_R$.

A *pseudo-R.A.O.* of P is defined by replacing “well-ordering” by “total order” in the preceding definition. A *simple R.A.O.* of P is an R.A.O. such that $(A, \leq_R)$ is either finite or isomorphic to $(\mathbb{N}, \leq)$.

A *recursive coatom ordering* (in short, R.C.O.) of P is an R.A.O. of $\tilde{P}$. A *shelling* of a pure m -(dimensional) polyhedral complex $\mathcal{P}$ is an R.C.O. of $\mathcal{P}$, ordered by set-inclusion and completed by a maximum element $\hat{1}$; simple shellings and pseudoshellings of $\mathcal{P}$ are defined accordingly.

The *lexicographic order of coordinates* is the total order $\leq_L$ defined on $\mathbb{R}^d$ by

$$(x_1, \dots, x_d) <_L (y_1, \dots, y_d) \Leftrightarrow (\exists j \in \{1, \dots, d\} : a_i = b_i \ (i < j), a_j < b_j).$$

It is invariant under translation, and, for every $z \in \mathbb{R}^d$, $\{x \in \mathbb{R}^d \mid x \leq_L z\}$, $\{x \in \mathbb{R}^d \mid z \leq_L x\}$ are convex cones with apex z . A *lexicographic order* on a finite-dimensional affine space is the lexicographic order of coordinates corresponding to some system of coordinates. The order induced on any affine subspace is also a lexicographic order.

In the following, $(H_i^+)_{i \in I}$ denotes a family of closed half-spaces H_i^+ of $\mathbb{R}^d$ with boundary H_i , respectively. The family is called *discrete* if, for every $x \in \bigcap_{i \in I} H_i^+$, there is a neighborhood V of x such that $\{i \in I \mid V \cap H_i \neq \emptyset\}$ is finite.

The half-space H_i^+ is *redundant* if $H_i^+ \supset \bigcap_{j \neq i} H_j^+$; $(H_i^+)_{i \in I}$ is *nonredundant* provided none of the H_i^+ 's is redundant. An *m-pseudopolyhedron* is a set in the form $P = \bigcap_{i \in I} H_i^+$, with $(H_i^+)_{i \in I}$ discrete and $\dim P = m$.

2. Basic Properties of Pseudopolyhedra

We give here the first properties of pseudopolyhedra needed in the next section. They are well known in the case of polyhedra (see Sections 2.6 and 3.1 of [8]) and we show briefly how the proofs can be adapted.

Lemma 2.1. *Let $(H_i^+)_{i \in I}$ be a discrete family of half-spaces, $P = \bigcap_{i \in I} H_i^+$ and $J \subset I$. Then*

$$P \cap \left(\bigcap_{i \in J} H_i^+ \right)^\circ = P \cap \left(\bigcap_{i \in J} H_i^{+\circ} \right).$$

In particular, $\dot{P} = \bigcup_{i \in I} (P \cap H_i)$.

Proof. Obviously, $P \cap \left(\bigcap_{i \in J} H_i^+ \right)^\circ \subset P \cap \left(\bigcap_{i \in J} H_i^{+\circ} \right)$. On the other hand, let $x \in P \cap \left(\bigcap_{i \in J} H_i^{+\circ} \right)$ and let V be a convex neighborhood of x such that $F = \{i \in J \mid V \cap H_i \neq \emptyset\}$ is finite. Then $V \cap \left(\bigcap_{i \in F} H_i^{+\circ} \right)$ is a neighborhood of x contained in $\bigcap_{i \in J} H_i^+$. $\square$

Proposition 2.2. *Let $P = \bigcap_{i \in I} H_i^+$ be a d -pseudopolyhedron of $\mathbb{R}^d$, with $(H_i^+)_{i \in I}$ discrete and $H_i^+ \neq H_j^+$ ($i \neq j$). Then H_i^+ is nonredundant in $(H_i^+)_{i \in I}$ if and only if $\dim(H_i \cap P) = d - 1$.*

Proof. Consider a convex neighborhood of some $x \in \text{relint}(H_i \cap P)$. $\square$

Theorem 2.3. *Let $P = \bigcap_{i \in I} H_i^+$ be an m -pseudopolyhedron in $\mathbb{R}^d$, with $(H_i^+)_{i \in I}$ discrete. Then there is a finite subset F of I such that $\bigcap_{i \in F} H_i^+ = \text{aff } P$, and a subset J of I such that $P = \bigcap_{i \in J} H_i^+$ and $(H_i^+)_{i \in J}$ is nonredundant.*

Proof. As to the first part, take $F = \{i \in I \mid P \subset H_i\}$, which is nonempty if $m < d$ (apply Lemma 2.1 to some $y \in \text{relint } P$). Then the second part reduces to the case $m = d$.

Let us assume $m = d$, $H_i^+ \neq H_j^+$ ($i \neq j$), and let $J = \{i \in I \mid \dim(H_i \cap P) = d - 1\}$, $Q = \bigcap_{i \in J} H_i^+ \supset P$. Since P is locally polyhedral it follows that, for any neighborhood V of any $x \in P$, $P \cap V = Q \cap V$. So $\dot{P}$ does not meet $\dot{Q}$. Thus $\dot{P} = \dot{Q}$ by convexity of $\dot{Q}$ hence $P = Q$. $\square$

Proposition 2.4. *Let $P = \bigcap_{i \in I} H_i^+$ be an m -pseudopolyhedron of $\mathbb{R}^d$ ($m \geq 1$), with $(H_i^+)_{i \in I}$ discrete and nonredundant. Then the maximal proper facets of P are the sets $H_i \cap P$, with $H_i \not\supset \text{aff } P$; they are $(m - 1)$ -pseudopolyhedra. $\square$*

Corollary 2.5. *If P is an m -pseudopolyhedron of $\mathbb{R}^d$ which contains no line, then $\mathcal{F}(P)$, ordered by set-inclusion, is a graded lattice of length m .*

Corollary 2.6. *The family of all facets of an m -pseudopolyhedron P is locally finite.*

Proof. By Proposition 2.4, all facets of P are in the form $(\bigcap_{i \in A} H_i) \cap P$, with A a finite subset of I . $\square$

Theorem 2.7. *Let P be an m -pseudopolyhedron in $\mathbb{R}^d$, and let v be a vertex of P (so P contains no line). Then there is a hyperplane H not containing v which separates strictly $\{v\}$ from the facets of P not containing v and meets every facet of P containing strictly $\{v\}$. The application $F \rightarrow F \cap H$ defines an isomorphism between the interval $[\{v\}, P]$ of $\mathcal{F}(P)$ and the facet lattice of the $(m-1)$ -convex polytope $H \cap P$.*

Proof. The proof is easily reduced to the case $m = d$. There is a closed ball B centered in v such that $\{i \in I \mid H_i \cap B \neq \emptyset\} = \{i \in I \mid v \in H_i\} = F$, hence $B \subset \bigcap_{i \in I \setminus F} H_i^{+^\circ}$ and F is finite. Let $K = \bigcap_{i \in F} H_i^+$; K is a d -polyhedral cone with apex v , and $K \cap B = P \cap B$. The compact convex set $\text{conv}(K \cap \bar{B}) \subset P$ does not contain the vertex v , because otherwise v would have to belong to $K \cap \bar{B}$. Then there is a hyperplane H separating strictly $\{v\}$ from $\text{conv}(K \cap \bar{B})$, hence from every facet of P not containing v (these are in the form $\bigcap_{i \in A} H_i \cap P$, $A \not\subset F$, so they are contained in $K \setminus B$). Moreover, H meets every facet F of P which contains $\{v\}$ strictly. Indeed, if D is a 1-facet of F containing v (see Corollary 2.5), then either D has a vertex $y \neq v$ and $[v: y]$ meets H , or D is a half-line with vertex v contained in K and meeting $\bar{B}$ thus H . Furthermore, H meets $\text{relint } F$, since otherwise F would be on one side of H and the nonempty facet $F \cap H$ of F would not be separated from v by H .

Consequently, $H \cap K = H \cap P \subset B$ is a $(d-1)$ -convex polytope. By Proposition 2.4 its facets are in the form $F \cap H$, with $v \in F \in \mathcal{F}(P)$. We also have that if $F, F' \in \mathcal{F}(P)$ with $v \in F \cap F'$, and $F \cap H \subset F' \cap H$, then $\text{relint } F$ meets F' hence $F \subset F'$. So the map $[\{v\}, P] \rightarrow \mathcal{F}(P \cap H): F \rightarrow F \cap H$ is an isomorphism. $\square$

3. Shelling Pseudopolyhedra

Theorem 3.1. *Let P be an m -pseudopolyhedron in $\mathbb{R}^d$ which contains no line. Then $\mathcal{F}(P)$ has a simple R.A.O.*

Proof. The closed convex cone P_∞ contains no line, so up to a change of coordinates we can assume $P_\infty \setminus \{0\} \subset \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_1 > 0\}$. We order the set $V(P)$ of vertices of P by lexicographic order of coordinates, and we prove by induction on m that this is an R.A.O. of P .

We show first that $(V(P), \leq_L)$ is either finite or isomorphic with $(\mathbb{N}, \leq)$. It is enough to show that, for every $a \in \mathbb{R}$, $X_a = \{x \in P \mid x_1 \leq a\}$ contains only

finitely many vertices of P . Indeed, X_a is a closed convex set such that $(X_a)_\infty \subset \{x \in \mathbb{R}^d | x_1 \leq 0\} \cap P_\infty = \{0\}$. So X_a is compact and since $V(P)$ is locally finite (Corollary 2.6), $X_a \cap V(P)$ is finite.

We show now that if v is a vertex of P , then $[\{v\}, P]$ has an R.A.O. in which the 1-facets having a vertex $v' <_L v$ come first. Let H be a hyperplane like in Theorem 2.7. The atoms $A_1, \dots, A_r$ of $[\{v\}, P]$ are in bijection with the vertices $y_1, \dots, y_r$ of $P \cap H$, and we can choose indices so that $A_i \cap H = \{y_i\}$ ($1 \leq i \leq r$), $y_1 <_L y_2 <_L \dots <_L y_r$. By the induction hypothesis, $y_1, \dots, y_r$ is an R.A.O. of $\mathcal{F}(P \cap H)$, so $A_1, \dots, A_r$ is an R.A.O. of $[\{v\}, P]$. If A_i has another vertex $v' <_L v$, then $y_i \in [v: v']$ thus $y_i <_L v$; if on the contrary $v <_L v'$, then similarly $v <_L y_i$. If A_i is a half-line with vertex v , then

$$A_i - \{v\} \subset P_\infty \subset \{x \in \mathbb{R}^d | x_1 > 0\} \cup \{0\} \subset \{x \in \mathbb{R}^d | 0 \leq_L x\}$$

hence $v <_L y_i$.

It remains to prove that if a facet F of P contains two vertices $v' <_L v$, then F has an edge with vertices v, v'' so that $v'' <_L v$. If $F \neq P$ this follows from the induction hypothesis applied to F . If $F = P$ we suppose on the contrary that such an edge does not exist. Then $A_i \subset \{x \in \mathbb{R}^d | v \leq_L x\}$ ($1 \leq i \leq r$). Indeed, if A_i is a half-line this is proved like above, and if A_i has a vertex $w \neq v$ this comes from $v <_L w$. So $P \cap H = \text{conv}\{y_1, \dots, y_r\} \subset \{x \in \mathbb{R}^d | v \leq_L x\}$, in contradiction with the fact that $[v: v']$ has to meet H in some point $z <_L v$. $\square$

Lemma 3.2. *Let P be a d -pseudopolyhedron in $\mathbb{R}^d$ containing no line and with bounded $(d-1)$ -facets. Let $P = \bigcap_{i \in I} H_i^+$, with $(H_i^+)_{i \in I}$ discrete and nonredundant. We assume, further, $0 \in \mathring{P}$ and we put $F_i = H_i \cap P$, $H_i^+ = \{x \in \mathbb{R}^d | \langle a_i, x \rangle \leq 1\}$. Then, for every $j, k \in I$,*

$$a_k - a_j \in \text{cone}\{a_i - a_j | F_i \cap F_j \supset F_k \cap F_j, \dim(F_i \cap F_j) = d-2\}.$$

Proof. Since P_∞ contains no line, there is a half-space K^+ bounded by a homogeneous hyperplane and such that $P_\infty \cap K^+ = \{0\}$ [9, Theorem 2.5]. Some translate M^+ of K^+ satisfies $M^{+\circ} \supset F_j$ and $(\bigcap_{i \neq k} H_i^+) \cap M^+ \not\subset H_k^+$. Then considering Theorem 2.3 and Proposition 2.4, it is enough to prove the result for the d -pseudopolyhedron $Q = P \cap M^+$, which is compact since $Q_\infty = P_\infty \cap K^+ = \{0\}$.

So assume that P is a convex polytope. The a_i 's are the vertices of the dual polytope P^* [8, Section 3.4]. Let G be the smallest face of P^* containing a_j and a_k . The lemma states that a_k is in the convex cone with apex a_j spanned by the edges of P^* containing a_j and contained in G . This is easily deduced from Theorem 2.7 applied to G and a_j . $\square$

Theorem 3.3. *Let P be a d -pseudopolyhedron of $\mathbb{R}^d$ containing no line, the $(d-1)$ -facets of which are bounded. If, moreover, $\mathbb{B}(P) \setminus \mathbb{B}(P)^\circ$ is convex (e.g., if P is parabolic or P_∞ is a half-line), then $\mathcal{B}(P)$ has a simple shelling.*

Proof. We take over the notations of Lemma 3.2, assuming $0 \in \mathring{P}$. We first prove by induction on d that every lexicographic order $\leq_L$ of the a_i 's gives

a pseudoshelling. The result is trivial for $d = 1$, so we assume $d \geq 2$. We order I by putting $i < j$ whenever $a_i <_L a_j$.

We first have to prove that if $k, j \in I$ satisfy $k < j$, then there is $i \in I$ such that $i < j$, $F_i \cap F_j \supset F_k \cap F_j$, and $\dim(F_i \cap F_j) = d - 2$. Indeed if this was not the case, then we would get

$$\{a_i - a_j | F_i \cap F_j \supset F_k \cap F_j, \dim(F_i \cap F_j) = d - 2\} \subset \{x \in \mathbb{R}^d | 0 \leq_L x\}$$

and thus the contradiction $a_k - a_j \in \{x \in \mathbb{R}^d | 0 \leq_L x\}$ by Lemma 3.2.

We prove next that, for every $j \in I$, $\mathcal{B}(F_j)$ has a shelling in which the $(d - 2)$ -facets of F_j contained in some F_i , $i < j$, come first. Let $S = \{i \in I | \dim(F_i \cap F_j) = d - 2\}$.

Let $z \in \text{relint } F_j$, so that $\langle a_j, z \rangle = 1$, $\langle a_i, z \rangle < 1$ ($i \in S$). It is easily verified that

$$\begin{aligned} F_j - \{z\} &= \bigcap_{i \in S} \{x \in H_j - \{z\} | \langle a_i, x + z \rangle \leq 1\} \\ &= \bigcap_{i \in S} \{x \in H_j - \{z\} | \langle b_i, x \rangle \leq 1\} \\ &= \bigcap_{i \in S} \{x \in H_j - \{z\} | \langle \pi(b_i), x \rangle \leq 1\}, \end{aligned}$$

where

$$b_i = \frac{a_i - \langle a_i, z \rangle a_j}{1 - \langle a_i, z \rangle} \in \{x \in \mathbb{R}^d | \langle x, z \rangle = 0\} = L$$

and

$$\pi: \mathbb{R}^d \rightarrow \{x \in \mathbb{R}^d | \langle a_j, x \rangle = 0\} = H_j - \{z\}$$

is the orthogonal projection. By the induction hypothesis applied to the $(d - 1)$ -convex polytope $F_j - \{z\}$ in $H_j - \{z\}$, with $(d - 2)$ -facets $(F_i \cap F_j) - \{z\}$ ($i \in S$), any lexicographic order of the $\pi(b_i)$'s gives a shelling of $\mathcal{B}(F_j)$. Since on the other hand $\pi|_L: L \rightarrow H_j - \{z\}$ is a vector isomorphism, a shelling of $\mathcal{B}(F_j)$ is given by

$$F_i \cap F_j \leq F_{i'} \cap F_j \Leftrightarrow b_i \leq_L b_{i'} \quad (i, i' \in S).$$

In this ordering, the $(d - 2)$ -facets $F_i \cap F_j$ such that $b_i \leq_L a_j$ come first. This last inequality is equivalent to $a_i \leq_L a_j$. This completes the first part of the proof.

Now we construct a lexicographic order of $\mathbb{R}^d$ which induces on $\{a_i | i \in I\}$ an order either finite or isomorphic to $(\mathbb{N}, \leq)$. If P is compact, then I is finite and any lexicographic order works. So we can assume $\mathbb{B}(P) \neq \mathbb{R}^d$. We first observe that $\mathbb{B}(P)$ is not contained in a homogeneous hyperplane $G = \{x \in \mathbb{R}^d | \langle x, c \rangle = 0\}$, because this would imply $*P_\infty = \overline{\mathbb{B}(P)} \subset G$ hence

$$c \in (**P_\infty) \cap (-**P_\infty) = P_\infty \cap (-P_\infty) = \{0\}.$$

So aff $\mathbb{B}(P) = \mathbb{R}^d$ and $\mathbb{B}(P) \subset \overline{\mathbb{B}(P)^\circ}$. Since the convex set $\mathbb{B}(P) \setminus \mathbb{B}(P)^\circ$ does not meet $\mathbb{B}(P)^\circ$, a standard separation theorem gives $u \in \mathbb{R}^d \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$\begin{aligned}\mathbb{B}(P) \setminus \mathbb{B}(P)^\circ &\subset \{x \in \mathbb{R}^d \mid \langle x, u \rangle \geq \alpha\}, \\ \mathbb{B}(P) &\subset \{x \in \mathbb{R}^d \mid \langle x, u \rangle \leq \alpha\}.\end{aligned}$$

Since $\mathbb{B}(P)$ is a convex cone with apex 0, we can assume $\alpha = 0$ thus $u \in {}^*\mathbb{B}(P) = P_\infty$, $\mathbb{B}(P) \setminus \mathbb{B}(P)^\circ \subset \{x \in \mathbb{R}^d \mid \langle x, u \rangle = 0\}$.

Let us take a lexicographic order of $\mathbb{R}^d$ corresponding to an orthogonal basis whose first element is $u/\|u\|$. We show that, for every $j \in I$, $\{i \in I \mid i < j\}$ is finite. Let us suppose on the contrary there exists a sequence $(i_n)_{n \geq 1}$ of $\{i \in I \mid i < j\}$, with $i_n \neq i_{n'}$ ($n \neq n'$). Let, for each n , $x_n \in F_{i_n}$; so $\langle a_{i_n}, x_n \rangle = 1$. Since any ball meets only finitely many facets of P (Corollary 2.6), $\|x_n\| \rightarrow \infty$. On the other hand, the sequence $(a_{i_n})_{n \geq 1}$ is contained in the closed convex set ${}^\Delta P = \{x \in \mathbb{R}^d \mid \forall p \in P: \langle p, x \rangle \leq 1\}$, which is compact since $0 \in \overset{\circ}{P}$ implies

$$({}^\Delta P)_\infty = \{x \in \mathbb{R}^d \mid \forall p \in P: \langle p, x \rangle \leq 0\} = \{0\}.$$

Consequently, extracting suitable subsequences if necessary, we may furthermore assume

$$a_{i_n} \rightarrow a \in {}^\Delta P \subset \mathbb{B}(P), \quad \frac{x_n}{\|x_n\|} \rightarrow y \in P_\infty \setminus \{0\},$$

thus

$$\langle a, y \rangle = \lim_{n \rightarrow \infty} \left\langle a_{i_n}, \frac{x_n}{\|x_n\|} \right\rangle = \lim_{n \rightarrow \infty} \frac{1}{\|x_n\|} = 0.$$

Since $\mathbb{B}(P)^\circ \subset \{x \in \mathbb{R}^d \mid \langle x, y \rangle < 0\}$ as a consequence of $y \in P_\infty \setminus \{0\}$, we have $a \in \mathbb{B}(P) \setminus \mathbb{B}(P)^\circ$ hence $\langle a, u \rangle = 0$. This yields a contradiction with

$$\langle a, u \rangle = \lim_{n \rightarrow \infty} \langle a_{i_n}, u \rangle \leq \langle a_j, u \rangle < 0,$$

which in turn comes from $a_{i_n} <_L a_j$, $u \in P_\infty$, and the fact that F_j contains no half-line.

Finally, the convexity of $\mathbb{B}(P) \setminus \mathbb{B}(P)^\circ$ is guaranteed when P is parabolic (because then $\mathbb{B}(P) \setminus \{0\}$ is an open set by (i) on p. 338 of [3]) and also when P_∞ is a half-line, let cone $\{w\}$. In this last instance

$$\overline{\mathbb{B}(P)} = {}^*P_\infty = \{x \in \mathbb{R}^d \mid \langle x, w \rangle \leq 0\}$$

hence $\mathbb{B}(P) \setminus \mathbb{B}(P)^\circ = \mathbb{B}(P) \cap \{x \in \mathbb{R}^d \mid \langle x, w \rangle = 0\}$ is convex. □

Remarks 3.4. 1. The idea, used in the proof of Theorem 3.3, that a shelling of P and a certain ordering of the vertex set of the dual body correspond to one another has been mentioned, for polytopes, in [10]—Note added in proof.

2. A shelling of $\mathcal{B}(P)$ which, like the one used in Theorem 3.3 or in [4], is compatible with the ordering of values of some linear function on the normal vectors of maximal facets, is often called a *line shelling*. Here is an example of a 3-pseudopolyhedron P in $\mathbb{R}^3$ such that no simple shelling of $\mathcal{B}(P)$ arises as a line shelling.

Take as vertices of P the points

$$\begin{aligned} u_i &= (\sqrt{i^2 - 1}, \sqrt{(i+1)^2 - 4}, i-2), \\ v_i &= (-\sqrt{i^2 - 1}, \sqrt{(i+1)^2 - 4}, i-2), \\ w_i &= (-\sqrt{i^2 - 1}, -\sqrt{(i+1)^2 - 4}, i-2), \\ x_i &= (\sqrt{i^2 - 1}, -\sqrt{(i+1)^2 - 4}, i-2) \end{aligned}$$

($i \in \mathbb{N}_0$). The 2-facets of P are, for $i \in \mathbb{N}_0$,

$$\begin{aligned} F_i^{(1)} &= \text{conv}\{u_i, x_i, u_{i+1}, x_{i+1}\} = \{x \in P \mid \langle x, a_i^{(1)} \rangle = 1\}, \\ F_i^{(2)} &= \text{conv}\{u_i, v_i, u_{i+1}, v_{i+1}\} = \{x \in P \mid \langle x, a_i^{(2)} \rangle = 1\}, \\ F_i^{(3)} &= \text{conv}\{v_i, w_i, v_{i+1}, w_{i+1}\} = \{x \in P \mid \langle x, a_i^{(3)} \rangle = 1\}, \\ F_i^{(4)} &= \text{conv}\{w_i, x_i, w_{i+1}, x_{i+1}\} = \{x \in P \mid \langle x, a_i^{(4)} \rangle = 1\}. \end{aligned}$$

Furthermore, $\lim_{i \rightarrow \infty} a_i^{(1)} = (\frac{1}{2}, 0, -\frac{1}{2})$, $\lim_{i \rightarrow \infty} a_i^{(2)} = (0, \frac{1}{3}, -\frac{1}{3})$, $\lim_{i \rightarrow \infty} a_i^{(3)} = (-\frac{1}{2}, 0, -\frac{1}{2})$, $\lim_{i \rightarrow \infty} a_i^{(4)} = (0, -\frac{1}{3}, -\frac{1}{3})$. Then for every nonzero linear function $x \rightarrow \langle u, x \rangle$, the set $\{\langle u, a_i^{(j)} \rangle \mid i \in \mathbb{N}_0, j \in \{1, 2, 3, 4\}\}$ has more than one cluster point and cannot be isomorphic, as a poset, with $(\mathbb{N}, \leq)$. However $\mathcal{B}(P)$ is simply shellable since its orthogonal projection on the plane $\mathbb{R}^2 \times \{0\}$ is a regular CW-decomposition of it (see Section 3 of [7]).

3. It is likely that some d -pseudopolyhedra with $d \geq 4$ are not simply shellable. Danzer pointed out a construction (see answer to Problem 8 in [5]) which leads to a polyhedral tiling of $\mathbb{R}^3$ he thinks is not shellable. We were unable to verify this because we do not know how to prove that an infinite complex is not shellable. It would be very interesting to settle this question. This, together with the methods of [7], could yield a nonshellable pseudopolyhedron.

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