On a Generalization of the Gallai–Sylvester Theorem*

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Abstract. It is shown that if $S \subset \mathbb{R}^d$, $\text{aff } S = \text{aff } \mathbb{R}^d$, and every hyperplane spanned by (a subset of) S misses fewer than k points of S ($k \geq 2$), then (a) $\#S \leq km$ if $d = 2m - 1$ is odd and (b) $\#S \leq km + 1$ if $d = 2m$ is even. We also fully describe the extreme sets for which equality holds in (a) or in (b). For odd d the proofs are later modified to purely algebraic ones, and carry over to $\mathbb{F}^d$, where $\mathbb{F}$ is an arbitrary field. For even d , (b) is generally not true when $\mathbb{F} \neq \mathbb{R}$, but we prove some weaker inequalities that do hold over arbitrary fields.

0. Introduction and Notation

Let $\mathbb{F}$ be a field (a division ring should suffice) and let $S \subset \mathbb{F}^d$ ($d > 1$) be a finite spanning set, i.e., $\text{aff } S = \mathbb{F}^d$. A flat H is (affinely) *spanned* by S if, $\exists S' \subset S$, $\text{aff } S' = H$. The set of one-dimensional flats (lines) spanned by (pairs of points in) S is denoted by $L(S)$. A $(d - 1)$ -dimensional flat is called a *hyperplane*. Let $k \geq 1$ be an integer. A spanned hyperplane is *k-avoiding* (relative to S) if $\#(S \setminus H) \geq k$. S is of *type k* if there is a *k-avoiding* hyperplane relative to S . Define, for $k, d \geq 1$,

$$f_{\mathbb{F}}(k; d) = \sup\{\#S : S \subset \mathbb{F}^d, \text{aff } S = \mathbb{F}^d, S \text{ is not of type } k\}. \quad (0.1)$$

Remark 0.1. (i) Usually we work in affine d -space $\mathbb{F}^d$, but occasionally we refer to the projective d -space $P(\mathbb{F}^d)$, and to the function $f_{P(\mathbb{F})}(k; d)$, obtained by replacing $\mathbb{F}^d$ by $P(\mathbb{F}^d)$ in (0.1).

(ii) Since any hyperplane spanned by a spanning set $S \subset \mathbb{F}^d$ is 1-avoiding, we convent $f_{\mathbb{F}}(1; d) = d$. Hence the case $k = 1$ is of no further interest.

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(iii) Sylvester–Gallai’s celebrated theorem asserts that any finite noncollinear point set in $\mathbb{R}^2$ spans a line passing through precisely two points of the set (see [K] for more historical information). It follows that $f_{\mathbb{R}}(k; 2) = k + 1$; hence the title of the paper.

1. On $f_{\mathbb{R}}(k; d)$ for $d = 2m$ Even

We deal with $f_{\mathbb{R}}(k; d)$ for even $d (= 2m)$ separately from the odd case $d = 2m - 1$ ($m \geq 1$) since this latter case falls neatly in the frame of the discussion on $f_{\mathbb{F}}(k; 2m - 1)$ for a general field $\mathbb{F}$ (see Theorem 2.2 below) ($f_{\mathbb{R}}(k; d = 2m - 1)$ could be dealt with also in the geometric spirit of this section, but, as mentioned, in view of later results this is superfluous); whereas later results show that the results of the present section *do not* hold in $\mathbb{F}^{2m}$ for unordered fields $\mathbb{F}$.

Theorem 1.1. *Let $d = 2m$ ($m \geq 1$) be an even positive integer, $k \geq 2$. Then:*

- (i) $f_{\mathbb{R}}(k; d) = mk + 1$.
- (ii) *If $S \subset \mathbb{R}^d$ is a spanning set of cardinality $mk + 1$, and if S is not of type (k) (i.e., if S does not span a k -avoiding hyperplane), then S consists of $m - 1$ mutually disjoint sets, each of cardinality k , contained in $m - 1$ skew lines respectively, plus a two-dimensional set (of cardinality $k + 1$), contained in a plane which is skew to the union of the lines.*

Proof. (i) The example described in part (ii) of the theorem shows that $f_{\mathbb{F}}(k; d) \geq mk + 1$ for all $k \geq 2$, $d \geq 1$ (over any field $\mathbb{F}$, and, in particular, for $\mathbb{F} = \mathbb{R}$). To prove $\leq$ for $\mathbb{F} = \mathbb{R}$ we proceed by induction on $d = 2m$. As noted in Remark 0.1(iii) the case $d = 2$ is just another formulation of Sylvester–Gallai theorem.

Induction Step $d - 2 \rightarrow d$. Assume $d \geq 4$ ($m \geq 2$), $k \geq 2$, $S \subset \mathbb{R}^d$, $\text{aff } S = \mathbb{R}^d$, and $\#S \geq mk + 2$; we have to prove that S spans a k -avoiding hyperplane. Hansen’s theorem (see [Ha]) assures the existence of an “elementary” hyperplane spanned by S , i.e., there is a spanned hyperplane H such that $S \cap H$ consists of a $(d - 2)$ -dimensional set and a single point q ; in other words, there is a $(d - 2)$ -dimensional flat $J \subset H$ (a hyperplane in H) and a point $q \in S \cap H$ such that $(S \cap H) \setminus \{q\} \subset J$. (This is a “dimensional” generalization of Sylvester’s theorem.)

If $\#(S \setminus H) \geq k$ we are done. If $\#(S \setminus H) < k$, then $\#(S \setminus J) = 1 + \#(S \setminus H) \leq k$, hence $\#(S \cap J) = \#S - \#(S \setminus J) \geq \#S - k = (m - 1)k + 2$. By the induction hypothesis there is a $(d - 3)$ -flat $K \subset J$, spanned by $S \cap J$ such that $\#((S \cap J) \setminus K) \geq k$. So S contains a subset $S \cap J$ which is of type (k) relative to J ; it follows easily that S is also of type (k) .

(ii) By induction on m ($d = 2m$). For $m = 1$ there is nothing to prove. Assuming the assertion was proved for every positive integer $< m$, we have to prove it for m ($m \geq 2$).

Let $S \subset \mathbb{R}^{2m}$ be a spanning set $\#S = km + 1$, such that, for every hyperplane H

spanned by S , $\#(S \setminus H) < k$. As before, choose an elementary hyperplane H and within it a flat J of dimension $2m - 2 = 2(m - 1)$, such that $\#(S \cap (H \setminus J)) = 1$ (Hansen's theorem). By assumption $\#(S \setminus H) \leq k - 1$, and by the considerations of the foregoing proof $\#(S \setminus H) = k - 1$ (if $\#(S \setminus H) \leq k - 2$, then an easy computation shows that $\#(S \cap J) \geq k(m - 1) + 2$, so by part (i) $S \cap J$ is of type (k) relative to J ; this is impossible). Hence $\#(S \cap J) = \#S - \#(S \setminus H) - 1 = km + 1 - (k - 1) - 1 = k(m - 1) + 1$.

Clearly, $S \cap J$ is not of type (k) in the $2(m - 1)$ -flat J , hence by our induction hypothesis it consists of $m - 2$ sets, each of cardinality k , lying on $m - 2$ skew lines $L_1, \dots, L_{m-2}$, and a two-dimensional set of $k + 1$ points in a plane $M (\subset J)$, which is skew to $L_1, \dots, L_{m-2}$. Choose an affine base for $\mathbb{R}^{2m}$ consisting of a set $\{a_0, b_0, c_0\}$ which spans M , $m - 2$ couples $\{a_i, b_i\}$, $1 \leq i \leq m - 2$, which span L_i , $1 \leq i \leq m - 2$, respectively, and one more couple $\{a_{m-1}, b_{m-1}\}$ where a_{m-1} is the single point in $S \cap (H \setminus J)$ and $b_{m-1} \in S \setminus H$ (see Fig. 1).

Let $x \in S \setminus H$ and let

$$x = \alpha_0 a_0 + \beta_0 b_0 + \gamma_0 c_0 + \sum_{i=1}^{m-1} (\alpha_i a_i + \beta_i b_i) \quad (1.1)$$

be an affine representation of x with this base. (The sum of the coefficients is 1, and the representation is unique.) Since $x \notin H$, $\beta_{m-1} \neq 0$.

Claim. $\alpha_i = \beta_i = 0$ for $1 \leq i \leq m - 2$.

Proof. Assume to the contrary that, say, $\alpha_1 \neq 0$. Consider the hyperplane H^* spanned by all the points appearing on the right-hand side of (1.1), excluding a_1 ; H^* does not contain x , and it also does not contain $k - 1$ points on the line L_1 (those which differ from b_1). Hence H^* is k -avoiding, contradicting our assumption.

Fig. 1

Now consider two complementary possibilities for the configuration of $M \cap S$:

Case I. All points of $S \cap M$ are collinear, excluding one point. Assume without loss of generality that the line is $L_0 = \text{aff}(a_0, b_0)$, and that the single point is c_0 . Clearly, $\#(L_0 \cap S) = k$, and the considerations used to prove the previous claim apply to show that $\alpha_0 = \beta_0 = 0$. Hence $x \in \text{aff}(a_{m-1}, b_{m-1}, c_0)$, i.e.,

$$S = \underbrace{\left(\bigcup_{i=0}^{m-2} (S \cap L_i) \right)}_{k \text{ points on each line}} \cup \underbrace{(S \cap \text{aff}(a_{m-1}, b_{m-1}, c_0))}_{k+1 \text{ points in one plane}}.$$

Case II. No line in M contains k points of S . Our aim is to prove that

$$S \setminus H \subset \text{aff}(a_{m-1}, b_{m-1}). \quad (1.2)$$

This will suffice since all other points of S are already on $m-2$ lines and a plane M which is skew to them. By the previous claim

$$S \setminus \bigcup_{i=1}^{m-2} L_i \subset \text{aff}(M, a_{m-1}, b_{m-1}) =: N. \quad (1.3)$$

(Clearly, N is a 4-flat.) By Sylvester's theorem we can choose $\{a_0, b_0, c_0\}$ such that $\text{aff}(a_0, b_0)$ is a simple (= elementary) line. Now if in (1.1) $\gamma_0 \neq 0$, then the hyperplane $H' = \text{aff}(\bigcup\{a_i, b_i: 0 \leq i \leq m-1\})$ avoids x and $k-1$ points in $S \cap M$ (all the points in $S \cap M$ excluding a_0, b_0), i.e., k points of S in all, contradiction. Hence $\gamma_0 = 0$. Repeating this argument for each simple line L in $S \cap M$ we get

$$S \setminus M \subset \text{aff}(L \cup (\bigcup\{a_i, b_i: i \leq m-1\})). \quad (1.4)$$

Since $S \setminus H \subset (S \setminus \bigcup_{i=1}^{m-2} L_i) \cap (S \setminus M)$, (1.3) and (1.4) imply

Intermediate Conclusion. If L is a simple line in M (relative to $S \cap M$), then

$$\begin{aligned} S \setminus H &\subset N \cap \text{aff}(L \cup (\bigcup\{a_i, b_i: i \leq m-1\})) \\ &\subset \text{aff}(L, a_{m-1}, b_{m-1}) \subset N. \end{aligned} \quad (1.5)$$

Note that to prove (1.2) it suffices to show that

$$\dim \text{aff}(a_{m-1}, S \setminus H) = 1 \quad (1.6)$$

(since $b_{m-1} \in S \setminus H$). Put $\Gamma := \text{aff}(a_{m-1}, S \setminus H)$. We have to show that $\dim \Gamma = 1$. It is more convenient to work from now on in a projective space. So, embed $\mathbb{R}^d$ in a natural way in a real projective space $P^d(\mathbb{R})$ ($\mathbb{R}^d$ in addition with infinity points). For each flat $A \subset \mathbb{R}^d$ denote by A^P the smallest projective space in P^d containing A , i.e., A^P is the projective closure of A (A^P consists of A and infinity points which form

a projective subspace of dimension $\dim A - 1$). Now to prove (1.6) it suffices to show that

$$\Gamma^P \cap M^P = \emptyset. \quad (1.7)$$

Explanation. $\dim \Gamma = \dim \Gamma^P$; if $\dim \Gamma^P \geq 2$, then, from $\dim M^P = 2$ and from $\Gamma^P, M^P \subset N^P$ (see, e.g., (1.3)), it would follow that $\Gamma^P \subset M^P \neq \emptyset$, since $\dim N^P = 4$ (only). (Any two subspaces of P^4 , each of dimension 2 at least, have a nonempty intersection. This is the reason why we prefer to work in projective space.)

By the intermediate conclusion above it follows that

$$L \text{ is a simple line in } S \cap M \Rightarrow \Gamma^P \subset (\text{aff}(L, a_{m-1}, b_{m-1}))^P \subset N^P, \quad (1.8)$$

hence if L is a simple line in $S \cap M$, then

$$\Gamma^P \cap M^P \subset (\text{aff}(L, a_{m-1}, b_{m-1}))^P \cap M^P = L^P. \quad (1.9)$$

(Explanation of the equality in (1.9): the intermediate term in (1.9) contains L^P , and it does not contain more, since if it contains more it would contain M^P *en bloc*; but $\dim N^P = 4$, hence it would follow that $\dim(\text{aff}(L, a_{m-1}, b_{m-1}))^P \geq 4$, which is impossible.)

The inclusion (1.9) is valid for every simple line L^P (in $M \cap S$), hence

$$\Gamma^P \cap M^P \subset \bigcap \{L^P: L^P \text{ is a simple line in } M^P \cap S\}. \quad (1.10)$$

So, to prove (1.7) it suffices to show that the right-hand side of (1.10) is empty. This is the essence of the following lemma:

Lemma. *If $V \subset \mathbb{R}^2$ is finite, and if there is no line which contains the whole of V , excluding one point, then the intersection of all the simple lines in V is empty (this being true even in a projective plane $P^2(\mathbb{R})$).*

Proof (based on [KM]). Embed $\mathbb{R}^2$ in a real projective plane $P^2(\mathbb{R})$. Denote by $L(V)$ the set of lines spanned by V . It is easy to see that $L(V)$ is not a “near pencil” (in the sense of [KM]), i.e., no point in $P^2(\mathbb{R})$ is incident with all lines of $L(V)$, excluding one. (Otherwise there would be a line containing all points of V , excluding one.) If $v \in V, L \in L(V)$ we say that L is a *neighbor* of v if L contains a segment on the boundary of the component of $P^2(\mathbb{R})$ minus the lines of $L(V)$ which are not incident to v , which contains v . In other words, L is a neighbor of v if $v \notin L$, and if there is a segment $[v, l]$ in $P^2(\mathbb{R})$ (possibly an infinity point) such that $l \in L$, and it is disjoint from any line in $L(V) \setminus \{L\}$ which does not pass through v (see Fig. 2).

$L(V)$ not being a near pencil it follows that v has at least three neighboring lines in $L(V)$ (this is not true in an affine plane; here v may have only two neighboring lines). Hence the intersection of the neighboring lines of v is empty (see Theorem 2.3 in [KM]).

Fig. 2

If there is a point $v \in V$ which is not on any simple line, then all the lines neighboring v are simple (Theorem 3.1 in [KM]), hence the intersection of all simple lines is empty (since there are at least three such lines neighboring v).

So we may assume from now on that every point $v \in V$ is incident with a simple line.

Now assume, contrary to the lemma, that all such simple lines in $L(A)$ pass through one point $p \in P^2(\mathbb{R})$. It follows that for all $v \in V \setminus \{p\}$ the line $\text{aff}(v, p) \in L(V)$ is simple (because there is a simple line through v , and it must pass through p by assumption). Possibly $p \in V$ and possibly not.

If $p \in V$, there is no simple line in $V \setminus \{p\}$, hence this is a collinear set (by Gallai-Sylvester's theorem), contradicting our assumptions. If $p \notin V$, the set $V \cup \{p\}$ has no simple line, contradicting the Gallai-Sylvester theorem. (Motzkin proved Gallai-Sylvester's theorem in projective plane; this also appears in [KM].) We arrived at a contradiction in every case, hence the intersection of all the simple lines is empty. $\square$

2. On $f_{\mathbb{F}}(k; d)$

Assume $\mathbb{F}$ is a field, $k, d \geq 1$.

Theorem 2.1. $f_{\mathbb{F}}(k; d) \leq [\frac{1}{2}k(d+1)]$.

Proof. Assume $S \subset \mathbb{F}^d$, $\text{aff } S = \mathbb{F}^d$, S is not of type k . Let $A \subset S$ be an affine basis of $\mathbb{F}^d$ and put $A = \{a_0, \dots, a_d\}$. For $x \in \mathbb{F}^d$ there are scalars $x_0, \dots, x_d \in \mathbb{F}$, $\sum_{i=0}^d x_i = 1$ such that

$$x = \sum_{i=0}^d x_i a_i. \quad (2.1)$$

For a fixed $0 \leq i \leq d$, $x_i \neq 0 \Leftrightarrow x \notin \text{aff}(A \setminus \{a_i\})$, hence $\#\{x \in S: x_i \neq 0\} < k$. Put $\mathcal{A}(S) = \{(i, x): 0 \leq i \leq d, x \in S, x_i \neq 0\}$ and $\alpha(S) = \#\mathcal{A}(S)$. It follows that $\alpha(S) \leq$

$(k - 1)(d + 1)$. On the other hand

$$\alpha(S) \geq \#A + 2 \cdot \#(S \setminus A) = 2 \cdot \#S - (d + 1). \quad (2.2)$$

(For each $a_i \in A$, $(i, a_i) \in \mathcal{A}(S)$, and for each $x \in S \setminus A$ there are at least two i 's, say i, j ($i \neq j$), such that $(i, x), (j, x) \in \mathcal{A}(S)$.) Hence $2 \cdot \#S - (d + 1) \leq (k - 1)(d + 1)$, i.e., $2 \cdot \#S \leq k(d + 1)$, i.e.,

$$\#S \leq \lceil \frac{1}{2}k(d + 1) \rceil. \quad (2.3)$$

Theorem 2.2. For $d = 2m - 1$ odd ($m \geq 1$) and $k \geq 2$:

- (i) $f_{\mathbb{F}}(k; d) = mk$.
- (ii) If $S \subset \mathbb{F}^d$, $\text{aff } S = \mathbb{F}^d$, $\#S = mk$, and if S is not of type k , then S consists of m mutually disjoint sets, each of cardinality k , contained in m skew lines in $\mathbb{F}^d$.

Proof. (i) By Theorem 2.1 $f_{\mathbb{F}}(k; 2m - 1) \leq \frac{1}{2}k(2m - 1 + 1) = km$. It is easy to check that the example described in part (ii) is not of type k ; hence $f_{\mathbb{F}}(k; 2m - 1) \geq mk$.

(ii) When is (2.3) satisfied as an equality? A necessary condition for this is that each point in $S \setminus A$ be an affine combination of precisely two points in A , i.e., it lies on one of the lines of $L(A)$, and there is no affine combination of three points in A which is in S . So assume that $\#S = mk$, and that S is included in the union of the lines of $L(A)$. Clearly, no two lines in $L(A)$ have a point in common, unless it is in A .

Claim. For any two intersecting lines in $L(A)$, at most one contains more than two points (of S).

Proof. By r.a.a., assume, without loss of generality, that each one of the lines a_0a_1 , a_0a_2 contains a point of $S \setminus \{a_0, a_1, a_2\}$. Let b_1, b_2 be points of $S \setminus \{a_0, a_1, a_2\}$ on the lines a_0a_1 , a_0a_2 , respectively. Consider the set $A' = \{b_1, a_1, a_2, \dots, a_{2m-1}\}$. Clearly, A' is an affinely independent set and $\text{aff } A' = \mathbb{F}^d$. By

Fig. 3

the foregoing considerations each point in S is an affine combination of the members of A' with at most two nonzero coefficients. But it is possible to represent

$$b_2 = \alpha_0 a_0 + \alpha_2 a_2, \quad \text{where } \alpha_0 + \alpha_2 = 1 \quad \text{and} \quad \alpha_0, \alpha_2 \neq 0,$$

and

$$a_0 = \beta_1 b_1 + \alpha_1 a_1, \quad \text{where } \beta_1 + \alpha_1 = 1 \quad \text{and} \quad \beta_1, \alpha_1 \neq 0,$$

hence

$$b_2 = \alpha_0 a_0 + \alpha_2 a_2 = \alpha_0(\beta_1 b_1 + \alpha_1 a_1) + \alpha_2 a_2 = \alpha_0 \beta_1 b_1 + \alpha_0 \alpha_1 a_1 + \alpha_2 a_2$$

with $\alpha_0 \beta_1 + \alpha_0 \alpha_1 + \alpha_2 = 1$ and all three coefficients are $\neq 0$. This contradiction proves the claim. $\square$

It follows from the claim that $S \setminus A$ is included on m skew lines from $L(A)$. Assume, without loss of generality, that $S \setminus A$ is contained in the union of the skew lines $a_0 a_1, a_2 a_3, \dots, a_{2m-2} a_{2m-1}$. No line $a_i a_{i+1}$, $0 \leq i \leq 2m-2 = d-1$, contains more than k points of S (e.g., if $\#(S \cap \text{aff}(a_0 a_1)) > k$, then the hyperplane spanned by $a_1, \dots, a_d$ is of type k). But $\#S = mk$, hence every line contains precisely k points of S . $\square$

Remark 2.3. Theorem 2.2 and its proof remain valid over a projective space $P(\mathbb{F}^d)$, i.e., $f_{P(\mathbb{F})}(k; 2m-1) = f_{\mathbb{F}}(k; 2m-1) = km$. For this we may also rely on Theorem 2.2 itself by the following standard argument: if $S \subset P(\mathbb{F}^d)$, find an hyperplane $H \subseteq P(\mathbb{F}^d)$ such that $S \cap H = \emptyset$; identify H with the hyperplane at infinity, and then S is contained in the affine part of $P(\mathbb{F}^d)$, where Theorem 2.2 applies.

Theorem 2.2 completely characterizes $f_{\mathbb{F}}(k; d)$ for odd d , hence this case is of no further interest. By Theorem 2.1 $f_{\mathbb{F}}(k; 2m) \leq [\frac{1}{2}k(2m+1)] = km + [k/2]$, and, as remarked in the beginning of the proof of Theorem 1.1, $f_{\mathbb{F}}(k; 2m) \geq km + 1$, thus

$$km + 1 \stackrel{(I)}{\leq} f_{\mathbb{F}}(k; 2m) \stackrel{(II)}{\leq} km + \left\lceil \frac{k}{2} \right\rceil \quad (2.4)$$

for all $m \geq 1, k \geq 2$. Recall that the proof of (2.4(I)) was based on $m-1$ sets, each set is collinear and of cardinality k , where the $m-1$ lines are skew as flats in $\mathbb{F}^{2m}$, plus a two-dimensional set (of cardinality $k+1$) whose spanned plane is skew to the affine span of the rest. This example is referred to as the *standard example*. The following result may be viewed as a generalization of Theorem 1.1 to arbitrary fields $\mathbb{F}$ for $k = 2, 3, 4$. Later we see that such a generalization for $k \geq 5$ is impossible without further assumptions, e.g., orderliness of $\mathbb{F}$.

Theorem 2.4

- (i) $f_{\mathbb{F}}(k; 2m) = km + 1$ for $k = 2, 3, 4$.
(ii) Let $2 \leq k \leq 4$ be fixed. Let $S \subset \mathbb{F}^{2m}$ be a spanning set, $\#S = km + 1$, and assume that S is not of type k . Then S is the standard example.

Proof. Regarding part (ii), only the proofs of cases $k = 2, 3$ are supplied here; the one for $k = 4$ being similar to that of $k = 3$, but much longer and tedious, and so we omit it. To prove part (i) note first that, for $k = 2, 3$, the extreme sides of (2.4) are equal, hence the validity of (i) for $k = 2, 3$. For $k = 4$, (2.4) gives

$$4m + 1 \stackrel{(i)}{\leq} f_{\mathbb{F}}(4; 2m) \stackrel{(ii)}{\leq} 4m + 2. \quad (2.5)$$

Hence it is enough to show that (2.5(II)) is satisfied as a strict inequality, i.e., that there can be no case of equality in (2.5(II)). Referring to the proof of Theorem 2.1, every point in $S \setminus A$ is an affine combination of not less than two points of A . If there is a point in S which is a strict affine combination of at least three points of A , then (2.2) would be satisfied as a strict inequality, hence (2.3) would be satisfied as a strict inequality, and (2.5(II)) would be satisfied also in a strict sense.

It follows that equality in (2.5(II)) is possible only if every point $x \in S \setminus A$ has exactly two nonzero coordinates relative to A , i.e., that $S \subset \bigcup L(A)$.

Remark. The same holds, of course, for any other affine base $A' \subset S$.

Each line in $L(A)$ contains at most four points of S (otherwise a spanned hyperplane which does not contain this line would avoid at least four points of S on this line).

Claim. The lines of $L(A)$ which contain points of $S \setminus A$ are mutually disjoint.

Proof. This is a rephrased version of a version of a claim which appears in the proof of Theorem 2.2(ii); the proof is the same. $\square$

Hence there are exactly m lines in $L(A)$ which are incident with $S \setminus A$, each line contains at most four points of S , and there is one more point—of A . This shows that $\#S \leq 4m + 1$.

(ii) The case $k = 2$ is trivial (extremal set S : any affine base for $\mathbb{F}^{2m}$). Assume now that $S \subset \mathbb{F}^{2m}$ is extremal for $f_{\mathbb{F}}(3; 2m)$, i.e., $\#S = 3m + 1$ and no spanned hyperplane avoids three (or more) points of S . Taking a glance back to the proof of Theorem 2.1 it follows that, for odd k (here $k = 3$) and even d (here $d = 2m$), (2.3) is satisfied as an equality iff $2 \cdot \#S = k(d + 1) - 1$, i.e.,

$$\#A + 2 \cdot \#(S \setminus A) = (k - 1)(d + 1) - 1, \quad (2.6)$$

equivalently

$$\#A + 2 \cdot \#(S \setminus A) + 1 = (k - 1)(d + 1). \quad (2.6')$$

The (equivalent) equalities (2.6), (2.6') can be satisfied in one of the following cases only:

Case 1 (see (2.6)). Each point of $S \setminus A$ lies on one of the lines of $L(A)$, on the one hand, and $\exists h: 0 \leq h \leq d$ and $H_h = \text{aff}(A \setminus \{a_h\})$ avoids $k - 2$ ($= 1$) points of S , and each other $H_i, 0 \leq i \leq d, i \neq h$, avoids $k - 1$ ($= 2$) points of S , on the other hand; hence $\alpha(S) =$ the left-hand side of (2.6), on the one hand, and $\alpha(S) = (k - 1)d + (k - 2) = (k - 1)(d + 1) - 1$, on the other hand (in accordance with (2.6)).

Case 2 (see (2.6')). There is precisely one point in $S \setminus A$ which is a strict affine combination of three points of A , and all other points of $S \setminus A$ are on the lines of $L(A)$, on the one hand, and each H_i avoids $k - 1$ ($= 2$) points of S , on the other hand. Hence $\alpha(S) =$ the left-hand side of (2.6'), on the one hand, and $\alpha(S) = (k - 1)(d + 1)$, on the other hand (in accordance with (2.6')).

Put $\delta_{ij} = \#(\text{aff}(a_i, a_j) \cap S) - 2$ for all $0 \leq i < j \leq d$. Clearly, $\delta_{ij} \geq 0$ and, for all $0 \leq i \leq d$ fixed,

$$1 + \sum \{\delta_{ij}: 0 \leq j \leq d, j \neq i\} \leq \#(S \setminus H_i) \leq k - 1 (= 2), \quad (2.7)$$

hence

$$\sum \{\delta_{ij}: 0 \leq j \leq d, j \neq i\} \leq k - 2 (= 1). \quad (2.7')$$

A Discussion of Case 1. Each point lies on such lines; by (2.7'), for each $0 \leq i \leq d$, there is at most one $0 \leq j \leq d$ such that $\delta_{ij} \neq 0$, and we get m skew lines, three points on each line, plus one point, skew to them (clearly, the skew point is a_h , all the other lines span H_h and $a_h \notin H_h$).

A Discussion of Case 2. $\exists x \in S \setminus A: x = \sum_{i=0}^2 x_i a_i, x_i \neq 0$ for $0 \leq i \leq 2$, and $\sum_{i=0}^2 x_i = 1$, and all the other points of S are on the lines of $L(A)$. Each H_i avoids $k - 1 = 2$ points of S , and $x \notin \bigcup L(A)$, but $x \in \bigcap_{i=3}^d H_i$.

Hence, for $3 \leq i \leq d$, both inequalities in (2.7) are satisfied as an equality, and the same for (2.7'). It follows that $\forall i: 3 \leq i \leq d$ there is a unique $0 \leq j \leq d$ such that $\delta_{ij} \neq 0$, hence $\delta_{ij} = 1$, i.e., for each one of the $2m - 2 = 2(m - 1)$ points $a_3, \dots, a_{2m}$ there is a unique line of $L(A)$ which passes through it, and contains another point from the set $S' = S \setminus (A \cup \{x\})$. But

$$\#S' = \#S \setminus \#(A \cup \{x\}) = 3m + 1 - ((2m + 1) + 1) = m - 1$$

and, since any two lines spanned by A have no common point unless it is in A , we conclude that every point of S' lies on exactly one line $a_i a_j, 3 \leq i \leq d, 0 \leq j \leq d$. Thus S is composed of $m - 1$ sets, each with three points, lying on $m - 1$ skew lines, and a two-dimensional set of $k + 1 = 4$ points in general position $\{a_0, a_1, a_2, x\}$.

We conclude the proof of part (ii) by omitting the one for case $k = 4$, which, as mentioned at the beginning, is a case analysis similar to that given above for $k = 3$, but much more painful and uninspiring. $\square$

Remark 2.5. In the next section we see that the range of k 's for which Theorem 2.4 is valid cannot be extended beyond $2 \leq k \leq 4$ (Fano plane furnishes a counterexample for $k = 5$), but a standard argument from first-order logic shows, by Theorem 1.1, that for ordered fields it is correct to replace “ $2 \leq k \leq 4$ ” by “ $2 \leq k$ ” in Theorem 2.4. Here is the argument: Assume $\mathbb{F}$ is ordered and assume $S \subset \mathbb{F}^d$ ($d = 2m$, even) satisfies:

- (i) $\text{aff } S = \mathbb{F}^d$,
- (ii) $\#S = mk + 2$, and
- (iii) S is not of type k ;

we have to contradict this. Let $\bar{\mathbb{F}}$ be the real closure of $\mathbb{F}$ ($\bar{\mathbb{F}} \supset \mathbb{F}$ is a real closed field); then $S \subset \bar{\mathbb{F}}^d$. The sentence “there is a spanning set in $\bar{\mathbb{F}}^d$ of cardinality $mk + 2$ which is not of type k ” is a first-order sentence, which, according to a theorem of Tarski, if true over any real closed field $\bar{\mathbb{F}}$, remains true over any other real closed field, in particular, over the reals $\mathbb{R}$. This contradicts Theorem 1.1.

3. Some More Results in $\mathbb{F}^2$

Proposition 3.1. *Let $S \subset \mathbb{F}^2$ (or $S \subset P^2(\mathbb{F})$) be a spanning set, and assume that every line in $L(S)$ contains at least $q + 1$ points of S . Then $\#S \geq q^2 + q + 1$.*

Remark. Equality occurs, e.g., when q is a prime power and $S = P^2(GF(q))$.

Proof. Let $a \in S$, $L \in L(S)$ such that $a \notin L$. Then $\#(L \cap S) \geq q + 1$, and there are at least $q + 1$ lines connecting a with points on L ; on each such line there are at least q points of S , in addition to A , hence $\#S \geq 1 + (q + 1)q = q^2 + q + 1$. $\square$

As usual the characteristic of a field $\mathbb{F}$ is denoted by $\chi(\mathbb{F})$. The following theorem shows that the range $2 \leq k \leq 4$ in Theorem 2.4 cannot be broadened even to $2 \leq k \leq 5$, at least for fields of characteristic $\chi(\mathbb{F}) = 2$.

Proposition 3.2. *For every field $\mathbb{F}$*

$$f_{\mathbb{F}}(5; 2) = \begin{cases} 7, & \chi(\mathbb{F}) = 2 \text{ and } \#\mathbb{F} \geq 8, \\ 6, & \mathbb{F} = GF(4) \text{ or } \chi(\mathbb{F}) > 2, \\ 4, & \mathbb{F} = \mathbb{Z}_2. \end{cases}$$

Proof. If $\mathbb{F} = \mathbb{Z}_2$, then $\#\mathbb{F}^2 = 2^2 = 4$, and *a priori* there can be no set of type 5 in $\mathbb{F}^2$, hence $f_{\mathbb{Z}_2}(5; 2) = 4$.

Fig. 4

If $\mathbb{F} \neq \mathbb{Z}_2$, then clearly $f_{\mathbb{F}}(5; 2) \geq 6$.

Assume now that $\mathbb{F}$ is a field for which $f_{\mathbb{F}}(5; 2) > 6$, i.e., $f_{\mathbb{F}}(5; 2) \geq 7$. Let $S \subset \mathbb{F}^2$ be a spanning set such that $\#S \geq 7$ and every line in $L(S)$ avoids at most four points of S . Hence every line in $L(S)$ contains at least $\#S - 4$ points of S . We prove that $\chi(\mathbb{F}) = 2$. Put $q = \#S - 5$. Every line spanned by S contains at least $\#S - 4 = (\#S - 5) + 1 = q + 1$ points of S , hence $\#S \geq q^2 + q + 1$ (Proposition 3.1), i.e., $\#S \geq (\#S - 5)^2 + (\#S - 5) + 1 = \#S^2 - 9 \cdot \#S + 21$ or

$$\#S^2 - 10 \cdot \#S + 21 \leq 0. \quad (3.1)$$

This implies $3 \leq \#S \leq 7$ (the solutions of the equation $x^2 - 10x + 21 = 0$ are $x = 3$ and $x = 7$). Since $\#S \geq 7$ (by assumption) we conclude that $\#S = 7$. It follows that S is a set of seven points in $\mathbb{F}^2$, every line spanned by which contains at least $\#S - 4 = 7 - 4 = 3$ points of S . It is easy to check that S and $L(S)$ form a Fano configuration (Fig. 4), i.e., S is combinatorially equivalent to $P^2(\mathbb{Z}_2)$ (every spanned line contains exactly three points).

It is known that Fano's configuration is embeddable in $P^2(\mathbb{F})$ only if $\chi(\mathbb{F}) = 2$ (see pp. 37–38 of [B] but see pp. 54–55 of [Har] or p. 424, Example 2.2, of [KN] for a better argument), and since $\mathbb{F}^2$ is embeddable in $P^2(\mathbb{F})$ it is clear that $\chi(\mathbb{F}) = 2$ is a necessary condition for Fano's configuration to be already embeddable in $\mathbb{F}^2$. Thus $\chi(\mathbb{F}) > 2 \Rightarrow f_{\mathbb{F}}(S; 2) = 6$.

It remains to handle the case $\chi F = 2$ and $\#F \geq 4$. We have to show that Fano's configuration is not embeddable in $(GF(4))^2$, but that it is embeddable in $(GF(8))^2$. Define $p(x) = x^3 + x + 1$; clearly, $p(x)$ does not factor over $\mathbb{Z}_2$, hence the minimal factorization field of $p(x)$ which contains $\mathbb{Z}_2$ is of order $2^3 = 8$. This field is obtained by adjunction of a root α of $p(x)$ and it is denoted by $\mathbb{Z}_2(\alpha)$. Since $\mathbb{Z}_2(\alpha)$ is of order 8, we have $\mathbb{Z}_2(\alpha) = GF(8)$. (It is easy to see that $\mathbb{Z}_2(\alpha)$ is actually a splitting field of $p(x)$, but this is of no use to us.)

These considerations show that in order to finish the proof of Proposition 3.2 it suffices to prove

Proposition 3.3. *Necessary and sufficient conditions for the embeddability of Fano's configuration in $\mathbb{F}^2$ are:*

- (i) $\chi(\mathbb{F}) = 2$, and
- (ii) $p(x) = x^3 + x + 1$ has a root in $\mathbb{F}$.

Fig. 5

Proof. (Necessity) Condition (i) is necessary as mentioned above. To prove that (ii) is also necessary it suffices to show that $(GF(4))^2$ does not contain a Fano configuration (this is sufficient because $p(x)$ does not factor over $GF(4)$). Let us denote by $\mathcal{F} = \{a, b, c, d, e, f, g, h\}$ the vertices of a Fano configuration with incidence relations as shown in Fig. 4, and assume that it is embedded in $(GF(4))^2$ (we have to contradict this assumption). We may assume, without loss of generality, that $a = (0, 0)$, $b = (0, 1)$, and $c = (1, 0)$ (affine transformation) (Fig. 5). Then $g = (\alpha, 0)$ and $e = (0, \beta)$ where $\alpha \neq 0, 1$, $\beta \neq 0, 1$, i.e., $\{\alpha, \beta\} = \mathbb{F} \setminus \{0, 1\}$ (since $\#\mathbb{F} = 4$). Put $h = (u, v)$. Clearly, $u, v \neq 0$ and

$$\begin{cases} \text{(I)} & u + \alpha v = \alpha & (\text{since } h \in \text{aff}(b, g)), \\ \text{(II)} & \beta u + v = \beta & (\text{since } h \in \text{aff}(c, e)). \end{cases} \quad (3.2)$$

Clearly, (u, v) is a *unique* solution of (3.2), hence

$$0 \neq \begin{vmatrix} 1 & \alpha \\ \beta & 1 \end{vmatrix} = 1 - \alpha\beta. \quad (3.3)$$

From the assumption that $\#\mathbb{F} = 4$ it follows that $\alpha = \beta$ (otherwise they are multiplicative reciprocals of each other since the reciprocal of α is not 0, not 1, and not α (the equation $x^2 = 1$ has a multiple root 1 in characteristic 2)). Summing (I) and (II) in (3.2) we obtain

$$(1 + \alpha)(u + v) = \alpha + \beta = 0. \quad (3.4)$$

Since $1 + \alpha \neq 0$ ($\alpha \neq 1$), (3.4) implies

$$u + v = 0. \quad (3.5)$$

Since $d \in \text{aff}(a, h)$, $\exists \mu \in \mathbb{F}: d = \mu(u, v)$, where $\mu \neq 0, 1$. By (3.5), $\mu u + \mu v = 0$; on the other hand, d being on the line $\text{aff}(b, c)$, whose equation is $x + y = 1$, $\mu u + \mu v = 1 \neq 0$, a contradiction.

(Sufficiency) Assume that $\chi(\mathbb{F}) = 2$ and that $p(x)$ factors over $\mathbb{F}$, i.e., $\exists \alpha \in \mathbb{F}: p(\alpha) = 0$. We construct a Fano configuration in $\mathbb{F}^2$ which satisfies the incidence

relation of Fig 4. Clearly, $\alpha \neq 0, 1$, hence $\alpha^2 \neq \alpha^3$. Put $\beta = \alpha + 1$, then $\beta \neq 0, 1$ and, since $\alpha^2 \neq \alpha^3$, $1 + \alpha\beta = 1 + \alpha(\alpha + 1) = 1 + \alpha + \alpha^2 \neq 1 + \alpha + \alpha^3 = 0$, i.e.,

$$1 + \alpha\beta \neq 0. \quad (3.6)$$

Now define $a = (0, 0)$, $b = (0, 1)$, $c = (1, 0)$, $g = (\alpha, 0)$, $e = (0, \beta)$, denote the left-hand side of (3.6) by Δ , and put

$$v = \frac{\begin{vmatrix} \alpha & \alpha \\ \beta & 1 \end{vmatrix}}{\Delta} = \frac{\alpha(1 + \beta)}{\Delta} = \frac{\alpha^2}{\Delta},$$

$$u = \frac{\begin{vmatrix} 1 & \alpha \\ \beta & \beta \end{vmatrix}}{\Delta} = \frac{\beta(1 + \alpha)}{\Delta} = \frac{\beta^2}{\Delta},$$

It is easy to check that $u \neq 0, 1$ and $v \neq 0, 1$, hence the point $h = (u, v)$ is different from a, b, c, e, g . Moreover, $h \in \text{aff}(b, g) \cap \text{aff}(c, e)$, $h \notin \text{aff}(a, b) \cup \text{aff}(a, c)$, and we still have to check that $h \notin \text{aff}(b, c)$, i.e., we have to check that $u + v \neq 1$, which means $u + v = (\alpha^2 + \beta^2)/\Delta \neq 1$ (please check). Put $d = (\Delta u, \Delta v)$; this makes $d \in \text{aff}(b, c) \cap \text{aff}(a, h)$ and $d \notin \{a, b, c, e, g, h\}$ (please check). So, in order for $\mathcal{F} = \{a, b, c, d, e, g, h\}$ to be a Fano configuration as described in Fig. 4, it remains to check that the points e, g , and d are collinear, i.e., that

$$\begin{vmatrix} 1 & \alpha & 0 \\ 1 & 0 & \beta \\ 1 & \Delta u & \Delta v \end{vmatrix} = 0.$$

This finishes the proof of Proposition 3.3, as well as that of Proposition 3.2. $\square$

Remark 3.4. Here is another example (Fig. 6), showing that the range $2 \leq k \leq 4$ in Theorem 2.4 cannot be broadened to $2 \leq k$. The example is again in the plane ($m = 1$), this time with $k = 7$ and $\mathbb{F} = \mathbb{Z}_3$; it consists of the affine plane $(\mathbb{Z}_3)^2$, which has nine points, spanning twelve lines, three points on each line. (See Fig. 6; the coordinates assigned to each point are in $\mathbb{Z}_3$, of course, and all the twelve lines sketched are straight lines in $(\mathbb{Z}_3)^2$ (including the four “curved” lines). The reader is urged to check the details.) Here $\#S = \#((\mathbb{Z}_3)^2) = 9 > 8 = 1 \cdot 7 + 1 = m \cdot k + 1$, and still there is no 7-avoiding line. It is well known (see Remark 3.7(iii) below) that this configuration can be embedded in the plane over the complexes, $\mathbb{C}^2$, implying $f_{\mathbb{C}}(7; 2) \geq 9$, and actually this is satisfied as an equality—by Proposition 3.6 below. As suggested implicitly in [K], we call this the *Sylvester–Gallai nine-point configuration*.

Staying in $\mathbb{F}^2$ ($\mathbb{F}$ being a general field), we now give an upper bound for $f_{\mathbb{F}}(k; 2)$ ($k \geq 2$) which is tight in the sense that there is an unbounded sequence of k 's for which this bound coincides with $f_{\mathbb{F}}(k; 2)$.

Fig. 6

Proposition 3.5. For every field $\mathbb{F}$, and for every integer $k \geq 2$,

$$f_{\mathbb{F}}(k; 2) \leq k - 1 + \lceil \sqrt{k} \rceil \quad (= k + \lceil \sqrt{k-1} \rceil) \quad (3.7)$$

($\lceil x \rceil$ = the upper integer part of x). Moreover, the set of integers k for which (3.7) is satisfied as an equality (for some $\mathbb{F}$) is unbounded. (The same is true for $P^2(\mathbb{F})$.)

Proof. Let $S \subset \mathbb{F}^2$ be a spanning set of cardinality $n := \#S$. Assume that S is not of type k , i.e., for every line $L \in L(S)$, $\#(S \setminus L) \leq k - 1$, i.e., $\#(S \cap L) \leq n - k + 1$. By Proposition 3.1 (with $q + 1 = n - k + 1$), $n \geq (n - k + 1)(n - k) + 1$, or, equivalently (by simple calculation), $n^2 - 2kn + k^2 - k + 1 \leq 0$. The range of solution of the quadratic inequality $x^2 - 2kx + k^2 - k + 1 \leq 0$ being $k - \sqrt{k-1} \leq x \leq k + \sqrt{k-1}$ (please check), while the effective range in our case being only positive integers we conclude that $n \leq k + \lceil \sqrt{k-1} \rceil$, which is equivalent to (3.7). This proves the first part of the theorem.

Let $k \geq 2$ be an integer such that $q = \sqrt{k-1}$ is a prime power (in particular, it is an integer). Then $S := P^2(GF(q))$ is a planar set of cardinality $\#S = q^2 + q + 1 = k - 1 + \lceil \sqrt{k} \rceil$, and every line spanned by S contains $q + 1$ points, hence avoids (exactly) $q^2 + q + 1 - (q + 1) = q^2 = k - 1$ points. This proves the second part for projective planes. The assertion for affine planes $\mathbb{F}^2$ is proved by noting that if $\#\mathbb{F}$ is big enough, then $P^2(GF(q))$ can be embedded in $\mathbb{F}^2$ (with $\chi(\mathbb{F}) = \chi(GF(q))$ of course). $\square$

Next, we discuss the situation for a field $\mathbb{F}$ with $\chi(\mathbb{F}) = 0$.

Proposition 3.6. Let $\mathbb{F}$ be a field with $\chi(\mathbb{F}) = 0$. Then

$$f_{\mathbb{F}}(k; 2) \leq k + 2. \quad (3.8)$$

For $k \equiv 1 \pmod{3}$ this is satisfied as an equality. (See also Remark 3.7(i) below concerning the possibility that (3.8) be satisfied as an equality for $k \equiv 2 \pmod{3}$.)

Proof. Assume first that $F = \mathbb{C}$. In this case (3.8) follows directly from the following theorem due to Hirzebruch (see p. 102, lines –13 and –14, of [K]; originally, it appears in p. 132 of [Hi] and we restate it in another form):

Theorem (Hirzebruch). *Let $S \subset \mathbb{C}^2$ be a spanning set such that every line $L \in L(S)$ contains at least three points of S . Then there is a line $L \in L(S)$ such that $\#(L \cap S) = 3$.*

The proposition being true for $F = \mathbb{C}$, it follows that the same holds for any field of characteristic 0, by the completeness of the first-order theory of algebraically closed fields (compare Remark 2.5 above which is based on a similar argument).

If $k \equiv 1 \pmod{3}$, then an example due to Motzkin [M, p. 461], presented in Remark 3.7(iii) below, proves equality in (3.8). (Note that in $\mathbb{C}$ there are roots of unity of any order.) $\square$

Remark 3.7. (i) As remarked in Remark 3.4, the Sylvester–Gallai nine-point configuration is embeddable in $\mathbb{C}^2$, this being a special case of the example of Motzkin that was mentioned in the proof of Proposition 3.6 above. (The example is given by Motzkin in $P^2(\mathbb{C})$ (in homogeneous three coordinates), but of course it is also embeddable in $\mathbb{C}^2$: by the standard argument of Remark 2.3, just take any line in $P^2(\mathbb{C})$ not passing through any point of the example, and declare it as “the line in infinity”; the rest lies in the affine part of $P^2(\mathbb{C})$.) It is worth mentioning that this example is unique in some sense (see Theorem 3.11, p. 433, of [KN]).

Finally, it is implicitly claimed on p. 347, line –9 of [BFK] that (3.8) is also satisfied as an equality for $k \equiv 2 \pmod{3}$, but this is false, at least for $k = 8$, because in Theorem 3.5, p. 426, of [KN] it is shown that if $\chi(F) \neq 3$, then any ten points spanning $\mathbb{F}^2$ must span a simple line (i.e., a line passing through exactly two points of the set).

(ii) There is no elementary proof of the above theorem of Hirzebruch (the one in [Hi] and [K] depends on a lemma from algebraic geometry), and, as remarked at the end of [K], there ought to be a simpler proof for it.

(iii) For the sake of completeness we repeat here Motzkin’s simple generalization of the Sylvester–Gallai nine-point configuration, which has been referred to twice in this paper (the example deserves to be better known, anyway). It is of a set $S \subset P^2(\mathbb{C})$, $\#S = 3m$ ($m \geq 3$), aff $S = P^2(\mathbb{C})$, such that $\forall L \in L(S)$: $\#(S \cap L) \geq 3$. Assume $m \geq 3$ and put $\zeta = e^{2\pi i/m}$, $U = \{\zeta^v : 0 \leq v \leq m-1\}$ (the set of roots of unity of order m), $S_1 = \{(0, -1, u) : u \in U\}$, $S_2 = \{(v, 0, 1) : v \in U\}$, $S_3 = \{(1, w, 0) : w \in U\}$, and $S = S_1 \cup S_2 \cup S_3$. S is a subset of $P^2(\mathbb{C})$ (in homogeneous coordinates) and $\#S = 3m \geq 9$.

Claim. *For any two points $p, q \in S$, aff(p, q) contains a third point of S .*

Proof. It is readily seen that S_i is a collinear set and $\#S_i = m \geq 3$ for $1 \leq i \leq 3$ (the determinant of any three points in S_i has a column of zeros). So assume,

without loss of generality, that p and q belong to different S_i 's, say $p = (0, -1, u) \in S_1$, $q = (v, 0, 1) \in S_2$ (a similar argument applies to $p \in S_i$, $q \in S_j$ for any $1 \leq i < j \leq 3$). Since $uv \in U$ and U is a (multiplicative) group, $\exists w \in U: uvw = 1$. Then

$$\begin{vmatrix} 0, & -1, & u \\ v, & 0, & 1 \\ 1, & w, & 0 \end{vmatrix} = -1 + uvw = 0,$$

i.e., $r := (1, w, 0) \in S_3$ is collinear with p and q . □

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