Convex Disks Can Cover Their Shadow

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Abstract. We give a short proof of the fact that given any two-dimensional convex disk in three-space and an orthogonal projection C' of C on a plane then there is a congruent copy of C which contains C' .

We give a short proof for the next theorem conjectured independently by Zalgaller [1] and Pach [2] and proved independently by Kowaljov [3] and Debrunner and Mani-Levitska [4].

Theorem. Any orthogonal projection C' of a two-dimensional convex compact set C in $\mathbb{R}^3$ onto a two-dimensional subspace S can be covered by some congruent copy of C .

Proof. It is clear that there is a congruent copy C_1 of C in the plane S and there is an orthogonal basis $\{x, y\}$ in S such that C' is the image of C_1 under a linear map which maps x to x and y to λy with some $\lambda \in [0, 1]$. We denote the image of point P under this map by P' .

It would suffice to find parallel lines e and f , tangent to C_1 at points E and F such that, for each point P of C_1 , $\angle PEF \geq \angle P'E'F'$ and $\angle PFE \geq \angle P'F'E'$. Indeed, in this case, any orientation preserving isometry i of $\mathbb{R}^2$ which maps the segment $E'F'$ onto the segment EF (with E' contained in the segment EF') will map C' onto C_1 . In fact, for each $P \in C_1$, $i(P') \in i(\triangle P'E'F') \subset \triangle PEF \subset C_1$ will hold (Fig. 1).

In order to find e and f let $\tilde{x}$ be the direction determined by x . Let $\angle \tilde{d}_1 \tilde{d}_2$ be the angle between directions $\tilde{d}_1$ and $\tilde{d}_2$ measured by rotating $\tilde{d}_1$ counterclockwise into $\tilde{d}_2$. Let $\alpha = \arctan(1/\sqrt{\lambda})$. For every direction $\tilde{d}$ with $|\angle \tilde{x} \tilde{d}| \leq \alpha$ define the direction $\tilde{d}^*$ with $\tan(\angle \tilde{x} \tilde{d}) \tan(\angle \tilde{x} \tilde{d}^*) = 1/\lambda$ (Fig. 2). Easy computation shows that $\angle \tilde{d} \tilde{d}^* = \angle \tilde{d}' \tilde{d}'^*$. Moreover, if $|\angle \tilde{x} \tilde{d}| \leq \alpha$ and $\tilde{d}_0$ is between $\tilde{d}$ and $\tilde{d}^*$ or between $\tilde{d}$ and $\tilde{d}^*$ (where $\tilde{d}^*$ is the direction opposite to $\tilde{d}^*$), then $\angle \tilde{d} \tilde{d}_0 \geq \angle \tilde{d}' \tilde{d}'_0$.

Fig. 1

So it suffices to find parallel lines e and f , tangent to C_1 such that $|\angle \vec{x} \overrightarrow{EF}| \leq \alpha$ and $\overrightarrow{EF}^* = \vec{e} = \vec{f}$ holds. $\square$

Define a decreasing continuous function g from $[-\alpha, \alpha]$ to $[0, \pi]$ such that $g(-\alpha) = \pi$, $g(\alpha) = 0$, and $g(\angle \vec{x} \vec{d}) = \angle \vec{d} \vec{d}^*$ for every $\vec{d}$ with $|\angle \vec{x} \vec{d}| \leq \alpha$. Let G be the graph of g (Fig. 3). Define $H \subset [-\alpha, \alpha] \times [0, \pi]$ such that $(\mu, \nu) \in H$ iff there

Fig. 2

Fig. 3

are tangents m and n to C_1 at points M and N such that $m \parallel n$, $|\angle \vec{x} \overline{MN}| \leq \alpha$, $\mu = \angle \vec{x} \overline{MN}$, and $\nu = \angle \overline{MN} m$, where $\angle \overline{MN} m$ is the angle between $\overline{MN}$ and m measured by rotating $\overline{MN}$ counterclockwise into m . For any $\mu \in [-\alpha, \alpha]$ there is at least one $\nu \in [0, \pi]$ satisfying $(\mu, \nu) \in H$. The graph of H is a continuous curve connecting the segments $[-\alpha] \times [0, \pi]$ and $[\alpha] \times [0, \pi]$. So $H \cap G \neq \emptyset$. Any pair of lines m, n and points M, N corresponding to an element of $H \cap G$ is a proper choice for the lines e and f and the points E and F , respectively.

References

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