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The Upper Envelope of Piecewise Linear Functions and the Boundary of a Region Enclosed by Convex Plates: Combinatorial Analysis*

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Abstract. Let $f_1, \ldots, f_m$ be (partially defined) piecewise linear functions of d variables whose graphs consist of n d-simplices altogether. We show that the maximal number of d-faces comprising the upper envelope (i.e., the pointwise maximum) of these functions is $O(n^d\alpha(n))$, where $\alpha(n)$ denotes the inverse of the Ackermann function, and that this bound is tight in the worst case. If, instead of the upper envelope, we consider any single connected component C enclosed by n d-simplices (or, more generally, $(d-1)$-dimensional compact convex sets) in $\mathbb{R}^{d+1}$, then we show that the overall combinatorial complexity of the boundary of C is at most $O(n^{d+1-\varepsilon(d+1)})$ for some fixed constant $\varepsilon(d+1) > 0$.

1. Introduction, Results

Given a collection of continuous functions $f_i: \mathbb{R} \to \mathbb{R}$, $1 \le i \le n$, let their *upper envelope* be defined as the pointwise maximum of the f_i's. Assume that the graphs of any two distinct functions of our collection intersect in at most s points, for some fixed integer s. Then the graph of their upper envelope consists of a finite number of arcs separated by some *intersection points* of the f_i's, i.e., by points belonging to the graph of more than one function. Let $\lambda_s(n)$ denote the maximum

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number of such intersection points which lie on the upper envelope. It is known that:

(i) $\lambda_1(n) = n$(trivial).
(ii) $\lambda_2(n) = 2n - 1$ [At].
(iii) $\lambda_3(n) = \Theta(n\alpha(n))$, where $\alpha(n)$ is the (extremely slowly growing) inverse of the Ackermann function [HS].
(iv) $\lambda_4(n) = \Theta(n \cdot 2^{\alpha(n)})$ [ASS].
$\lambda_{2s}(n) = O(n \cdot 2^{O(\alpha(n)^{s-1})})$ for $s > 2$ [ASS].
$\lambda_{2s+1}(n) = O(n \cdot \alpha(n)^{O(\alpha(n)^{s-1})})$ for $s \geq 2$ [ASS].
$\lambda_{2s}(n) = \Omega(n \cdot 2^{\Omega(\alpha(n)^{s-1})})$ for $s > 2$ [ASS].
(See also [Sh1] and [Sh2] for earlier bounds.)

Note that all of these functions are about an order of magnitude smaller than the naive bound $O(sn^2)$ on the number of *all* intersection points of the functions.

An important corollary of (iii) is that the upper envelope of *n partially defined linear functions* (i.e., n straight line segments) consists of at most $O(n\alpha(n))$ subsegments (see Fig. 1). Indeed, we can easily extend our segments to total functions (defined for every $x \in \mathbb{R}$) so that any two of them have at most three intersection points, and the upper envelope of the extended functions consists of at least as many pieces as the upper envelope of the original segments. Recently, it was shown [WS] that this bound is tight, i.e., there exist collections of n segments in the plane whose upper envelopes consist of $\Omega(n\alpha(n))$ subsegments (a simplified version of this construction has recently been given in [Sho]). For a strengthening of these theorems, see [EGH*]. Numerous applications for motion planning and other geometric problems are discussed in [SCKLPS].

The aim of this paper is to generalize the above results to multivariate functions. In contrast to the univariate case, the combinatorial structure of the upper envelope of functions with d variables has been very little studied and appears to be substantially more difficult. The only previous attempts in this direction were made in [SL] and [SS].

We restrict our attention to the case of *piecewise linear* (or, in other words, *polyhedral*) functions f_i with d variables, i.e., to functions whose graphs consist of a finite number of d-simplices in $\mathbb{R}^{d+1}$. In fact, by splitting the graph of every f_i into pieces, we may assume that the graph of each (now only partially defined) function is a d-dimensional simplex. The *upper envelope* of these *d-simplices* is

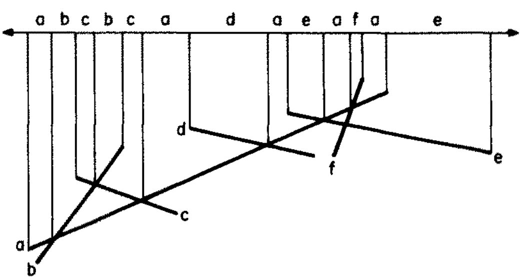

Fig. 1. The upper envelope of segments in the plane.

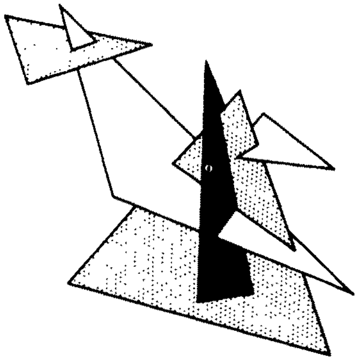

Fig. 2. The upper envelope of triangles (2-simplices) in 3-space.

defined as the graph of the upper envelope of the corresponding functions. See Fig. 2 for an illustration. Hence the upper envelope of n d-simplices in $\mathbb{R}^{d+1}$ is a polyhedral surface.

In Section 2 of this paper we show that for $d=2$ the following is true.

Theorem 1. *The upper envelope of n triangles scattered in $\mathbb{R}^3$ has at most $O(n^2\alpha(n))$ faces, and this bound is asymptotically tight.*

The total number of all i-dimensional faces of a polyhedral surface σ over all $i \geq 0$ is called the *combinatorial* (or *total*) *complexity* of σ.

Using Euler's formula for planar maps, we can easily deduce the following slightly stronger form of Theorem 1.

Theorem 1′. *The combinatorial complexity of the upper envelope of n triangles scattered in $\mathbb{R}^3$ is at most $O(n^2\alpha(n))$, and this bound is asymptotically tight.*

In Section 3 we generalize Theorem 1 for every $d \geq 2$. We establish

Theorem 2. *The upper envelope of n d-simplices scattered in $\mathbb{R}^{d+1}$ has at most $O(n^d\alpha(n))$ facets (i.e., d-dimensional faces), and this bound is asymptotically tight for every $d \geq 2$.*

This immediately implies that, given any collection of piecewise linear functions with d variables, whose graphs contain n d-simplices altogether, the number of facets in the graph of their upper envelope is at most $O(n^d\alpha(n))$.

The proof of the upper bound in Theorems 1 and 2 is by induction on d. We divide our collection of d-simplices into two subcollections of nearly equal sizes, and then we obtain a recurrence relation between the number of facets of the upper envelope M of the entire collection and the number of facets of the upper envelopes M_1 and M_2 of the two subcollections. In this recurrence we show that the number of extra facets of M not "accounted for" by faces of M_1 or M_2 is related to the complexity of the envelope M when restricted over some $(d-1)$-dimensional hyperplanes. To bound this number, we apply the induction hypothesis (whose basis is provided by the results of [HS]).

Theorem 2 gives an upper bound only on the number of *facets* of the upper envelope. For $d \geq 3$ we have been unable to establish the same bound on the *combinatorial complexity* of the upper envelope, i.e., to prove an analogue of Theorem 1′. This gap has recently been closed by Edelsbrunner [Ed] who, using a nice counting argument, has extended Theorem 1′ to arbitrary dimensions $d \geq 3$.

Section 4 of this paper is devoted to the following more general problem. Let S be a collection of n d-dimensional compact convex sets (so-called *plates*) in $\mathbb{R}^{d+1}$. If we delete from $\mathbb{R}^{d+1}$ all points belonging to at least one of these plates, then the space may split up into a number of connected components. Determine or estimate the maximal possible combinatorial complexity of the boundary of such a component, if d is fixed and n tends to infinity.

For $d=1$ it was shown in [PSS] that this maximal complexity is $\Theta(n\alpha(n))$. In the case $d \geq 2$ we prove the following result.

Theorem 3. *For every $d+1 \geq 3$ there exists a constant $\varepsilon(d+1)>0$ such that, given any collection* S *of n d-dimensional convex plates arranged in $\mathbb{R}^{d+1}$, the combinatorial complexity of the boundary of any connected component of $\mathbb{R}^{d+1} - \bigcup \mathrm{S}$ is at most $O(n^{d+1-\varepsilon(d+1)})$.*

The proof of Theorem 3 is based on a combinatorial result of Erdös [Er], which gives an upper bound for the number of hyperedges of a uniform hypergraph containing no complete subhypergraph of a given size.

In Section 5 we reformulate Erdös's theorem in a slightly stronger form, and this enables us to prove that Theorem 3 and the corollary are valid with $\varepsilon(3)=\frac{1}{49}$. (See Theorem 4 in Section 5.)

The results obtained in this paper have many applications in discrete and computational geometry. In a companion paper [EGS] we describe an efficient algorithm for the calculation of the upper envelope of n triangles in $\mathbb{R}^3$, whose time complexity is $O(n^2\alpha(n))$, and is thus optimal in the worst case. We also describe various applications of these results, including translational motion planning for a polyhedral object in $\mathbb{R}^3$ amidst a collection of polyhedral obstacles, combinatorial analysis and algorithms for the calculation of the set of all common transversal planes of a set of polyhedra in $\mathbb{R}^3$, hidden surface elimination of intersecting polyhedral surfaces, and generalized Voronoi diagrams of point clusters in the plane.

2. The Upper Envelope of Triangles in 3-space

In this section we prove Theorem 1. We use the following notation. Let $\mathrm{S}=\{S_1,\ldots,S_n\}$ be a collection of n nonvertical triangles (2-simplices) in $\mathbb{R}^3$, and let $S_1^*,\ldots,S_n^*$ denote their orthogonal projections onto the x-y plane. Assume that these triangles are in *general position*, i.e., no two of them overlap one another, no vertex of one of them lies on another triangle, no two edges of distinct S_i's intersect, no edge of one triangle meets the intersection of any other two triangles, and the edges of the projections S_i^* do not overlap one another. Call such a collection of triangles *regular*. Each S_i can be regarded as the graph of a partially defined function $S_i(x, y)$, and in the rest of this section, without the danger of confusion, S_i will stand both for the triangle and the corresponding function.

Let $M_{\mathbf{S}} = M$ denote the *upper envelope* of the S_i's, i.e., for each (x, y), $M(x, y)$ is the z-coordinate of the highest point of intersection of the vertical line through (x, y) with any of the S_i's (if there is no such intersection, we put $M(x, y) = -\infty$). Let $\mathbf{S}(x, y) \subset \mathbf{S}$ denote the set of those triangles S_i which *attain* $M(x, y)$ (i.e., $(x, y, M(x, y)) \in S_i$). Let M^* be the orthogonal projection of M to the x-y plane, i.e., M^* is a straight-edge planar map formed by the maximal connected regions over which $\mathbf{S}(x, y)$ is constant. We denote by $N(\mathbf{S})$ the number of two-dimensional faces in M^*, and refer to it as the *face-complexity* of the upper envelope M. Let $\psi(n)$ denote the maximum value of $N(\mathbf{S})$ for any regular collection $\mathbf{S}$ of n triangles in 3-space. We can now restate Theorem 1 as follows.

Theorem 1. $\psi(n) = O(n^2\alpha(n))$.

Proof. Let $\mathbf{S} = \{S_1, \ldots, S_n\}$ be a regular collection of triangles in 3-space, and consider their x-y projections $S_1^*, \ldots, S_n^*$. Partition this collection into two disjoint subcollections $\mathbf{S}_1, \mathbf{S}_2$, each consisting of nearly $n/2$ triangles. Let M_1, M_2 denote the upper envelopes of the triangles in $\mathbf{S}_1$ and $\mathbf{S}_2$, respectively. The number of two-dimensional faces of M_1 and of M_2 are both $\leq \psi(n/2)$, by definition.

However, the complexity of the overall upper envelope M can in general be larger than the sum of the complexities of the "subenvelopes" M_1, M_2. The reason is that a face F of M_i can be split into several faces in M due to the addition of the other subenvelope. To overcome this difficulty, consider one of these subenvelopes, say M_1, and superimpose the $3n$ lines containing the edges of all the projections $S_1^*, \ldots, S_n^*$ on the map M_1^*, to produce a refined planar map $\bar{M}_1$. See Fig. 3 for an illustration.

Lemma 2.1. *Let F be a face of $\bar{M}_1$ which is contained in the projection of some face of M_1 which is part of some triangle S_i. Then the portion of F over which S_i attains the overall envelope M is connected.*

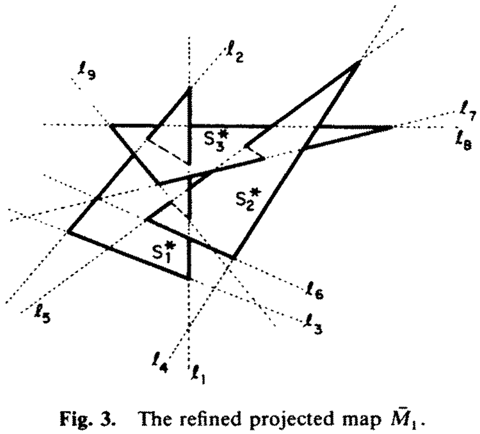

Fig. 3. The refined projected map $\bar{M}_1$.

Proof. Let F' be that portion of F. The $3n$ added lines partition the x–y plane into a collection of openly disjoint convex polygonal "base cells," so that no edge of any of the triangles in $\mathbf{S}$ projects into the interior of any of these cells. Let Q be the base cell containing F, and let $\mathbf{S}_Q$ denote the subcollection of all triangles S_j whose projections S_j^* contain Q. Note that, when restricted over Q, the upper envelope M is the same as the upper envelope of the planes containing the triangles in $\mathbf{S}_Q$. In particular, the portion F' over which S_i attains M is convex and thus connected. □

Lemma 2.1 implies that the sum of the number of faces of $\bar{M}_1$ and of the corresponding refined map $\bar{M}_2$ is an upper bound for the face-complexity of M.

Lemma 2.2. *The number $\bar{t}$ of faces in $\bar{M}_1$ is at most the number t^* of faces of M_1^* plus $O(n^2\alpha(n))$.*

Proof. Let F be a face of M_1^* which splits into k_F subfaces by the addition of the lines $l_1, \ldots, l_{3n}$ containing the edges of the projections $S_1^*, \ldots, S_n^*$ of all triangles in $\mathbf{S}$. Suppose that F is the projection of a connected portion of some triangle $S_i \in \mathbf{S}_1$ which appears on the upper envelope M_1. For each $1 \le i \le 3n$, let $p_i(F)$ denote the number of connected portions of $F \cap l_i$, and let $q(F)$ denote the number of intersection points of the lines l_i inside F. It is then easily checked that $k_F \le 1 + q(F) + \sum_i p_i(F)$. (This is best seen by adding the lines l_i one at a time, and is illustrated schematically in Fig. 4.) Hence, if we sum these inequalities over all faces F of M_1^*, we obtain

$$\bar{t} \le t^* + \sum_F q(F) + \sum_{i,F} p_i(F).$$

But clearly $\sum_F q(F) = O(n^2)$. As to the other sum, note that for each i the sum $\sum_F p_i(F)$ is just the complexity of the upper envelope M_1 restricted over the line l_i. Since each of the $n/2$ triangles in F_1, when restricted over l_i, becomes a straight segment, it follows from standard Davenport–Schinzel theory (see [HS]) that

$$\sum_F p_i(F) \le \lambda_3(n/2) \le \lambda_3(n) = O(n\alpha(n)).$$

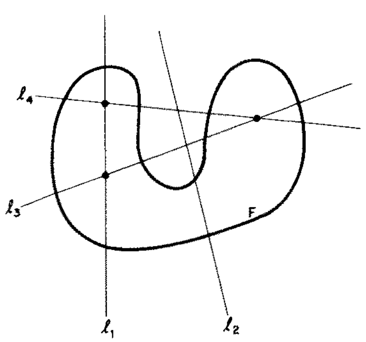

Fig. 4. Dissection of a face F by the lines l_i.

Thus, summing over all lines l_i, we obtain

$$\bar{t} \leq t^* + O(n^2) + O(n^2\alpha(n)) = t^* + O(n^2\alpha(n)). \qquad \square$$

Since a similar inequality applies to the map $\bar{M}_2$, we can now obtain the desired recurrence formula for ψ:

$$\psi(n) \leq 2\psi(n/2) + O(n^2\alpha(n)).$$

The solution of this formula is readily seen to be $\psi(n) = O(n^2\alpha(n))$. This completes the proof of our theorem. $\square$

If, instead of the face-complexity, we wish to bound the total (i.e., combinatorial) complexity of M, then we also need to estimate the number of edges and vertices in the upper envelope M (or in its projection M^*). To do so, note that our assumptions about general position of the triangles in **S** imply that each vertex v of M^* is the projection of either the intersection of the interiors of exactly three triangles (and thus has degree 3), or a vertex of one of the triangles (and thus has degree 2), or the intersection of an edge of one triangle with the interior of another (and thus has degree 2), or else v is the intersection of the projections of two edges of distinct triangles in **S** (in which case v has degree 3). Let V_2, V_3 denote respectively the number of vertices of M^* having degree 2, 3, and let E, Ψ denote the number of edges and faces of M^*, respectively. Then we obtain $2V_2 + 3V_3 = 2E$, and, using Euler's formula, we have

$$V_2 + V_3 + \Psi \geq E + 2 = V_2 + \tfrac{3}{2}V_3 + 2.$$

Hence

$$V_3 \leq 2\Psi - 4 \leq 2\psi(n) - 4 = O(n^2\alpha(n)).$$

To estimate V_2, note that there are only $O(n)$ degree-2 vertices in M^* that are projections of the vertices of the triangles in **S**, and only $O(n^2)$ possible intersections between edges of triangles and interiors of other triangles in **S**. Hence $V_2 = O(n^2)$.

Finally, we need to discuss the case in which the triangles in **S** are not in general position. (In particular, when taking the graph of an arbitrary polyhedral function and decomposing it into a collection of triangles, these triangles will definitely not be in general position.) However, here we can make use of the following general observation. Given a collection **S** of triangles, we can perturb slightly each of them so as to obtain a collection **S**$'$ of the same number of triangles which are now in general position, such that the complexity of the upper envelope of the triangles in **S**$'$ is at least the complexity of the upper envelope of the triangles in **S**. Hence the same bounds obtained above also apply to collections of triangles not lying in general position.

It is also worth noting that the bound obtained in Theorem 1 is tight in the worst case. This follows from the recent result of [WS] that constructs a collection of $n/2$ line segments in, say, the x-z plane, whose upper envelope consists of $\Omega(n\alpha(n))$ subsegments. By taking the Cartesian product of each of these segments with a large interval on the y axis, we obtain a collection of $n/2$ rectangles, to which we add $n/2$ sharp and narrow wedges whose upper edges are all parallel to the x-z plane, and are all at the same height, so that they cut through the entire upper envelope of the first $n/2$ rectangles. It is easy to extend our rectangles and wedges to triangles, so that their upper envelope has complexity $\Omega(n^2\alpha(n))$.

In summary, we have

Theorem 1′. *The total number of vertices, edges, and faces (i.e., the combinatorial complexity) of the upper envelope of any collection of n (nonvertical) triangles in 3-space is at most $O(n^2\alpha(n))$, and this bound is asymptotically tight in the worst case.*

3. The Upper Envelope of Simplices in Higher Dimensions

Theorem 1 can be easily generalized to polyhedral functions in higher dimensions.

Theorem 2. *The maximal number of d-dimensional faces in the upper envelope of n d-simplices in $\mathbb{R}^{d+1}$ (and thus also of any collection of polyhedral functions from $\mathbb{R}^d$ to $\mathbb{R}$ having n simplicial d-faces altogether) is $O(n^d\alpha(n))$, and this bound is asymptotically tight in the worst case. The constants appearing in the upper and lower bounds increase and decrease, respectively, exponentially with d.*

Proof. By induction on d. For $d \le 2$ the assertion has already been proved. Suppose the theorem holds for all $d' < d$, and let $\mathbf{S} = \{S_1, \ldots, S_n\}$ be a collection of n d-simplices in $\mathbb{R}^{d+1}$. As before, we can assume, without loss of generality, that these simplices are in general position. Let $S_1^*, \ldots, S_n^*$ denote the projections of these simplices on the hyperplane $x_{d+1} = 0$, and let $P_1, \ldots, P_{(d+1)n}$ denote the $(d-1)$-planes containing the $(d-1)$-faces of these projections. Our assumptions on general position ensure in particular that these P_i's are also in general position, i.e., no $d+1$ of them have a point in common.

Partition $\mathbf{S}$ into two subcollections $\mathbf{S}_1, \mathbf{S}_2$ of $n/2$ simplices each, and let M_1 and M_2 denote the upper envelopes of $\mathbf{S}_1$ and $\mathbf{S}_2$, respectively. Let M_1^*, M_2^* denote the projections of these subenvelopes on $x_{d+1} = 0$. Superimpose the $(d-1)$-planes P_i on each of these projections to obtain two respective refined polyhedral partitions $\bar{M}_1, \bar{M}_2$ of the hyperplane $x_{d+1} = 0$.

Lemmas 2.1 and 2.2 can now be generalized as follows.

Lemma 3.1. *Let F be a d-cell of $\bar{M}_1$ which is contained in the projection of some d-face of M_1 which is part of a simplex S_i. Then the portion of F, over which S_i attains the overall envelope M, is connected.*

Proof. Essentially identical to that of Lemma 2.1. □

Lemma 3.2. *The number $\bar{t}$ of d-cells in $\bar{M}_1$ is at most the number t^* of d-cells of M_1^* plus $O(n^d\alpha(n))$.*

Proof. Let F be a d-cell of M_1^* which is split into k_F subcells by the addition of all the $(d+1)n$-planes P_i. Suppose that F is the projection of a connected portion of some simplex $S_i \in \mathbf{S}_1$ which appears on the upper envelope M_1. For each nonempty subset I of the indices of the $(d-1)$-planes P_i of size $\le d$, let $p_I(F)$ denote the number of connected portions of $F \cap (\bigcap_{i \in I} P_i)$. It is then easily checked that $k_F \le 1 + \sum_I p_I(F)$. (This can be shown by adding the planes P_i one at a time and use induction on their number.) Hence, if we sum these inequalities over all d-cells F of M_1^*, we obtain

$$\bar{t} \le t^* + \sum_{I,F} p_I(F).$$

For each I (of size $s \le d$) the sum $\sum_F p_I(F)$ is just the number of $(d-s)$-faces in the upper envelope M_1 restricted over the intersection $P_I = \bigcap_{i \in I} P_i$. Since each of the $n/2$ simplices in $\mathbf{S}_1$, when restricted over P_I, becomes a $(d-s)$-dimensional convex polytope with some constant number of facets, we can decompose it into a constant number of $(d-s)$-simplices, and then apply the induction hypothesis to conclude that $\sum_F p_I(F) = O(n^{d-s}\alpha(n))$. The number of distinct subsets I of size s is $O(n^s)$, whence

$$\bar{t} \le t^* + O(n^d\alpha(n)).$$ □

Thus, denoting by $\psi_d(n)$ the maximum number of d-faces in the upper envelope of n d-simplices in $\mathbb{R}^{d+1}$, we conclude that

$$\psi_d(n) \le 2\psi_d(n/2) + O(n^d\alpha(n)),$$

and thus $\psi_d(n) = O(n^d\alpha(n))$.

The lower bound can be proved, using induction on d, in exactly the same way as for the case $d = 2$. That is, take a collection of $n/2$ $(d-1)$-simplices in $\mathbb{R}^d$ whose upper envelope consists of $\Omega(n^{d-1}\alpha(n))$ facets, and extend each of them to a d-prism in $\mathbb{R}^{d+1}$ by translating it in the additional dimension. Add $n/2$ "sharp hyperwedges" to this collection, which cut through the entire upper envelope of the $n/2$ prisms. It is easily checked that the upper envelope of the resulting collection has $\Omega(n^d\alpha(n))$ simplicial d-faces.

The calculations showing the last statement of the theorem are left to the reader. □

4. The Boundary of a Region Enclosed by Convex Plates

Let $\mathbf{S} = \{S_1, \ldots, S_n\}$ be a collection of n $(d-1)$-dimensional compact convex sets (plates) in $\mathbb{R}^d$. If we delete from $\mathbb{R}^d$ all points belonging to at least one of these plates, then $\mathbb{R}^d$ may split up into a number of connected components. Let C denote such a component. The aim of this section is to give an upper bound on the maximal possible combinatorial complexity of C. We are going to prove

Theorem 3. *For every $d \geq 3$ there exists a constant $\varepsilon(d) > 0$ such that, given any collection $\mathbf{S} = \{S_1, \ldots, S_n\}$ of $(d-1)$-dimensional convex plates arranged in $\mathbb{R}^d$, the combinatorial complexity of the boundary of any single connected component C of $\mathbb{R}^d - \bigcap_{i=1}^n S_i$ is at most $O(n^{d-\varepsilon(d)})$.*

Assume without loss of generality, as we did in the previous section, that the plates are in general position. Then any vertex of the given component C belongs to exactly d plates, and using the fact that any d plates have at most one point in common, we obtain that C has at most $\binom{n}{d} = O(n^d)$ vertices.

On the other hand, it is easy to see that the total combinatorial complexity of C (i.e., the number of all i-dimensional faces over all $0 \leq i \leq d$) is proportional to the number of its vertices.

Hence, it is sufficient to prove

Theorem 3′. *For every $d \geq 3$ there exists a constant $\varepsilon(d) > 0$ such that, given any collection $\mathbf{S} = \{S_1, \ldots, S_n\}$ of $(d-1)$-dimensional convex plates arranged in $\mathbb{R}^d$ in general position, the number of points belonging to d members of $\mathbf{S}$ and lying on the boundary of a given component C of $\mathbb{R}^d - \bigcup \mathbf{S}$* is at most $O(n^{d-\varepsilon(d)})$. *($\bigcup \mathbf{S}$ is the shorthand for $\bigcup_{i=1}^n S_i$.)*

We need some preparation. A *d-uniform hypergraph* is a set system whose members (the so-called *hyperedges*) are d-element sets.

Definition 4.1. Let $H = H(\mathbf{S})$ be a d-uniform hypergraph whose vertex set is $\mathbf{S}$ and whose hyperedges are those d-tuples $\{S^{(1)}, \ldots, S^{(d)}\} \subset \mathbf{S}$ for which $\bigcup_{i=1}^d S^{(i)}$ lies on the boundary of the given component C of $\mathbb{R}^d - \bigcup \mathbf{S}$.

Let $K^{(r)}(m_1, m_2, \ldots, m_r)$ denote an r-uniform hypergraph with $m_1 + m_2 + \cdots + m_r$ vertices, whose vertex set is $V_1 \cup V_2 \cup \cdots \cup V_r$, $|V_i| = m_i$ $(1 \leq i \leq r)$, and whose edge set is $V_1 \times V_2 \times \cdots \times V_r$, i.e., consists of all r-tuples containing exactly one element from each V_i. If $m_1 = m_2 = \cdots = m_r = m$, then we will write $K^{(r)}(m)$ for $K^{(r)}(m, m, \ldots, m)$.

We want to apply the following well-known combinatorial result of Erdös [Er] to the hypergraph defined in Definition 4.1.

Theorem (Erdös). *Let H be an r-uniform hypergraph on n vertices containing no subhypergraph isomorphic to $K^{(r)}(m)$. Then $|H| \leq n^{r-(1/m)^{r-1}}$.*

We also make use of the following little "side-trip" to elementary plane geometry, which is perhaps also of some independent interest.

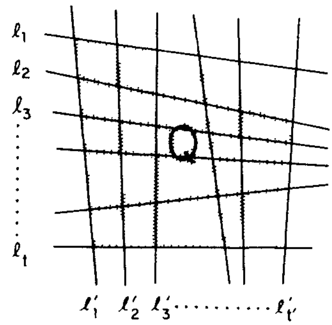

Fig. 5. Two systems of lines meeting regularly.

Definition 4.2. Two systems of straight lines in the plane $\{l_1, l_2, \ldots, l_t\}$ and $\{l'_1, l'_2, \ldots, l'_{t'}\}$ are said to *meet regularly* if there is a convex quadrilateral Q whose sides are segments of $l_1, l'_1, l_t, l'_{t'}$ (in this cyclic order) and

(i) $l_i \cap l'_j \cap Q \neq \emptyset$ for every $1 \leq i \leq t$, $1 \leq j \leq t'$.
(ii) $l_i \cap l_j \cap Q = l'_i \cap l'_j \cap Q = \emptyset$ for every $i \neq j$.

Roughly speaking, two systems of straight lines meet regularly if they form a "grid-like" configuration (see Fig. 5).

Lemma 4.3. *For any natural number t there exists an $f = f(t)$ such that for any two systems $\mathbf{L}$ and $\mathbf{L}'$ of f straight lines in $\mathbb{R}^2$ (in general position) we can find two t-element subsystems $\mathbf{L}_0 \subset \mathbf{L}$ and $\mathbf{L}'_0 \subset \mathbf{L}'$ which meet regularly.*

Proof. Let us supply $\mathbb{R}^2$ with the usual rectilinear coordinate system (x, y). For any straight line l, define $\alpha(l)$ (the angle of inclination of l) as the minimum α such that a counterclockwise turn of the x axis around the origin results in a line parallel to l.

It is easily seen that we can choose $0 < \beta < \pi$ and two subsystems $\mathbf{L}_1 \subset \mathbf{L}$ and $\mathbf{L}'_1 \subset \mathbf{L}'$ with $|\mathbf{L}_1|, |\mathbf{L}'_1| \geq |\mathbf{L}|/2 = |\mathbf{L}'|/2 = f/2$ and such that either $0 < \alpha(l) < \beta$ for every $l \in \mathbf{L}_1$ and $\beta < \alpha(l') < \pi$ for every $l' \in \mathbf{L}'_1$, or $0 < \alpha(l') < \beta$ for every $l' \in \mathbf{L}'_1$ and $\beta < \alpha(l) < \pi$ for every $l \in \mathbf{L}_1$.

Let $\mathbf{L}_2$ be any k-element subsystem of $\mathbf{L}_1$ with $k = t^5$. The lines belonging to $\mathbf{L}_2$ determine at most $\binom{k}{2}$ pairwise intersection points, and the set P of these intersection points can be divided into two parts by a line in at most $4\binom{\binom{k}{2}}{2} < k^4$ different ways. Thus, we can pick a t-element subset $\mathbf{L}'_0 \subset \mathbf{L}'_1$ such that every line of $\mathbf{L}'_0$ divides P into two parts in the same way, provided that $|\mathbf{L}'_1| \geq f/2 \geq tk^4 = t^{21}$. Similarly, the set P' of all pairwise intersection points of the lines belonging to $\mathbf{L}'_0$ can be cut by a line in at most t^4 different ways, hence we can find an $|\mathbf{L}_1|/t^4 = t$-element subset $\mathbf{L}_0 \subset \mathbf{L}_2$ each of whose lines represents the same bipartition of P'.

It is not difficult to show now that $\mathbf{L}_0$ and $\mathbf{L}_0'$ meet regularly. □

Next we extend the notion of regularly meeting lines recursively to higher dimensions.

Definition 4.4. Let $\mathbf{P}^{(1)}, \mathbf{P}^{(2)}, \ldots, \mathbf{P}^{(d)}$ by systems of hyperplanes in $\mathbb{R}^d$ in general position, $d \geq 3$. We say that they *meet regularly* if for every $1 \leq i \leq d$ and for every $P \in \mathbf{P}^{(i)}$ the plane-systems

$$\{P \cap P^{(j)}\colon\ P^{(j)} \in \mathbf{P}^{(j)}\}, \qquad j = 1, \ldots, i-1, i+1, \ldots, d,$$

meet regularly in P.

The following assertions are trivially true for $d = 2$, and in general can be proved by a straightforward induction.

Proposition 4.5. *Let $d \geq 2$ and let $\mathbf{P}^{(1)}, \mathbf{P}^{(2)}, \ldots, \mathbf{P}^{(d)}$ be systems of hyperplanes in $\mathbb{R}^d$ which meet regularly. Further, we fix two distinct elements $P_a^{(i)}, P_b^{(i)} \in \mathbf{P}^{(i)}$ for each $1 \leq i \leq d$. Then*:

(i) $\mathbf{P}_0^{(1)}, \mathbf{P}_0^{(2)}, \ldots, \mathbf{P}_0^{(d)}$ *meet regularly for any subsystems* $\mathbf{P}_0^{(i)} \subset \mathbf{P}^{(i)}$, $1 \leq i \leq d$.

(ii) *The polytope*

$$C(P_a^{(1)}, P_b^{(1)}; \ldots; P_a^{(d)}, P_b^{(d)}) = \operatorname{conv}\left[\bigcap_{i=1}^{d} P_{\alpha_i}^{(i)}\colon \alpha_1, \ldots, \alpha_d \in \{a, b\}\right]$$

is combinatorially equivalent to the d-dimensional cube. In particular, it has $2d$ facets, which can be expressed as

$$\operatorname{conv}\left[P_{\alpha_k}^{(k)} \cap \left(\bigcap_{i \neq k} P_{\alpha_i}^{(i)}\right)\colon \alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_d \in \{a, b\}\right],$$

where $k = 1, \ldots, d$ and $\alpha_k \in \{a, b\}$. □

By the help of Proposition 4.5, we can easily grab the most striking property of the "grid-like" structure of regularly meeting systems of hyperplanes. That is, we can show that for each i there is a natural linear ordering of the hyperplanes belonging to $\mathbf{P}^{(i)}$.

This can be done as follows:

Let us first define an order "$<$" on $\mathbf{P}^{(1)}$. Let $P_a^{(1)} < P_b^{(1)}$ and assume that the elements of a subsystem $\mathbf{P}_0^{(1)} \subset \mathbf{P}^{(1)}$ have already been ordered. We will determine the position of a "new" element $P^{(1)} \in \mathbf{P}^{(1)} - \mathbf{P}_0^{(1)}$ relative to $\mathbf{P}_0^{(1)}$.

Let $P_c^{(1)} < P_d^{(1)}$ be two elements of $\mathbf{P}_0^{(1)}$. Then the following cases can occur:

(i) If the set $C(P_c^{(1)}, P_d^{(1)}; P_a^{(2)}, P_b^{(2)}; \ldots; P_a^{(d)}, P_b^{(d)})$ contains the set $C(P_c^{(1)}, P^{(1)}; P_a^{(2)}, P_b^{(2)}; \ldots; P_a^{(d)}, P_b^{(d)})$, then let $P_c^{(1)} < P^{(1)} < P_d^{(1)}$.

(ii) If the set $C(P_c^{(1)}, P_d^{(1)}; P_a^{(2)}, P_b^{(2)}; \ldots; P_a^{(d)}, P_b^{(d)})$ is contained in the set $C(P_c^{(1)}, P^{(1)}; P_a^{(2)}, P_b^{(2)}; \ldots; P_a^{(d)}, P_b^{(d)})$, then let $P_c^{(1)} < P_d^{(1)} < P^{(1)}$.

(iii) If the set $\operatorname{int} C(P_c^{(1)}, P_d^{(1)}; P_a^{(2)}, P_b^{(2)}; \ldots; P_a^{(d)}, P_b^{(d)})$ and the set $\operatorname{int} C(P_c^{(1)}, P^{(1)}; P_a^{(2)}, P_b^{(2)}; \ldots; P_a^{(d)}, P_b^{(d)})$ have empty intersection, then let $P^{(1)} < P_c^{(1)} < P_d^{(1)}$.

It can easily be checked that there are no other feasible cases, and the above rules define indeed a total order on $\mathbf{P}^{(1)}$. For any other i, a total order on $\mathbf{P}^{(i)}$ can be defined analogously.

Corollary 4.6. *Let $\mathbf{P}^{(1)}, \mathbf{P}^{(2)}, \ldots, \mathbf{P}^{(d)}$ be regularly meeting systems of hyperplanes in $\mathbb{R}^d$, and let $P_a^{(i)} < P_c^{(i)} < P_b^{(i)}$ be three elements of $\mathbf{P}^{(i)}$ for every i $(1 \le i \le d,\ d \ge 2)$. Then*

$$\bigcap_{i=1}^{d} P_c^{(i)} \in \operatorname{int} C(P_a^{(1)}, P_b^{(1)}; \ldots; P_a^{(d)}, P_b^{(d)}). \qquad \square$$

The following simple Ramsey-type result plays a key role in the proof of Theorem 3′.

Lemma 4.7. *Given any natural numbers $t, d \ge 2$, there exists an $F = F(t, d)$ with the property that in $\mathbb{R}^d$ any d F-element systems of hyperplanes $\mathbf{P}^{(1)}, \mathbf{P}^{(2)}, \ldots, \mathbf{P}^{(d)}$ contain some t-element subsystems $\mathbf{P}_0^{(1)} \subset \mathbf{P}^{(1)}$, $\mathbf{P}_0^{(2)} \subset \mathbf{P}^{(2)}, \ldots, \mathbf{P}_0^{(d)} \subset \mathbf{P}^{(d)}$ which meet regularly.*

Proof. For $d = 2$ this is the same as Lemma 2.6. For larger values of d we can use a trivial inductional argument. $\square$

We are now in a position to prove Theorem 3′.

Proof. Let us consider the d-uniform hypergraph $H = H(\mathbf{S})$ defined in Definition 4.1. If $H \supset K^{(d)}(F(3, d))$, where $F(t, d)$ denotes the same as in Lemma 4.7, then there are some $F(3, d)$-element subsystems $\mathbf{S}^{(1)}, \ldots, \mathbf{S}^{(d)} \subset \mathbf{S}$ such that $\bigcap_{i=1}^{d} S^{(i)}$ lies on the boundary of the given component of $\mathbb{R}^d - \bigcup \mathbf{S}$ for every choice $S^{(1)} \in \mathbf{S}^{(1)}, \ldots, S^{(d)} \in \mathbf{S}^{(d)}$.

Let $\mathbf{P}^{(i)}$ denote the system of hyperplanes containing the elements of $\mathbf{S}^{(i)}$, $1 \le i \le d$. Applying Lemma 4.7, we can choose 3-element subsystems $\mathbf{P}_0^{(1)} = \{P_a^{(1)}, P_b^{(1)}, P_c^{(1)}\} \subset \mathbf{P}^{(1)}, \ldots, \mathbf{P}_0^{(d)} = \{P_a^{(d)}, P_b^{(d)}, P_c^{(d)}\} \subset \mathbf{P}^{(d)}$, which meet regularly. See Fig. 6.

Assume without loss of generality that $P_a^{(i)} < P_c^{(i)} < P_b^{(i)}$ (according to the total order on $\mathbf{P}^{(i)}$) for every $1 \le i \le d$. Then, by Corollary 4.6,

$$\bigcap_{i=1}^{d} P_c^{(i)} \in \operatorname{int} \operatorname{conv}\left[\bigcap_{i=1}^{d} P_{\alpha_i}^{(i)} : \alpha_1, \ldots, \alpha_d \in \{a, b\}\right].$$

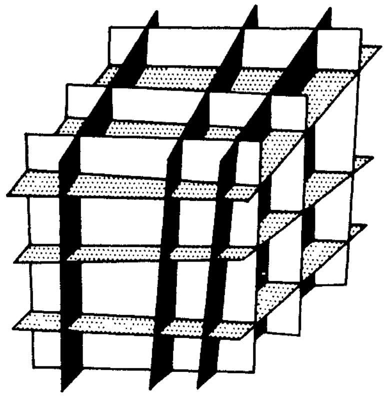

Fig. 6. The subsystems $\mathbf{P}_0^{(i)}$ meeting regularly.

Let $S_a^{(i)}, S_b^{(i)}, S_c^{(i)} \in \mathbf{S}^{(i)}$ denote the corresponding sets contained in the hyperplanes $P_a^{(i)}, P_b^{(i)}$, and $P_c^{(i)}$, respectively. Obviously, $\bigcap_{i=1}^d S_{\alpha_i}^{(i)} = \bigcap_{i=1}^d P_{\alpha_i}^{(i)}$ for any $\alpha_1, \ldots, \alpha_d \in \{a, b, c\}$. Hence, in view of Proposition 4.5(ii), $\bigcap_{i=1}^d S_c^{(i)}$ and $\{\bigcap_{i=1}^d S_{\alpha_i}^{(i)} : \alpha_1, \ldots, \alpha_d \in \{a, b\}\}$ cannot all lie on the boundary of the same component of $\mathbb{R}^d - \bigcup \mathbf{S}$.

This contradiction proves that H does not contain $K^{(d)}(F(3, d))$, and in this case we can deduce from the Erdös theorem that

$$|H| \le n^{d-(1/F(3,d))^{d-1}},$$

as desired. □

5. Regions Enclosed by Convex Plates in 3-space

In what follows we would like to improve the methods of the previous section for $d = 3$, to obtain an explicit value for $\varepsilon(3)$ in Theorems 3 and 3′.

More precisely, we will establish

Theorem 4. *Given any collection* $\mathbf{S} = \{S_1, \ldots, S_n\}$ *of two-dimensional convex plates scattered in* $\mathbb{R}^3$, *the combinatorial complexity of the boundary of any given connected component of* $\mathbb{R}^3 - \bigcup \mathbf{S}$ *is at most* $O(n^{3-1/49})$.

As in the case of Theorem 3, it is sufficient to bound the number of *vertices* of the given component. We can also assume that the S_i's are in general position.

The proof is somewhat technical, so some of the details will be omitted.

Lemma 5.1. *Let* $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ *and* $\Sigma' = \{\sigma_1', \sigma_2', \ldots, \sigma_t'\}$ *be two systems of straight line segments in* $\mathbb{R}^2$ *such that*:

(i) $\sigma_i \cap \sigma_j' \ne \emptyset$ *for every* $1 \le i \le 3, 1 \le j \le t$.

(ii) *All intersection points* $p_{ij} = \sigma_i \cap \sigma_j'$ *are on the boundary of the same connected component of* $\mathbb{R}^2 - \bigcup \Sigma - \bigcup \Sigma'$.

Then $t \le 6$ *and this bound cannot be improved, as is demonstrated by Fig.* 7.

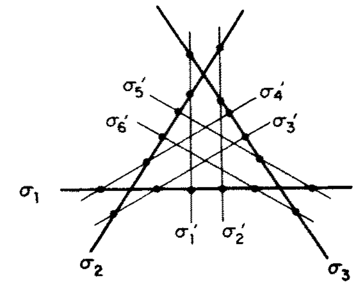

Fig. 7. A 3-line system and a 6-line system satisfying the conditions of Lemma 5.1.

Proof. $p_{ij} = \sigma_i \cap \sigma_j'$ is called an *exposed* point of σ_j', if p_{ij} is not contained in the interior of the segment conv$[p_{i'j}: 1 \le i' \le 3]$. Thus, each σ_j' has exactly two exposed points, and the total number of exposed points is $2t$. To establish the lemma, it is sufficient to show that

$$\text{every } \sigma_i \text{ contains at most four exposed points.} \qquad (*)$$

From here $2t \le 3 \times 4 = 12$, hence $t \le 6$ follows immediately.

The proof of $(*)$ may consist of the following steps:

Let l_i and l_j' denote the straight lines containing σ_i and σ_j', respectively. Further, set $q_{ij} = l_i \cap l_j$ $(1 \le i \ne j \le 3)$. Assume without loss of generality that $i = 1$.

Fact 1. The interval $[q_{12}, q_{13}]$ contains at most two exposed points.

Let p_{11}, p_{12}, p_{13}, say, be three such points on $[q_{12}, q_{13}]$. Then all points p_{ij} $(1 \le i, j \le 3)$ must lie in the same closed halfplane bounded by l_1, in which q_{23} can be found. (See Fig. 8.)

Furthermore, every σ_j' $(1 \le j \le 3)$ intersects the arc $q_{12}q_{23}q_{13}$. Let p, p', and p'' be these intersection points, listed in the order as they appear on $q_{12}q_{23}q_{13}$. It is easy to see now that p' $(= p_{i'j'}$ for some $1 \le i', j' \le 3)$ is completely "enclosed" by the segments σ_i and σ_j' $(1 \le i, j \le 3)$. In particular, it is impossible that all points p_{ij} $(1 \le i, j \le 3)$ are on the boundary of the same connected component of $\mathbb{R}^2 - \bigcup \sigma_i - \bigcup \sigma_j'$. This contradicts condition (ii) of the lemma.

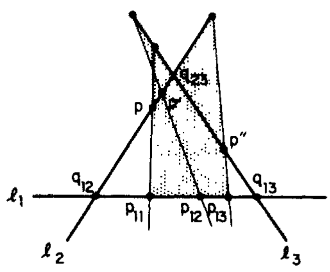

Fig. 8. The proof of Lemma 5.1.

Let $(-\infty, q_{12}]$ and $[q_{13}, +\infty)$ denote the halflines obtained from l_1 after the deletion of the open interval (q_{12}, q_{13}).

Fact 2. None of $(-\infty, q_{12}]$ and $[q_{13}, +\infty)$ contains more than two exposed points.

Fact 3. If both $(-\infty, q_{12}]$ and $[q_{13}, +\infty)$ contain exposed points, and at least one of them contains two, then there cannot be any exposed point on $[q_{12}, q_{13}]$.

The proof of the last two facts is very similar to that of Fact 1, the details are left to the reader. Putting Facts 1-3 together, we obtain (*). □

We make use of the following generalization of the Erdös theorem, which can be obtained by a fairly straightforward modification of the original proof (see [Er]).

Theorem 5.2. *Given any natural numbers r, $m \geq 2$, $M \geq m$, there exists a constant $C(r, m, M) = \hat{C}$ such that the number of hyperedges of any r-uniform hypergraph on n vertices, which does not contain a subhypergraph isomorphic to $K^{(r)}(m, \ldots, m, M)$, is at most $Cn^{r-(1/m)^{r-1}}$.*

Definition 5.3. Given any system Π of (two-dimensional) planes in $\mathbb{R}^3$, and two planes P_1 and P_2 in general position, we say that P_1 and P_2 are *equivalent with respect to* Π if there is a single rotation or translation which takes P_1 to P_2 so that during the motion the plane:

(i) Never passes through any point belonging to three members of Π.
(ii) Is never parallel to the intersection line of any two members of Π.

Lemma 5.4. *Definition 5.3 yields an equivalence relation on the family of all planes which are in general position with respect to Π, and the number of equivalence classes is at most $|\Pi|^9$.*

Proof. For any point x and for any plane P in the three-dimensional projective space, let $\bar{x}$ and $\bar{P}$ denote the plane dual to x and the point dual to P, respectively. Further, let P_∞ denote the plane of infinity.

Consider now the set T_Π of all triple intersection points of members of $\Pi \cup \{P_\infty\}$. Clearly,

$$|T_\Pi| \leq \binom{|\Pi|+1}{3}.$$

Set $\bar{T}_\Pi = \{\bar{y}: y \in T_\Pi\}$. Then the planes belonging to $\bar{T}_\Pi$ divide the projective space into at most

$$\binom{|T_\Pi|}{3} + |T_\Pi| < |\Pi|^9$$

cells, and two planes P_1 and P_2 are equivalent with respect to Π if and only if $\bar{P}_1$ and $\bar{P}_2$ are in the same cell. □

Now we are in a position to prove Theorem 4.

Let $H(\mathbf{S})$ be a 3-uniform hypergraph whose vertex set is $\mathbf{S}$ and whose hyperedges are those triples $\{S, S', S''\} \subset \mathbf{S}$ for which $S \cap S' \cap S''$ lies on the boundary of the given component (say, the unbounded component) of $\mathbb{R}^3 - \bigcup \mathbf{S}$. In view of Theorem 5.2, it is sufficient to show that $H(\mathbf{S})$ cannot contain a subhypergraph isomorphic to $K^{(3)}(7, 7, M)$ for some integer M independent of n. In fact, we are going to prove the following somewhat stronger result.

Lemma 5.5. *$H(\mathbf{S})$ does not contain a subhypergraph isomorphic to*

$$K^{(3)}(3, 7, 2 \times 10^9 + 1).$$

Proof. Assume, in order to obtain a contradiction, that there are three subsystems $\mathbf{T}, \mathbf{T}', \mathbf{T}'' \subset \mathbf{S}$ such that:

(i) $|\mathbf{T}| = 3, |\mathbf{T}'| = 7, |\mathbf{T}''| = 2 \times 10^9 + 1$.
(ii) $S \cap S' \cap S'' \neq \emptyset$ and lies on the boundary of the unbounded component of $\mathbb{R}^3 - \bigcup \mathbf{S}$ for every $S \in \mathbf{T}, S' \in \mathbf{T}', S'' \in \mathbf{T}''$.

Let $\mathbf{P}$, $\mathbf{P}'$, and $\mathbf{P}''$ denote the systems of planes containing the plates belonging to $\mathbf{T}, \mathbf{T}'$, and $\mathbf{T}''$, respectively. Applying Lemma 5.4 with $\Pi = \mathbf{P} \cup \mathbf{P}'$, we obtain that there exist three plates $S_1'', S_2'', S_3'' \in \mathbf{T}''$ such that the corresponding planes P_1'', P_2'', P_3'' are pairwise equivalent with respect to $\mathbf{P} \cup \mathbf{P}'$.

For any $S \in \mathbf{T}$, let

$$\tilde{S} = \operatorname{conv}[S \cap S' \cap S_i'' : S' \in \mathbf{T}', 1 \le i \le 3] \subseteq S,$$

and similarly, for any $S' \in \mathbf{T}'$, let

$$\tilde{S}' = \operatorname{conv}[S \cap S' \cap S_i'' : S \in \mathbf{T}, 1 \le i \le 3] \subseteq S'.$$

Further, let

$$\Sigma_i = \{\tilde{S} \cap S_i'' : S \in \mathbf{T}\}, \qquad \Sigma_i' = \{\tilde{S}' \cap S_i'' : S' \in \mathbf{T}'\}, \qquad 1 \le i \le 3.$$

In view of the fact that the planes P_i'' are pairwise equivalent with respect to $\mathbf{P} \cup \mathbf{P}'$, the intersection structures of the segment-systems $\Sigma_i \cup \Sigma_i'$, $1 \le i \le 3$, are combinatorially the same.

Applying Lemma 5.1 to Σ_1 and Σ_1', say, there exist $\sigma \in \Sigma_1$, $\sigma' \in \Sigma_1'$ such that the point $\sigma \cap \sigma'$ is completely enclosed by a simple closed polygon p, all of whose sides are portions of some segments belonging to $\Sigma_1 \cup \Sigma_1'$. (As a matter of fact, p can be chosen to be the boundary of the unbounded component of $P_1'' - \bigcup \Sigma_1 - \bigcup \Sigma_1'$.) In other words, there exist $S \in \mathbf{T}, S' \in \mathbf{T}'$ and a finite sequence $S_0, S_1, \ldots, S_k = S_0 \in \mathbf{T} \cup \mathbf{T}'$ such that, for every $1 \le i \le 3$, the points $\tilde{S}_j \cap \tilde{S}_{j+1} \cap S_i'' = S_j \cap S_{j+1} \cap S_i''$, $j = 0, 1, \ldots, k-1$ (in this cyclic order), form a simple closed polygon which contains $S \cap S' \cap S_i''$ in its interior.

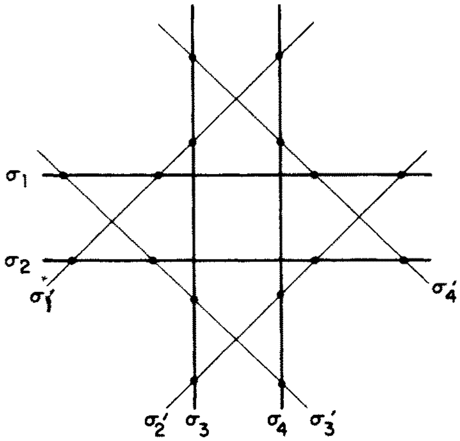

Fig. 9. Two 4-line systems satisfying the conditions of Lemma 5.1.

Assume without loss of generality that

$$S \cap S' \cap S'' \in \operatorname{conv}[S \cap S' \cap S_1'', S \cap S' \cap S_3''].$$

Then S_1'', S_3'' and $\tilde{S}_0, \tilde{S}_1, \ldots, \tilde{S}_{k-1}$ enclose a bounded polyhedral region containing $S \cap S' \cap S_2''$ in its interior. This contradicts (ii). □

Note, however, that the hypergraph $H(\mathbf{S})$ associated with our system of plates may contain a subhypergraph isomorphic to $K^{(3)}(4, 4, M)$ for any large integer M. This follows from the fact that we can find two systems of straight line segments Σ and Σ' such that $|\Sigma| = |\Sigma'| = 4$ and they satisfy both conditions in Lemma 5.1 (see Fig. 9).

Conjecture 5.6. *There exists a constant* M *(independent of* $\mathbf{S}$*) such that* $H(\mathbf{S})$ *does not contain a subhypergraph isomorphic to* $K^{(3)}(5, 5, M)$.

According to Theorem 5.2, this conjecture would imply that Theorem 4 is valid with $O(n^{3-1/25})$.

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