Helly-Type Theorems for Spheres

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Dedicated to Professor Itiro Tamura on his 60th birthday

Abstract. We prove that (i) a family $\mathbf{F}$ of at least $n+3$ spheres in E^n has nonempty intersection if each $n+1$ spheres of $\mathbf{F}$ have nonempty intersection, and (ii) if a family $\mathbf{F}$ of spheres in E^n has nonempty intersection, then there exist $n+1$ or fewer spheres in $\mathbf{F}$ whose intersection coincides with the intersection of all spheres of $\mathbf{F}$.

1. Introduction

Helly's theorem is one of the most important results in the study of combinatorial geometry of convex sets in Euclidean space E^n .

Theorem (Helly [2]). *Let $\mathbf{F}$ be a finite family of at least $n+1$ convex sets in E^n . If every $n+1$ members of $\mathbf{F}$ have a point in common, then there is a point common to all members of $\mathbf{F}$.*

Here we present similar results for families of spheres in E^n . Let us say that a family $\mathbf{F}$ of sets has the *Helly- n -property* if the intersection of $\mathbf{F}$ is nonempty or there are $n+1$ or fewer members of $\mathbf{F}$ which have empty intersection. Then Helly's theorem states that every finite family of convex sets in E^n has the Helly- n -property. We will show that a family $\mathbf{F}$ of spheres in E^n whose centers are all lying on an m -dimensional flat has the Helly- $(m+1)$ -property (Theorem 1), and if $\mathbf{F}$ has nonempty intersection then there exist $m+1$ or fewer spheres in $\mathbf{F}$ whose intersection coincides with the intersection of all spheres in $\mathbf{F}$ (Theorem 2). Further, if a family of spheres in E^n contains at least $n+3$ distinct spheres then the family has the Helly- n -property (Theorem 3).

2. Spheres with Centers on a Flat

A sphere S of radius r in Euclidean n -space E^n consists of all points at distance r from some fixed point called the center of S . If $r=1$ then S is called a unit sphere. We admit the case of $r=0$ (then S consists of a single point). The intersection $S \cap L$ of a sphere S and a flat (affine subspace) L in E^n is called a sphere in L , provided that $S \cap L$ is nonempty. The intersection of all sets in a family F is denoted by $\bigcap F$.

Theorem 1. *Let F be a family of spheres in E^n with centers on a flat L of dimension m . Suppose that every subfamily consisting of at most $m+2$ spheres in F has nonempty intersection. Then F has nonempty intersection.*

Remark. the number $m+2$ cannot be reduced, see Fig. 1 in which $m=1$, $n=2$, and each two circles have two common points, but all circles have no point in common.

Proof. Since a family of compact sets with the finite intersection property has nonempty intersection, and since every sphere is compact, it is enough to prove the theorem when F is finite. So, we may assume that F is finite. First we consider the case $m=1$. Suppose $y \in S_i \cap S_j$ ($S_i, S_j \in F$), and let H_y be the hyperplane passing through y and perpendicular to L at z . Then $S_i \cap S_j$ is the sphere in the hyperplane H_y with center z and radius yz . Hence $S_1 \cap S_2 \cap S_3 \neq \emptyset$ ($S_i \in F$, $i=1, 2, 3$) implies that $S_1 \cap S_2 = S_1 \cap S_2 \cap S_3$. Therefore every sphere of F contains, say, $S_1 \cap S_2$, and hence $\bigcap F \neq \emptyset$. Now the general case. We proceed by induction on n . If $n=1$ then $m=1$ and the theorem is true. Let $n>1$ and suppose that the theorem is true for dimension $<n$. We may assume that there is a sphere S_0 of positive radius in F (otherwise, the theorem is trivial). Let $G = \{S_0 \cap S : S \in F, S \neq S_0\}$. Then every subfamily of G of size at most $m+1$ has nonempty intersection, and we need only show that $\bigcap G$ is nonempty. Since F is finite, and assuming $m \geq 2$, we can take a point x on $S_0 \cap L$ which does not lie on any other sphere in F . Let H be the hyperplane tangent to S_0 at the point antipodal to x . By the stereographic projection f of S_0 from x to H , each member T of G is carried onto a sphere in H . The centers of the spheres of

Fig. 1

Fig. 2

$f(\mathbf{G}) := \{f(T) : T \in \mathbf{G}\}$ lie on the $(m-1)$ -dimensional flat $H \cap L$, and every subfamily of $f(\mathbf{G})$ of size at most $m+1$ has nonempty intersection. Since H is $(n-1)$ -dimensional, $\bigcap f(\mathbf{G})$ must be nonempty by the inductive hypothesis. Hence $\bigcap \mathbf{G}$ is nonempty. $\square$

Corollary 1. *Let $\mathbf{F}$ be a family of at least $n+2$ spheres in E^n . If every $n+2$ spheres in $\mathbf{F}$ have a point in common, then $\mathbf{F}$ has nonempty intersection.*

Theorem 2. *Let $\mathbf{F}$ be a family of spheres in E^n with nonempty intersection. Suppose that the centers of the spheres in $\mathbf{F}$ all lie on a flat of dimension m . Then there exist $m+1$ or fewer spheres in $\mathbf{F}$ whose intersection is equal to the intersection $\bigcap \mathbf{F}$.*

Remark. The number $m+1$ cannot be reduced, see Fig. 2 in which $m = n = 2$ and any two circles have two points in common, but $\bigcap \mathbf{F}$ consists of one point. It is also impossible to drop the assumption $\mathbf{F} \neq \emptyset$, see Fig. 3.

Proof. By induction on n . If $n = 1$ then the theorem is obvious. Suppose that the theorem is true for $n-1$ ($n > 1$), and let us consider the case of dimension n . Since the case $m = 1$ is clear, we may further suppose $m \geq 2$. First we show that there is a finite subfamily $\mathbf{K}_0$ of $\mathbf{F}$ such that $\bigcap \mathbf{K}_0 = \bigcap \mathbf{F}$. To see this let $\mathbf{K}_0$ be a finite subfamily of $\mathbf{F}$ which attains the minimum value of $\dim(\bigcap \mathbf{K})$ among all finite-subfamilies $\mathbf{K}$ of $\mathbf{F}$. (Here $\dim(X)$ means the dimension of the flat spanned by X , $\dim(\text{one point}) = 0$, and $\dim(\emptyset) = -1$.) Then clearly $\bigcap \mathbf{K}_0$ is a

Fig. 3

sphere in some flat. Suppose $\bigcap \mathbf{K}_0 \neq \bigcap \mathbf{F}$. Then there must be a sphere S in $\mathbf{F} \setminus \mathbf{K}_0$ such that $\dim(\bigcap \mathbf{K}_0 \cap S) < \dim(\bigcap \mathbf{K}_0)$, which contradicts the choice of $\mathbf{K}_0$. Thus $\bigcap \mathbf{K}_0 = \bigcap \mathbf{F}$. Now, we may assume that there is a sphere S_0 of radius > 0 in $\mathbf{K}_0$. Let $\mathbf{G} = \{S_0 \cap S : S \in \mathbf{K}_0, S \neq S_0\}$. Then $\bigcap \mathbf{F} = \bigcap \mathbf{G}$. Since $\mathbf{K}_0$ is finite and $m \geq 2$, there is a point x on $S_0 \cap L$ which does not lie on any other sphere in $\mathbf{K}_0$. Let f be the stereographic projection of S_0 from x to the hyperplane H tangent to S_0 at the point antipodal to x . Then $f(\mathbf{G}) := \{f(T) : T \in \mathbf{G}\}$ has nonempty intersection, and the centers of the spheres in $f(\mathbf{G})$ all lie on the $(m-1)$ -dimensional flat $H \cap L$. Since $f(\mathbf{G})$ is a family of spheres in $(n-1)$ -dimensional flat H and $f(\mathbf{G})$ has nonempty intersection, it follows from the inductive hypothesis that there are spheres $S_1, \dots, S_k$ ($k \leq m$) in $\mathbf{K}_0$ such that the intersection of $f(S_0 \cap S_i)$, $i = 1, \dots, k$, is equal to $\bigcap f(\mathbf{G})$. Therefore the intersection of S_i , $i = 0, 1, \dots, k$, is equal to $\bigcap \mathbf{F}$. $\square$

Corollary 2. *Let $\mathbf{F}$ be a family of spheres in E^n with nonempty intersection. Then there exist $n+1$ or fewer spheres in E^n whose intersection coincides with the intersection of all spheres in $\mathbf{F}$.*

3. A Family of at Least $n+3$ Spheres

Theorem 3. *Let $\mathbf{F}$ be a family of at least $n+3$ different spheres in E^n . If any $n+1$ spheres in $\mathbf{F}$ have a common point, then all spheres in $\mathbf{F}$ have a common point.*

Remark. The number $n+1$ cannot be replaced by n , see Proposition 1 below. It is also impossible to reduce the number $n+3$ to $n+2$, see Fig. 3.

First we prove the following lemma.

Lemma. *Let $\mathbf{G} = \{S_i : i = 1, \dots, n+2\}$ be a family of $n+2$ different spheres in E^n such that*

$$\bigcap (\mathbf{G} \setminus \{S_i\}) \ni x_i, \quad i = 1, \dots, n+2, \quad \text{and} \quad \bigcap \mathbf{G} = \emptyset.$$

Let $X = \{x_i : i = 1, \dots, n+2\}$ and $Z = \{z_i : i = 1, \dots, n+2\}$, where z_i is the center of S_i . Then for each i , both of $Z \setminus \{z_i\}$ and $X \setminus \{x_i\}$ span E^n . Furthermore, for $1 \leq i \neq j \leq n+2$,

$$\bigcap (\mathbf{G} \setminus \{S_i\}) = \{x_i\} \quad \text{and} \quad \bigcap (\mathbf{G} \setminus \{S_i, S_j\}) = \{x_i, x_j\}.$$

Proof. The case $n=1$ is obvious, so we assume $n \geq 2$. Suppose that $Z \setminus \{z_i\}$ is contained in a flat of dimension $m < n$. Then, by Theorem 2, there are $m+1$ ($\leq n$) spheres in $\mathbf{G} \setminus \{S_i\}$ whose intersection coincides with $\bigcap (\mathbf{G} \setminus \{S_i\})$. These $m+1$ spheres and the sphere S_i have a common point because any $n+1$ spheres of $\mathbf{G}$ have a common point. Hence $\bigcap (\mathbf{G} \setminus \{S_i\})$ and S_i have a common point, that is, $\bigcap \mathbf{G} \neq \emptyset$, a contradiction. Therefore $Z \setminus \{z_i\}$ must span E^n . Suppose now x_i lies

on the flat spanned by $X \setminus \{x_i, x_j\}$, $i \neq j$. then, since $X \setminus \{x_i\}$ is spherical (i.e., lying on the sphere S_i), every sphere passing through all the points of $X \setminus \{x_i, x_j\}$ also passes through x_i . Hence S_i (which passes through all points of $X \setminus \{x_i\}$) must pass through x_i . This means $\bigcap \mathbf{G} \ni x_i$, a contradiction. Hence $X \setminus \{x_i\}$ spans E^n . Finally, to show $\bigcap (\mathbf{G} \setminus \{S_i\}) = \{x_i\}$, suppose that $\bigcap (\mathbf{G} \setminus \{x_i\})$ contains another point $y_i \neq x_i$. Then the set $Z \setminus \{z_i\}$ must lie on the hyperplane perpendicularly bisecting the line segment $x_i y_i$, which is a contradiction. Thus $\bigcap (\mathbf{G} \setminus \{S_i\}) = \{x_i\}$. Similarly, since $Z \setminus \{z_i, z_j\}$ spans a hyperplane, it follows that $\bigcap (\mathbf{G} \setminus \{S_i, S_j\})$ consists of at most two points. And since

$$\bigcap (\mathbf{G} \setminus \{S_i, S_j\}) = (\bigcap (\mathbf{G} \setminus \{S_i\})) \cup (\bigcap (\mathbf{G} \setminus \{S_j\})) = \{x_i, x_j\},$$

it follows that $\bigcap (\mathbf{G} \setminus \{S_i, S_j\}) = \{x_i, x_j\}$. □

Proof of Theorem 3. We will show that all $n+2$ spheres in $\mathbf{F}$ have a point in common. Then, by Corollary 1, all spheres in $\mathbf{F}$ have a point in common. Suppose that there is a subfamily $\mathbf{G} = \{S_i: i = 1, \dots, n+2\}$ of $n+2$ spheres of $\mathbf{F}$ with empty intersection. Let $X = \{x_i: i = 1, \dots, n+2\}$ and $Z = \{z_i: i = 1, \dots, n+2\}$ be as in the lemma. Then $\bigcap (\mathbf{G} \setminus \{S_i\}) = \{x_i\}$ and $\bigcap (\mathbf{G} \setminus \{S_i, S_j\}) = \{x_i, x_j\}$. Take a sphere S_{n+3} from $\mathbf{F} \setminus \mathbf{G}$. Then

$$\emptyset \neq \bigcap (\mathbf{G} \setminus \{S_i, S_j\}) \cap S_{n+3} = \{x_i, x_j\} \cap S_{n+3}, \quad 1 \leq i \neq j \leq n+2.$$

Hence S_{n+3} contains at least $n+1$ points of X . Without loss of generality, we may assume that $X \setminus \{x_1\} \subset S_{n+3}$. Then, since $X \setminus \{x_1\} \subset S_1$ and $X \setminus \{x_1\}$ spans E^n , S_{n+3} must be identical with S_1 , which is a contradiction. □

Combining Corollary 2 with Theorem 3, we have the following.

Corollary 3. *Let $\mathbf{F}$ be a family of at least $n+3$ spheres in E^n . Then there exist $n+1$ or fewer spheres in $\mathbf{F}$ whose intersection coincides with the intersection of all spheres in $\mathbf{F}$.*

Now we show that the number $n+1$ in Theorem 3 cannot be replaced by n .

Proposition 1. *For any $N > n$, there exists a family $\mathbf{F}$ of N different unit spheres in E^n such that all n spheres in $\mathbf{F}$ have a common point, but no $n+1$ spheres in $\mathbf{F}$ have a common point.*

Proof. Choose N points on a sphere of radius $\frac{1}{2}$ in E^n in such a way that no $n+1$ points of them lie on a hyperplane (that is, these N points are in general position). This is clearly possible because no finite number of hyperplanes can cover a sphere of positive radius. Let $\mathbf{F}$ be the family of the N unit spheres centered at these N points. Then any n spheres of $\mathbf{F}$ have exactly two points in common, but no $n+1$ spheres in $\mathbf{F}$ have a point in common. □

4. A Family of Unit Spheres

In Theorem 3 we cannot reduce the size of F to $n+2$, since for every n there exists a family G of $n+2$ spheres in E^n such that any $n+1$ spheres of G have a common point, but $\bigcap G = \emptyset$. The plane case ($n=2$) is depicted in Fig. 3. This plane case is of special interest: if some three circles of G are of unit radius, then the rest is also a unit circle. (Proof is not difficult.) Thus, even if we restrict ourselves to unit spheres, it is generally impossible to reduce the least size $n+3$ of F in Theorem 3 to $n+2$. But how about the case $n \neq 2$? Is there a family G of $n+2$ unit spheres in E^n ($n \neq 2$) such that any $n+1$ spheres of G have a common point and $\bigcap G = \emptyset$? We consider this in some detail.

Let Δ be an n -simplex in E^n . By the c -center and the c -radius (denoted by $R(\Delta)$) of Δ , we mean the center and the radius of the circumscribed sphere of Δ . Let w be the c -center of Δ and let y_i , $i=1, \dots, n+1$, be the feet of the perpendicular from w to the $n+1$ hyperplanes determined by the $n+1$ faces of Δ . Let us denote by Δ' the (possibly degenerate) simplex spanned by y_i , $i=1, \dots, n+1$. If Δ' is degenerate then we set its c -radius $R(\Delta') = \infty$. A simplex is said to be *critical* if its c -center lies on the hyperplane determined by one of its faces, otherwise the simplex is said to be *general*.

Proposition 2. *If Δ is a critical simplex then $R(\Delta')/R(\Delta) = \frac{1}{2}$.*

Proof. Suppose that Δ is a critical n -simplex with vertices z_i , $i=1, \dots, n+1$, in E^n whose c -center w lies on the hyperplane determined by the face opposite z_{n+1} . Then w is a vertex of Δ' . Let y_i , $i=1, \dots, n$, be the other vertices of Δ' . Then, since the angle $\angle wy_i z_{n+1} = 90^\circ$ for $i=1, \dots, n$, the $n+1$ points $w, y_1, \dots, y_n$ lie on the sphere with diameter wz_{n+1} . Hence $2R(\Delta') = wz_{n+1} = R(\Delta)$. $\square$

Proposition 3. *The following two statements are equivalent:*

- (1) *There exists a family $G = \{S_i; i=1, \dots, n+2\}$ of unit spheres in E^n such that $\bigcap (G \setminus \{S_i\}) \neq \emptyset$ for each i , but $\bigcap G = \emptyset$.*
- (2) *There exists a general n -simplex Δ in E^n such that $R(\Delta) = 2R(\Delta')$.*

Proof. (1) $\rightarrow$ (2). Let X and Z be as in the lemma, and let Δ be the simplex spanned by $z_1, \dots, z_{n+1}$. Then x_{n+2} is the c -center of Δ , and since $\bigcap (G \setminus \{S_i, S_{n+2}\}) = \{x_i, x_{n+2}\}$, $i < n+2$, the hyperplane determined by $Z \setminus \{z_i, z_{n+2}\}$ perpendicularly bisects the line segment $x_i x_{n+2}$ at, say, y_i . Since $x_{n+2} \neq y_i$, $i < n+2$, Δ is a general simplex. Further, the simplex Δ'' spanned by $x_1, \dots, x_{n+1}$ has unit c -radius (because all vertices lie on S_{n+2}), and Δ'' is similar to the simplex Δ' spanned by y_i 's with center of similitude x_{n+2} and ratio 2. Hence $R(\Delta') = \frac{1}{2}$.

(2) $\rightarrow$ (1). Suppose there exists a general n -simplex Δ in E^n of unit c -radius such that $R(\Delta') = \frac{1}{2}$. Let z_i , $i=1, \dots, n+1$, be the vertices of Δ and x_{n+2} be the c -center of Δ . For each $i < n+2$, let x_i be the reflection of x_{n+2} with respect to the hyperplane determined by the face of Δ opposite the vertex z_i . Then, since Δ is general, the points x_i , $i=1, \dots, n+1$, are all different, and the simplex Δ'' spanned by these points is similar to Δ' . So $R(\Delta'') = 1$. Let z_{n+2} be the c -center

of Δ^n . Let G be the family of unit spheres S_i centered at z_i , $i = 1, \dots, n+2$. Then it is easy to check that

$$\bigcap (G \setminus \{S_i\}) = \{x_i\} \quad \text{for } i = 1, \dots, n+2, \quad \text{but } \bigcap G = \emptyset. \quad \square$$

In the plane case $n = 2$, $R(\Delta')/R(\Delta) = \frac{1}{2}$ holds for any triangle Δ . However this is not the case in higher dimensions. It is not difficult to see that $R(\Delta')/R(\Delta) = 1/n$ for a regular n -simplex Δ of unit c -radius in E^n . In the case $n > 2$, if we move a vertex z_1 of a regular n -simplex Δ along the perpendicular to the opposite face of z_1 (keeping the other vertices fixed), then $R(\Delta')/R(\Delta)$ varies from $1/n$ to ∞ . And $R(\Delta')/R(\Delta)$ takes the value $\frac{1}{2}$ only when Δ becomes critical (this is also not difficult to check). This suggests the following conjecture.

Conjecture. *Let F be a family of at least $n+2$ unit spheres in E^n . If $n \neq 2$ and any $n+1$ spheres in F have a common point, then all spheres in F have a common point.*

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