An Algorithm for Reconstructing Convex Bodies from Their Projections*

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Abstract. An algorithm is described for the approximative reconstruction of a plane convex body from its projections in a finite number of directions. *A priori* and *a posteriori* error estimates are given, and the convergence of certain sequences of an approximative solution of the reconstruction problem to the exact solution is proven. Finally, it is shown that, after small modifications, the algorithm can be applied to reconstruct convex bodies from discrete projectional data.

The algorithm consists in an approximation of the convex body, to be reconstructed, by recursively defined cores and envelopes, following the ideas of Kuba [6] for the reconstruction of binary patterns.

1. Introduction

Let K be a plane *convex body*, i.e., compact convex and with nonempty interior, and ϑ be a *direction* (an angle) in $\mathbb{R}^2$, $0 \leq \vartheta < \pi$. We define the ϑ -*directional projection* (ϑ -*projection*) of K by

$$[P_{\vartheta}K](u) = \int_{-\infty}^{\infty} \chi_K(x, y) \, dv, \quad (1)$$

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Fig. 1. ϑ -directional projections $P_{\vartheta}K$ of a ϑ -convex body K with border functions k_1 and k_2 .

where $\chi_K(x, y)$ denotes the characteristic function of K , and u, v are the variables of the coordinate system rotated by ϑ (Fig. 1):

$$\begin{aligned} u &= x \cos \vartheta + y \sin \vartheta, \\ v &= -x \sin \vartheta + y \cos \vartheta. \end{aligned}$$

Hammer [5] proposed the following question: How many projections of a convex body have to be taken to permit its exact reconstruction?

There are papers which show the existence of a set of directions determining K uniquely (see Gardner [2], Gardner and McMullen [3], and Giering [4]). On the other hand, some special methods were published to reconstruct certain subclasses of planar bodies, from two given projections (see Chang and Chow [1], and Volčič [8]).

This paper gives a general reconstruction method by which it is possible, at least theoretically, to reconstruct a convex body, to any required degree of accuracy, from its projections. The procedure is similar to the reconstruction algorithm which was used by Kuba [6], in the case of two-directionally connected binary patterns.

Formally, let

$$p_i(u) = [P_{\vartheta_i}K](u), \quad i = 1, 2, \dots, M,$$

where $\vartheta_1, \vartheta_2, \dots, \vartheta_M$ ($M \geq 2$) are distinct directions. We describe a procedure which permits the reconstruction of K (and all of the convex bodies having the same projections as K) from the given functions $p_1, p_2, \dots, p_M$. Furthermore, if p_i , $i = 1, 2, \dots, M$, are arbitrary nonnegative functions, this method would tell us, after infinitely many steps, whether or not there is a convex body having $p_1, p_2, \dots, p_M$ as projections.

After giving the necessary definitions (Section 2), we describe the construction rules (Section 3) and an algorithm (Section 4, Algorithm E) by which we can

reconstruct, from exact projections, the so-called “approximative solutions” of the reconstruction problem. In Section 5 we present an *a priori* and two *a posteriori* estimates to measure the difference between the exact and the approximative solution. In Section 6 we show that certain sequences of approximative solutions converge to the exact solution. Finally, in Section 7, we prove that our method, with small modifications (Algorithm ED), is suitable to reconstruct convex bodies from exact discrete projectional data. Implementation of this algorithm as a computer program is in preparation.

2. Definitions

The line in direction ϑ having distance u from the origin, we denote by

$$l(u, \vartheta) = \{(x, y) \mid x \cos \vartheta + y \sin \vartheta = u\}.$$

We say that a set $K (\subset \mathbb{R}^2)$ is *convex in direction ϑ* (ϑ -convex) if all of the lines in direction ϑ intersect K in a segment (Fig. 1). In other words, for all values of u and for arbitrary points Q_1 and Q_2 , from $Q_1, Q_2 \in K \cap l(u, \vartheta)$ it follows that $[Q_1, Q_2] \subseteq K$. It is clear that K is convex if and only if K is ϑ -convex for every $\vartheta \in [0, \pi)$.

Let a compact set K be convex in direction ϑ . Then the functions

$$\begin{aligned} k_1(u, \vartheta) &= \min\{v \mid (u, v) \in K\}, \\ k_2(u, \vartheta) &= \max\{v \mid (u, v) \in K\} \end{aligned}$$

are called the *border functions* of K (Fig. 1). They have the following properties:

- (i) k_1 and k_2 are defined on the same compact domain.
- (ii) k_1 is lower semicontinuous, k_2 is upper semicontinuous on their common domain.
- (iii) k_1 and k_2 are bounded.
- (iv) $k_1(u, \vartheta) \leq k_2(u, \vartheta)$ on the common domain.

Conversely, if there are two functions, $k_1(u, \vartheta)$ and $k_2(u, \vartheta)$, such that (i)–(iv) are satisfied, then

$$K = \{(u, v) \mid k_1(u, \vartheta) \leq v \leq k_2(u, \vartheta)\}$$

is a compact, ϑ -convex set with border functions k_1 and k_2 in direction ϑ .

Before giving the definition of the reconstruction problem, we summarize some properties of the projections of a Lebesgue measurable set $H \subseteq \mathbb{R}^2$:

- (i) $[P_\vartheta H](u)$ exists for almost all u for every ϑ .
- (ii) $[P_\vartheta H](u) \geq 0$.
- (iii) $[P_\vartheta H](u) = \lambda_1(H \cap l(u, \vartheta))$, for almost all u , where λ_1 denotes the one-dimensional Lebesgue measure. When H is ϑ -convex, the projection gives the lengths of chords in direction ϑ .

$$(iv) \quad \lambda_2(H) = \int_{-\infty}^{\infty} [P_{\vartheta}H](u) du, \quad (2)$$

where $\lambda_2(\cdot)$ denotes the two-dimensional Lebesgue measure.

(v) If $\lambda_2(H_1), \lambda_2(H_2) < \infty$, then

$$H_1 \subseteq H_2 \Rightarrow [P_{\vartheta}H_1](u) \leq [P_{\vartheta}H_2](u).$$

(vi) Let K be ϑ -convex, and k_1, k_2 its border functions in direction ϑ . Then

$$[P_{\vartheta}K](u) = \begin{cases} k_2(u, \vartheta) - k_1(u, \vartheta), & \text{if } k_1 \text{ and } k_2 \text{ are defined at } (u, \vartheta), \\ 0, & \text{otherwise.} \end{cases}$$

(vii) For a convex body K , the support of $[P_{\vartheta}K](u)$ is an interval. On this interval, $[P_{\vartheta}K](u)$ is a concave function (in u).

The *reconstruction problem* (RP): suppose that a finite number of distinct directions, $\mathfrak{D} = (\vartheta_1, \vartheta_2, \dots, \vartheta_M)$,

$$0 \leq \vartheta_1 < \vartheta_2 < \dots < \vartheta_M < \pi \quad (M \geq 2),$$

and the same number of functions, $\mathbf{p} = (p_1, p_2, \dots, p_M)$, are given, with

$$p_i(u) \geq 0, \quad i = 1, 2, \dots, M.$$

We want to find a convex body K such that

$$p_i(u) = [P_{\vartheta_i}K](u) \quad (3)$$

for every $i = 1, 2, \dots, M$. If there is such a set K , we say that K is a *solution* of the RP. Henceforth, we shall use the shorter notation $P_iK = P_{\vartheta_i}K$.

Remarks. (a) It is not assumed that there is a convex body K corresponding to the functions $\mathbf{p}$.

(b) On the other hand, if the set of directions $\mathfrak{D} = \{\vartheta_1, \vartheta_2, \dots, \vartheta_M\}$ does not satisfy the condition of Theorem 1 of Gardner and McMullen [3], i.e., if S is linearly equivalent to a subset of directions of diagonals of a regular polygon, then a convex body may not be uniquely determined by $\mathfrak{D}$. In this case, it can even happen that there exist continuously many convex bodies which are equichordal with K , with respect to $\mathfrak{D}$ (see Theorem 2.2 of Volčič [9]).

In the description of our reconstruction algorithm we say that a set $H(\subset \mathbb{R}^2)$ is a *core* of the convex body K to be reconstructed if $H \subseteq K$. A set $H(\subset \mathbb{R}^2)$ is an *envelope* of K if $H \supseteq K$.

The *convex hull* of a set $H(\subset \mathbb{R}^2)$ is defined as

$$\text{conv}(H) = \bigcap_{H' \in \mathcal{K}(H)} H',$$

where $\mathcal{C}(H) = \{H' \mid H \subseteq H', H' \text{ convex}\}$. It has the following properties:

$$(i) \text{ conv}(H) \text{ is a convex set.} \quad (4)$$

$$(ii) H \subseteq \text{conv}(H). \quad (5)$$

$$(iii) \text{ Let } K \text{ be a convex set. If } H \subseteq K \text{ then } \text{conv}(H) \subseteq K. \quad (6)$$

Now, we define the set-function inconv . Let C and E be subsets of $\mathbb{R}^2$. If $\text{conv}(C) \not\subseteq E$, then let

$$\text{inconv}(C, E) = \emptyset.$$

If $\text{conv}(C) \subseteq E$, then let

$$\text{inconv}(C, E) = \bigcup_{H \in \mathcal{T}(C, E)} H,$$

where $\mathcal{T}(C, E) = \{H \mid C \subseteq H \subseteq E, H \text{ convex}\} = \{H \mid \text{conv}(C) \subseteq H \subseteq E, H \text{ convex}\}$ (Fig. 2). It has the following properties:

$$(i) \text{ inconv}(C, E) \subseteq E. \quad (7)$$

$$(ii) C \subseteq \text{inconv}(C, E) \text{ if and only if } \text{conv}(C) \subseteq E. \quad (8)$$

$$(iii) \text{ If } \text{inconv}(C, E) \neq \emptyset, \text{ then } \text{conv}(C) \subseteq E. \quad (9)$$

$$(iv) \text{ Let } K (\subset \mathbb{R}^2) \text{ be a convex set. If } C \subseteq K \subseteq E, \text{ then } K \subseteq \text{inconv}(C, E). \quad (10)$$

$$(v) \text{ In particular, } \text{inconv}(\emptyset, E) = E.$$

In Sections 5, 6, and 7 we use a *lattice* L , consisting of a finite number of distinct lines in the given directions $\mathfrak{D} = (\vartheta_1, \vartheta_2, \dots, \vartheta_M)$ such that for every ϑ_i , $i = 1, 2, \dots, M$, there are two lines of L , $l(u'_i, \vartheta_i)$ and $l(u''_i, \vartheta_i)$, such that

$$[u'_i, u''_i] \supseteq \text{supp}(p_i),$$

where $\text{supp}(p_i)$ denotes the support of the function p_i . The *lattice-points*, denoted by $\underline{L}$, are the points of intersection of lines of L . The points Q_1 and Q_2 are *adjacent* lattice-points in L if they are on the same line of L and between them there is no lattice-point. The *density* of lattice L , d_L , is defined as $d_L = \max\{\lambda_1([Q_1, Q_2]) \mid Q_1 \text{ and } Q_2 \text{ adjacent lattice-points in } L\}$.

Fig. 2. Construction of the set $\text{inconv}(C, E)$ from the sets C and E .

Let $K(\subset \mathbb{R}^2)$ be a set and let H be a subset of $\mathbb{R}^2$ containing a finite number of points. Then we denote the intersection of K and H by $\underline{K}_H$:

$$\underline{K}_H = K \cap H.$$

We shall denote by $\underline{K}_L$ the set of lattice-points belonging to K .

There are a given finite number of directions $\mathfrak{D} = (\vartheta_1, \vartheta_2, \dots, \vartheta_M)$, the same number of functions $\mathbf{p} = (p_1, p_2, \dots, p_M)$, and a set H containing a finite number of points of $\mathbb{R}^2$. We say that (C, E) is an *approximative solution* of the RP on the set H if:

- (i) $C \subseteq E \subset \mathbb{R}^2$.
- (ii) $\underline{E} \setminus \underline{C}_H = \emptyset$.
- (iii) $P_i C \leq p_i \leq P_i E$, $i = 1, 2, \dots, M$.

It is clear that if K is a solution of the RP then (K, K) is an approximative solution of the RP on an arbitrary set H .

A *stack memory* SM will be used in our algorithm to reconstruct approximative solutions of the RP on a given set H . Each element of SM consists of three parts, (C, E, Q) , where C and E are subsets of $\mathbb{R}^2$, $C \subseteq E$, and Q is a point such that $Q \in \underline{E} \setminus \underline{C}_H$. We denote the writing of the element (C, E, Q) into the uppermost cell of SM by

$$(C, E, Q) \rightarrow SM,$$

and the reading from SM by

$$SM \rightarrow (C, E, Q).$$

3. Construction Rules

To reconstruct approximative solutions, we give an iterative procedure in which, starting with an initial core set and with an initial envelope set of the convex body K to be reconstructed, we can increase the core and decrease the envelope successively in each iterative step, whereby K is approached.

If the given input data do not distinguish K from the other convex bodies, we also find cores and envelopes of the bodies which are equichordal with K in directions ϑ_i ($i = 1, 2, \dots, M$). To do this, we present construction rules which give new core and envelope sets from the old ones, and also a method for finding initial core and envelope sets.

(a) *Construction of initial core and envelope* corresponding to the directions $\mathfrak{D}$ and to the functions $\mathbf{p}$. Let $[a_i, b_i]$ denote the support of the function p_i , $i = 1, 2, \dots, M$ (if the support of p_i is not an interval, there is not a convex body corresponding to p_i , see property (vii) of projection). We define the strip determined by $l(a_i, \vartheta_i)$ and $l(b_i, \vartheta_i)$ as

$$S_i = \{(u, v) \mid a_i \leq u \leq b_i\}$$

Fig. 3. Construction of the initial envelope polygon $A_1, A_2, A_3, A_4, A_5, A_6$, and the initial core polygon $B_1, B_2, B_3, B_4, B_5, B_6$.

for $i = 1, 2, \dots, M$. It is clear that if there is a convex body K with projections $p_1, p_2, \dots, p_M$ then $S_i \supseteq K$, $i = 1, 2, \dots, M$. Therefore we can select

$$E^{(0)} = \bigcap_{i=1}^M S_i$$

as the initial envelope.

The set $E^{(0)}$ is a polygon with vertices $A_1, A_2, \dots, A_{2M}$ (Fig. 3), where

$$\begin{aligned} A_i &\in l(a_i, \vartheta_i) \cap l(a_{i+1}, \vartheta_{i+1}), & i = 1, 2, \dots, M-1, \\ A_M &\in l(a_M, \vartheta_M) \cap l(b_1, \vartheta_1), \\ A_{M+i} &\in l(b_i, \vartheta_i) \cap l(b_{i+1}, \vartheta_{i+1}), & i = 1, 2, \dots, M-1, \\ A_{2M} &\in l(b_M, \vartheta_M) \cap l(a_1, \vartheta_1) \end{aligned}$$

(some of them possibly coincide). Consider the polygon $B_1, B_2, \dots, B_{2M}$, where

$$\begin{aligned} B_1 &\in A_{2M}A_2 \cap A_1A_3, \\ B_i &\in A_{i-1}A_{i+1} \cap A_iA_{i+2}, & i = 2, 3, \dots, 2M-2, \\ B_{2M-1} &\in A_{2M-2}A_{2M} \cap A_{2M-1}A_1, \end{aligned}$$

and

$$B_{2M} \in A_{2M-1}A_1 \cap A_{2M}A_2.$$

(If $A_{i-1} = A_{i+1}$ for some i , let $B_i = A_{i-1}$). It is easy to see, since each of the sides of the polygon $A_1, A_2, \dots, A_{2M}$ contains at least one point of a convex body K , that the polygon $B_1, B_2, \dots, B_{2M}$ is in K . That is, we can use the polygon $B_1, B_2, \dots, B_{2M}$ as an initial core $C^{(0)}$. (If $M = 2$, then $B_1 = B_2 = B_3 = B_4$.)

We have

$$C^{(0)} \subseteq K \subseteq E^{(0)} \quad (11)$$

for all convex bodies.

Remark. Since in our algorithm we shall assume only (11) about $C^{(0)}$ and $E^{(0)}$, other choices are also possible. For instance, instead of the proposal given above, we can use the trivial initial core $C^{(0)} = \emptyset$.

(b) *Construction of the envelope set $E(C, p_i)$ corresponding to a ϑ_i -convex set C (core) and to a nonnegative function p_i (projection).* Let $c_1(u, \vartheta_i)$ and $c_2(u, \vartheta_i)$ be the border functions of C in direction ϑ_i (they are defined on a common domain). Then let

$$e_1(u, \vartheta_i) = \begin{cases} c_2(u, \vartheta_i) - p_i(u), & \text{if } u \text{ is in the common domain,} \\ -\infty, & \text{otherwise,} \end{cases}$$

$$e_2(u, \vartheta_i) = \begin{cases} c_1(u, \vartheta_i) + p_i(u), & \text{if } u \text{ is in the common domain,} \\ +\infty, & \text{otherwise.} \end{cases}$$

If $e_1 \leq e_2$, then there is a ϑ_i -convex set (see the definition of border functions) denoted by $E(C, p_i)$ with the border functions $e_1(u, \vartheta_i)$ and $e_2(u, \vartheta_i)$ (Fig. 4). If $e_1 \not\leq e_2$, we have a contradiction, and it means that there is no convex body corresponding to the given projections. In this case, let $E(C, p_i) = \emptyset$.

The envelope constructed in such a way has the following properties:

- (i) $E(\emptyset, p_i) = \mathbb{R}^2$.

Fig. 4. Construction of the envelope set $E(C, p_i)$.

- (ii) $E(C, p_i) \supseteq C$ if and only if $P_i C \leq p_i$.
- (iii) $P_i E(C, p_i) > p_i$ if and only if $P_i C < p_i$. $P_i E(C, p_i) = p_i$ if and only if $P_i C = p_i$.
- (iv) If $C \cap l(u, \vartheta_i) \neq \emptyset$, then $[P_i E(C, p_i)](u) \leq 2p_i(u)$.
- (v) Let K be a ϑ_i -convex set.

If $C \subseteq K$ and $P_i K \leq p_i$, then

$$K \subseteq E(C, p_i), \quad (12)$$

because $E(C, p_i)$ contains every ϑ_i -convex set not having bigger projection from direction ϑ_i than p_i and containing C . Thus, if C is a core set of K , then $E(C, p_i)$ is an envelope set of K .

(c) *Construction of the core set $C(E, p_i)$ corresponding to a given set E (envelope) and to a nonnegative function p_i (projection).* Let $l(u, \vartheta_i)$ be an arbitrary line intersecting E , and consider the set $E \cap l(u, \vartheta_i)$. For a fixed value of u , the set $E \cap l(u, \vartheta_i)$ consists of disjoint subsets (segments, points) of the line $l(u, \vartheta_i)$. Let the functions $e'(u, \vartheta_i)$ and $e''(u, \vartheta_i)$ be defined for those values of u , where among the disjoint subsets of $E \cap l(u, \vartheta_i)$ there is only one segment having length $\geq p_i(u)$ and this segment has length $\leq 2p_i(u)$. Let $e'(u, \vartheta_i)$ and $e''(u, \vartheta_i)$ be the coordinates of the endpoints of this segment, i.e.,

$$p_i(u) \leq e''(u, \vartheta_i) - e'(u, \vartheta_i) \leq 2p_i(u). \quad (13)$$

Now let

$$c_1(u, \vartheta_i) = \begin{cases} e''(u, \vartheta_i) - p_i(u), & \text{if } e''(u, \vartheta_i) \text{ is defined,} \\ \text{undefined,} & \text{if } e''(u, \vartheta_i) \text{ is undefined,} \end{cases}$$

and

$$c_2(u, \vartheta_i) = \begin{cases} e'(u, \vartheta_i) + p_i(u), & \text{if } e'(u, \vartheta_i) \text{ is defined,} \\ \text{undefined,} & \text{if } e'(u, \vartheta_i) \text{ is undefined.} \end{cases}$$

The functions $c_1(u, \vartheta_i)$ and $c_2(u, \vartheta_i)$ are defined on the same values of u . Because of (13)

$$c_1(u, \vartheta_i) \leq c_2(u, \vartheta_i)$$

for these values of u . Therefore, there is a ϑ_i -convex set, denoted by $C(E, p_i)$, with border functions $c_1(u, \vartheta_i)$ and $c_2(u, \vartheta_i)$ (Fig. 5).

The set $C(E, p_i)$ has the following properties:

- (i) $C(E, p_i) \subseteq E$.
- (ii) $P_i C(E, p_i) \leq p_i$.
- (iii) Let K be a ϑ_i -convex set.

If $K \subseteq E$ and $p_i \leq P_i K$, then

$$C(E, p_i) \subseteq K, \quad (14)$$

Fig. 5. Construction of the core set $C(E, p_i)$.

because $C(E, p_i)$ is contained in every ϑ_i -convex subset of E , which does not have smaller projections from direction ϑ_i than p_i . Thus, if E is an envelope set of K , then $C(E, p_i)$ is a core set of K .

4. Algorithm E

Algorithm E is an iterative procedure by which the approximative solutions of the RP are reconstructed.

The *input data* of Algorithm E are $(M, \mathfrak{D}, \mathbf{p}, H)$, and the *input predicate* (Manna [7], p. 161) of Algorithm E is $\varphi(M, \mathfrak{D}, \mathbf{p}, H)$, where:

- (a) $M \geq 2$ is the number of directions.
- (b) $\mathfrak{D} = (\vartheta_1, \vartheta_2, \dots, \vartheta_M)$, $0 \leq \vartheta_1 < \vartheta_2 < \dots < \vartheta_M < \pi$, are the directions.
- (c) $\mathbf{p} = (p_1, p_2, \dots, p_M)$ are the functions such that

$$0 \leq p_i(u), \quad i = 1, 2, \dots, M, \quad (15)$$

for all u , and there are i and u such that

$$0 < p_i(u).$$

- (d) H is a finite subset of $\mathbb{R}^2$.

Algorithm E is given by Chart 1.

The *output data* of Algorithm E are the ordered pairs of certain cores and envelopes, $(C^{(k)}, E^{(k)})$. The *output predicate* of Algorithm E is $\psi(M, \mathfrak{D}, \mathbf{p}, H, \{(C^{(k)}, E^{(k)})\})$, where:

- (a) $C^{(0)} \subseteq C^{(k)} \subseteq E^{(k)} \subseteq E^{(0)}$.
- (b) $\overline{E^{(k)}} \setminus C_H^{(k)} = \emptyset$.
- (c) $P_i C^{(k)} \leq p_i \leq P_i E^{(k)}$, $i = 1, 2, \dots, M$.

Chart 1. Flowchart of Algorithm E.

Statements (a)–(c) can be proved in the following way: we can arrive at instruction E_{13} only through instructions $E_4, \dots, E_{10}$; that is,

$$p_i \leq p_i E^{(k)}, \quad i = 1, 2, \dots, M \quad (16)$$

(instruction E_9). From (15), (16), and instruction E_7 follows

$$E^{(k)} = \text{inconv}(C^{(k)}, E) \neq \emptyset.$$

Because of (9), $C^{(k)} \subseteq E$, from where, using (8), we have

$$C^{(k)} \subseteq \text{inconv}(C^{(k)}, E),$$

that is

$$C^{(k)} \subseteq E^{(k)}.$$

It is easy to see that in every iteration step

$$C^{(0)} \subseteq C^{(k)}, \quad k = 1, 2, \dots$$

(instructions E_5 , E_{12} , E_{16} , E_{17} , and property (5)), and

$$E^{(k)} \subseteq E^{(0)}$$

(instructions E_6 , E_7 , E_{16} , E_{17} , and property (7)). Furthermore,

$$\underline{E^{(k)} \setminus C_H^{(k)}} = \emptyset$$

(instruction E_{10}), and from instruction E_9

$$P_i C^{(k)} \leq p_i \leq P_i E^{(k)}, \quad i = 1, 2, \dots, M.$$

That is, the output data of Algorithm E are the approximative solutions of the RP on the set H .

Now, we show that Algorithm E is *totally correct* with respect to φ and ψ (Manna [7], p. 165); that is, if $\varphi(M, \mathfrak{D}, \mathbf{p}, H)$ is true, the computation of the program terminates (Theorem 1E) and $\psi(M, \mathfrak{D}, \mathbf{p}, H, \{(C^{(k)}, E^{(k)})\})$ is true (as shown above).

We need the following lemma:

Lemma 1E. *If in the k th iteration step of Algorithm E we have $\underline{E_H^{(k-1)}} \neq \underline{E_H^{(k)}}$, then*

- (a) $E^{(k+1)} \subseteq E^{(k)}$ and
- (b) $C^{(k+1)} \supseteq C^{(k)}$.

Proof. Part (a) of Lemma 1E follows from instructions E_4 , E_6 , and E_7 and from property (7) of *inconv*:

$$E^{(k+1)} = \text{inconv}(C^{(k+1)}, E) \subseteq E = E^{(k)} \cap \bigcap_{i=1}^M E(C^{(k)}, p_i) \subseteq E^{(k)}.$$

Part (b) follows from instructions E_4 and E_5 and from property (5) of *conv*:

$$C^{(k+1)} = \text{conv}\left(C^{(k)} \cup \bigcup_{i=1}^M C(E^{(k)}, p_i)\right) \supseteq C^{(k)} \cup \bigcup_{i=1}^M C(E^{(k)}, p_i) \supseteq C^{(k)}. \quad \square$$

Theorem 1E. *If, for a given input data, $\varphi(M, \mathfrak{D}, \mathbf{p}, H)$ is true, the computation of Algorithm E terminates after a finite number of steps.*

Proof. In other words, we have to show that there is no loop in Algorithm E which could be repeated infinitely many times.

From Lemma 1E it is clear that during each iteration in loop $E_4, E_5, E_6, E_7, E_8, E_4$,

$$E_H^{(k+1)} \subset E_H^{(k)} \quad \text{and} \quad C^{(k+1)} \supseteq C^{(k)} \quad (17)$$

hold, while at instruction E_8 we have $E_H^{(k-1)} = E_H^{(k)}$ for some k . After a finite number of iteration steps $E_H^{(k-1)} = E_H^{(k)}$ necessarily occurs due to the monotonicity (17) and due to the finiteness of the set H .

Consider the loop $E_4, E_5, E_6, E_7, E_8, E_9, E_{10}, E_{11}, E_{12}, E_4$. Similarly as in the case of output data, from $P_i E^{(k)} \supseteq p_i, i = 1, 2, \dots, M, C^{(k)} \subseteq E^{(k)}$ follows. Furthermore, if there is a point of H in $E^{(k)} \setminus C^{(k)}$, then $C_H^{(k)} \subset C_H^{(k+1)}$ due to instructions E_{12}, E_4 , and E_5 , and, due to (5),

$$C^{(k+1)} = \text{conv} \left(\text{conv} \left(C^{(k)} \cup \bigcup_{i=1}^M C(E, p_i) \right) \cup \{Q\} \right) \supset C^{(k)}$$

holds, where $Q \notin C^{(k)}$ and $Q \in H$. After a finite number of iteration steps $E^{(k)} \setminus C_H^{(k)} = \emptyset$ for some k due to the finiteness of the set H (or $P_i C^{(k)} \not\supseteq p_i$ or $P_i E^{(k)} \not\supseteq p_i$ for some i).

The other two loops, $E_4, E_5, E_6, E_7, E_8, E_9, E_{14}, E_{15}, E_{16}, E_{17}, E_4$ and $E_4, E_5, E_6, E_7, E_8, E_9, E_{10}, E_{13}, E_{14}, E_{15}, E_{16}, E_{17}, E_4$, will also be finished after a finite number of steps, because of the fact that a point $Q \in H$ cannot be in the stack memory SM more than once. If $(C^{(k)}, E^{(k)}, Q) \in SM$ and $(C^{(k')}, E^{(k')}, Q') \in SM$ ($k < k'$) at the same time, $Q \in C^{(k)}$ (instruction E_{12}) and later only such a Q' may be selected such that $Q' \in E^{(k')} \setminus C_H^{(k')} \subset E^{(k)} \setminus C_H^{(k)}$, i.e., $Q' \notin C^{(k)}$, so $Q \neq Q'$. $\square$

Corollary 1E. *The output of Algorithm E is a finite number of approximative solutions on the set H : $(C_1, E_1), (C_2, E_2), \dots, (C_t, E_t)$. The case $t=0$ is also possible because the input data are not necessarily the projections of a convex body (cf. Corollary 2E). The reconstructed approximative solutions have the following properties:*

- (i) $C_j, j = 1, 2, \dots, t$, are convex sets (see instruction E_5 and (4)).
- (ii) $C_{jH} \neq C_{nH}$ and $E_{jH} \neq E_{nH}$ if $j \neq n$ ($1 \leq j, n \leq t$). This follows from the way point Q is selected (see the proof of Theorem 1E).
- (iii) $C_i = E_i$ iff $P_i C_i = p_i = P_i E_i$ for every $i = 1, 2, \dots, M$.

Theorem 2E. *Let K be a convex body. Let $(M, \mathfrak{D}, \mathbf{p}, H)$ be the input data of Algorithm E satisfying the input predicate φ . If*

$$p_i = P_i K, \quad i = 1, 2, \dots, M, \quad (18)$$

then among the approximative solutions reconstructed by Algorithm E there is exactly one (C^, E^*) , such that*

$$C^* \subseteq K \subseteq E^*. \quad (19)$$

Proof. On the basis of Theorem 1E we know that Algorithm E terminates after a finite number of iteration steps. That is, we have two finite sequences of sets, $\{C^{(k)}\}$ and $\{E^{(k)}\}$, where $C^{(k)}$ and $E^{(k)}$ denote the core and envelope sets in the k th iteration steps, respectively.

First, we show that from the sequences $\{C^{(k)}\}$ and $\{E^{(k)}\}$ we can select the subsequences

$$C^{(1_0)}, C^{(1_1)}, \dots, C^{(1_n)}, \\ E^{(1_0)}, E^{(1_1)}, \dots, E^{(1_n)},$$

in which

$$C^{(1_0)} \subseteq C^{(1_1)} \subseteq \dots \subseteq C^{(1_n)} \subseteq K, \\ E^{(1_0)} \supseteq E^{(1_1)} \supseteq \dots \supseteq E^{(1_n)} \supseteq K. \quad (20)$$

According to Lemma 1E and instructions E_5 , E_6 , E_7 , and E_{12} ,

$$C^{(0)} \subseteq C^{(1)} \subseteq \dots \subseteq C^{(q)}, \\ E^{(0)} \supseteq E^{(1)} \supseteq \dots \supseteq E^{(q)}$$

are fulfilled while $P_i C^{(k)} \leq p_i \leq P_i E^{(k)}$, for every $i = 1, 2, \dots, M$ (instruction E_9), and $\underline{E^{(k)}} \setminus C_H^{(k)} \neq \emptyset$ (instruction E_{10}).

Suppose that q is the first index for which $P_i C^{(q)} \not\leq p_i$ or $p_i \not\leq P_i E^{(q)}$ (for some i) or $\underline{E^{(q)}} \setminus C_H^{(q)} = \emptyset$. There are two cases:

- (a) If we did not use the stack memory SM in the iterations $k = 0, 1, \dots, q-1$, (loop E_4 , E_5 , E_6 , E_7 , E_8 , E_4), then from (11), (14), (6), (12), and (10) follows inductively

$$C^{(k+1)} \subseteq K \subseteq E^{(k+1)}$$

in each iterative step $k = 0, 1, \dots, q-1$. In this case we have subsequences (20) directly.

- (b) If we used the stack memory SM (instructions E_{11} and E_{16}) one or more times in the iterations $k = 0, 1, \dots, q-1$, then $C^{(k)} \subseteq K$ is not necessarily true for every $k = 0, 1, \dots, q$. Since there is a convex body K corresponding to the input data, one of the possibilities $Q \in K$ or $Q \notin K$, where $Q \in \underline{E^{(k)}} \setminus C_H^{(k)}$, is proved to be true. If $Q \in K$ (instruction E_{11}) is the right decision, $C^{(k)} \subseteq K$ remains valid because of (6). Similarly as in case (a) $K \subseteq E^{(k)}$ holds, too. If $Q \notin K$, the members of sequences $\{C^{(k)}\}$, $\{E^{(k)}\}$ are to be omitted until the iteration in which the procedure will be continued with the decision $Q \notin K$ (instruction E_{17}). This selection method can be applied whenever necessary. In this manner we get subsequences (20).

In subsequences (20) we have

$$C^{(1_n)} \subseteq K \subseteq E^{(1_n)}. \quad (21)$$

From (21) and (18) follow

$$P_i C^{(1_n)} \leq p_i \leq P_i E^{(1_n)}.$$

It is clear that

$$\overline{E^{(1_n)} \setminus C_{H^n}^{(1_n)}} = \emptyset.$$

This means that $(C^{(1_n)}, E^{(1_n)})$ is an approximative solution on the set H .

Furthermore, it is clear from the selection procedure that only the members of subsequences (20) have the property $C^{(k)} \subseteq K \subseteq E^{(k)}$. $\square$

Corollary 2E.

- (a) *If, by the application of Algorithm E, we do not get an approximative solution, there is no convex body to the given input data.*
- (b) *It is possible to begin Algorithm E with arbitrary $C^{(0)}$ and $E^{(0)}$ ($C^{(0)} \subseteq E^{(0)}$), which have the property $C^{(0)} \subseteq K \subseteq E^{(0)}$. Theorem 2E also remains true in this case.*

Remark. Algorithm E can be modified in order to reconstruct approximative solutions $(C^{(k)}, E^{(k)})$, where $C^{(k)}$ is ϑ_i -convex, for $i = 1, 2, \dots, M$, but not convex. In this case, instead of operations conv and inconv , we have to use conv_M and inconv_M . If $H \subseteq \mathbb{R}^2$, then let

$$\text{conv}_M(H) = \bigcap_{H' \in \mathcal{C}_M(H)} H',$$

where $\mathcal{C}_M(H) = \{H' \mid H \subseteq H', H' \text{ } \vartheta_i\text{-convex, for } i = 1, 2, \dots, M\}$. Let C and E be two subsets of $\mathbb{R}^2$. If $\text{conv}_M(C) \not\subseteq E$, then let

$$\text{inconv}_M(C, E) = \emptyset.$$

If $\text{conv}_M(C) \subseteq E$, then let

$$\text{inconv}_M(C, E) = \bigcup_{H \in \mathcal{F}_M(C, E)} H,$$

where $\mathcal{F}_M(C, E) = \{H \mid C \subseteq H \subseteq E, H \text{ } \vartheta_i\text{-convex, for } i = 1, 2, \dots, M\} = \{H \mid \text{conv}_M(C) \subseteq H \subseteq E, H \text{ } \vartheta_i\text{-convex, for } i = 1, 2, \dots, M\}$.

5. Error Estimates

Let $(C_j, E_j), j = 1, 2, \dots, t$, be the approximative solutions of the RP reconstructed by Algorithm E. We can estimate the measure of the difference between a solution K and the core set $C^* \in \{C_j \mid j = 1, 2, \dots, t\}$ having property (19) in the following ways:

Estimate 1. Because $\bigcup_{j=1}^t E_j \supseteq K$ and $C^* \supseteq \bigcap_{j=1}^t C_j$, we can write

$$\lambda_2(K \setminus C^*) \leq \lambda_2\left(\bigcup_{j=1}^t E_j \setminus \bigcap_{j=1}^t C_j\right).$$

Or, because $\bigcup_{j=1}^I E_j \supseteq E^*$ and $\bigcap_{j=1}^I C_j \subseteq K$,

$$\lambda_2(E^* \setminus K) \leq \lambda_2\left(\bigcup_{j=1}^I E_j \setminus \bigcap_{j=1}^I C_j\right).$$

Estimate 2. Since $C^* \subseteq K$, we have from (2) and (3)

$$\begin{aligned} \lambda_2(K \setminus C^*) &= \int_{-\infty}^{\infty} ([P_i K](u) - [P_i C^*](u)) du \\ &= \int_{-\infty}^{\infty} (p_i(u) - [P_i C^*](u)) du \\ &\leq \max_{1 \leq j \leq I} \left\{ \int_{-\infty}^{\infty} (p_i(u) - [P_i C_j](u)) du \right\} \end{aligned} \quad (22)$$

for arbitrary $i = 1, 2, \dots, M$.

Estimates 1 and 2 can be used if each of the approximative solutions are reconstructed (*a posteriori* estimates).

It would be interesting to give an *a priori* estimate of $\lambda_2(E_j \setminus C_j)$ which can be used before the reconstruction. It is clear that $\lambda_2(E_j \setminus C_j)$ also depends on the set H . We shall show that using a more dense set H , we get a smaller measure of $\lambda_2(E_j \setminus C_j)$. To do this, we need the following remarks.

Remark 1E. Because of the convexity of the set K to be reconstructed, we can assume that the support of the function $p_i(u)$ is an interval, and on this interval $p_i(u)$ is a concave function for every $i = 1, 2, \dots, M$ (see the properties of P_θ in Section 2).

Remark 2E. We shall use, instead of an arbitrary set H , the lattice-points of a lattice L defined in Section 2.

Taking the union of the input predicate φ and the new conditions (Remarks 1E and 2E), we get the *new input predicate* $\varphi'(M, \mathfrak{D}, \mathbf{p}, \underline{L})$, where:

- (a) $M \geq 2$ is the number of projections.
- (b) $\mathfrak{D} = (\vartheta_1, \vartheta_2, \dots, \vartheta_M)$, $0 \leq \vartheta_1 < \vartheta_2 < \dots < \vartheta_M < \pi$, are the directions.
- (c) $\mathbf{p} = (p_1, p_2, \dots, p_M)$ are the functions such that

$$0 \leq p_i(u), \quad i = 1, 2, \dots, M,$$

$p_i(u)$ is a concave function on its support $[a_i, b_i]$, for every i , and there are i and u such that

$$0 < p_i(u).$$

- (d) L is a lattice defined in Section 2.

Theorem 3E. Let $(M, \mathfrak{D}, \mathbf{p}, \underline{L})$ be input data satisfying the input predicate φ' . Then, for every approximative solution (C_j, E_j) , $j = 1, 2, \dots, t$, reconstructed by Algorithm E, we have

$$\int_{a_i}^{b_i} (p_i(u) - [P_i C_j](u)) du \leq 2[(b_i - a_i) + z_i] d_L, \quad (23)$$

where $[a_i, b_i] = \text{supp}(p_i)$, $z_i = \max_u \{p_i(u)\}$, $i = 1, 2, \dots, M$, and d_L denotes the density of lattice L .

Proof. Let (C_j, E_j) be an arbitrary approximative solution on $\underline{L}$ reconstructed by Algorithm E from the given input data $(M, \mathfrak{D}, \mathbf{p}, \underline{L})$. Let ϑ_i be an arbitrary direction.

Let us enumerate the values of u_n corresponding to the lines $l(u_n, \vartheta_i)$ of L so that $u_n < u_{n+1}$, u_1 being the first value such that

$$p_i(u_1) \geq 2d_L,$$

and u_k the last such that

$$p_i(u_k) \geq 2d_L$$

($k \geq 1$, see Fig. 6). (If there is no line of L for which $p_i(u_n) \geq 2d_L$, then (23) is obviously true.) Because of the concavity of p , it is clear that

$$p_i(u_n) \begin{cases} \geq 2d_L, & \text{if } n = 1, 2, \dots, k, \\ < 2d_L, & \text{otherwise.} \end{cases}$$

Since (C_j, E_j) is an approximative solution,

$$l(u_n, \vartheta_i) \cap C_j \neq \emptyset, \quad n = 1, 2, \dots, k,$$

Fig. 6. Notations used in the Proof of Theorem 3E.

and

$$p_i(u_n) - [P_i C_j](u_n) \leq 2d_L \quad (24)$$

for every $n = 1, 2, \dots, k$.

Let us now estimate the integral in three parts:

$$\begin{aligned} & \int_{a_i}^{b_i} (p_i(u) - [P_i C_j](u)) \, du \\ &= \int_{a_i}^{u_1} + \int_{u_1}^{u_k} + \int_{u_k}^{b_i} (p_i(u) - [P_i C_j](u)) \, du \\ &\leq \int_{a_i}^{u_1} p_i(u) \, du + \int_{u_1}^{u_k} (p_i(u) - [P_i C_j](u)) \, du + \int_{u_k}^{b_i} p_i(u) \, du. \end{aligned} \quad (25)$$

Let

$$\bar{m}_n = \max_{u_n \leq u \leq u_{n+1}} \{p_i(u)\}, \quad n = 0, 1, \dots, k \quad (26)$$

(Fig. 6), then

$$\int_{a_i}^{u_1} p_i(u) \, du \leq 2(u_1 - a_i)d_L + \bar{m}_0 d_L \quad (27)$$

and

$$\int_{u_k}^{b_i} p_i(u) \, du \leq 2(b_i - u_k)d_L + \bar{m}_k d_L, \quad (28)$$

due to the concavity of p_i and due to the definition of u_1 and u_k .

Let

$$G_n = p_i(u_n) - 2d_L, \quad n = 1, 2, \dots, k, \quad (29)$$

and let

$$g(u) = G_n + \frac{G_{n+1} - G_n}{u_{n+1} - u_n} (u - u_n), \quad \text{if } u_n \leq u \leq u_{n+1}, \quad (30)$$

for $n = 1, 2, \dots, k-1$ (see Fig. 6). Because of (24) and the concavity of $P_i C_j$ (see Corollary 1E(i) and property (vii) of the projection), we have

$$g(u) \leq [P_i C_j](u) \quad (31)$$

for all $u \in [u_1, u_k]$.

From (31), (26), and (30)

$$\begin{aligned}
 & \int_{u_1}^{u_k} (p_i(u) - [P_i C_j](u)) \, du \\
 & \leq \int_{u_1}^{u_k} (p_i(u) - g(u)) \, du \\
 & = \sum_{n=1}^{k-1} \int_{u_n}^{u_{n+1}} (p_i(u) - g(u)) \, du \\
 & \leq \sum_{n=1}^{k-1} \int_{u_n}^{u_{n+1}} (\bar{m}_n - g(u)) \, du \\
 & = \sum_{n=1}^{k-1} \int_{u_n}^{u_{n+1}} \left(\bar{m}_n - G_n - \frac{G_{n+1} - G_n}{u_{n+1} - u_n} (u - u_n) \right) \, du \\
 & = \sum_{n=1}^{k-1} \left(\bar{m}_n - \frac{G_n + G_{n+1}}{2} \right) (u_{n+1} - u_n)
 \end{aligned} \tag{32}$$

follows.

Depending on the position of the maximum of $p_i(u)$, there are three cases:

Case 1. If $z_i = \bar{m}_0$, then

$$\bar{m}_n = p_i(u_n), \quad n = 1, 2, \dots, k. \tag{33}$$

From (32), (33), and (29), we have

$$\begin{aligned}
 & \int_{u_1}^{u_k} (p_i(u) - [P_i C_j](u)) \, du \\
 & \leq \sum_{n=1}^{k-1} \left(p_i(u_n) - \frac{G_n + G_{n+1}}{2} \right) (u_{n+1} - u_n) \\
 & = 2(u_k - u_1)d_L + \sum_{n=1}^{k-1} \frac{p_i(u_n) - p_i(u_{n+1})}{2} (u_{n+1} - u_n) \\
 & \leq 2(u_k - u_1)d_L + \sum_{n=1}^{k-1} \frac{p_i(u_n) - p_i(u_{n+1})}{2} d_L \\
 & = 2(u_k - u_1)d_L + \frac{p_i(u_1) - p_i(u_k)}{2} d_L.
 \end{aligned} \tag{34}$$

Using (25), (27), (34), (28), and (33) we get

$$\begin{aligned}
 & \int_{a_i}^{b_i} (p_i(u) - [P_i C_j](u)) du \\
 & \leq 2(u_1 - a_i)d_L + z_i d_L + 2(u_k - u_1)d_L \\
 & \quad + \frac{p_i(u_1) - p_i(u_k)}{2} d_L + 2(b_i - u_k)d_L + p_i(u_k)d_L \\
 & = 2(b_i - a_i)d_L + z_i d_L + \frac{p_i(u_1) + p_i(u_k)}{2} d_L \\
 & \leq 2(b_i - a_i)d_L + 2z_i d_L.
 \end{aligned} \tag{35}$$

Case 2. If $z_i = \bar{m}_q$, where $1 \leq q \leq k-1$, then

$$\bar{m}_n = p_i(u_{n+1}), \quad n = 0, 1, \dots, q-1, \tag{36}$$

and

$$\bar{m}_n = p_i(u_n), \quad n = q+1, q+2, \dots, k. \tag{37}$$

From (32), (36), and (37) we have

$$\begin{aligned}
 & \int_{u_1}^{u_k} (p_i(u) - [P_i C_j](u)) du \\
 & \leq \sum_{n=1}^{q-1} \left(\bar{m}_n - \frac{G_n + G_{n+1}}{2} \right) (u_{n+1} - u_n) + \left(\bar{m}_q - \frac{G_q + G_{q+1}}{2} \right) (u_{q+1} - u_q) \\
 & \quad + \sum_{n=q+1}^{k-1} \left(\bar{m}_n - \frac{G_n + G_{n+1}}{2} \right) (u_{n+1} - u_n) \\
 & \leq \left[2(u_q - u_1)d_L + \frac{p_i(u_q) - p_i(u_1)}{2} d_L \right] \\
 & \quad + \left[z_i d_L - \frac{p_i(u_q) + p_i(u_{q+1})}{2} d_L + 2(u_{q+1} - u_q)d_L \right] \\
 & \quad + \left[2(u_k - u_{q+1})d_L + \frac{p_i(u_{q+1}) - p_i(u_k)}{2} d_L \right] \\
 & = 2(u_k - u_1)d_L + z_i d_L - \frac{p_i(u_1) + p_i(u_k)}{2} d_L.
 \end{aligned} \tag{38}$$

Using (25), (27), (38), (28), (36), and (37) we get

$$\begin{aligned}
 & \int_{a_i}^{b_i} (p_i(u) - [P_i C_i](u)) du \\
 & \leq 2(u_1 - a_i)d_L + p_i(u_1)d_L + 2(u_k - u_1)d_L + z_i d_L \\
 & \quad - \frac{p_i(u_1) + p_i(u_k)}{2} d_L + 2(b_i - u_k)d_L + p_i(u_k)d_L \\
 & = 2(b_i - a_i)d_L + z_i d_L + \frac{p_i(u_1) + p_i(u_k)}{2} d_L \\
 & \leq 2(b_i - a_i)d_L + 2z_i d_L.
 \end{aligned} \tag{39}$$

Case 3. If $z_i = \bar{m}_k$, then

$$\bar{m}_n = p_i(u_{n+1}), \quad n = 0, 1, \dots, k-1. \tag{40}$$

From (32), (40), and (29) we have

$$\begin{aligned}
 \int_{u_1}^{u_k} (p_i(u) - [P_i C_i](u)) du & \leq \sum_{n=1}^{k-1} \left(p_i(u_{n+1}) - \frac{G_n + G_{n+1}}{2} \right) (u_{n+1} - u_n) \\
 & \leq 2(u_k - u_1)d_L + \frac{p_i(u_k) - p_i(u_1)}{2} d_L.
 \end{aligned} \tag{41}$$

Using (25), (27), (41), (28), and (40) we get

$$\begin{aligned}
 & \int_{a_i}^{b_i} (p_i(u) - [P_i C_i](u)) du \\
 & \leq 2(u_1 - a_i)d_L + p_i(u_1)d_L + 2(u_k - u_1)d_L + \frac{p_i(u_k) - p_i(u_1)}{2} d_L + 2(b_i - u_k)d_L + z_i d_L \\
 & = 2(b_i - a_i)d_L + z_i d_L + \frac{p_i(u_k) + p_i(u_1)}{2} d_L \\
 & \leq 2(b_i - a_i)d_L + 2z_i d_L.
 \end{aligned} \tag{42}$$

Summarizing (35), (39), and (42) we have relation (23). $\square$

Corollary 3E. Let (C_j, E_j) be an arbitrary approximative solution of the RP on the set $\underline{L}$ reconstructed by Algorithm E. If

$$\int_{a_i}^{b_i} p_i(u) du = \int_{a_k}^{b_k} p_k(u) du, \tag{43}$$

for every $i, k = 1, 2, \dots, M$, the values

$$\int_{a_i}^{b_i} (p_i(u) - [P_i C_j](u)) du$$

are the same, for every $i = 1, 2, \dots, M$. That is, we can use the relation

$$\int_{a_i}^{b_i} (p_i(u) - [P_i C_j](u)) du \leq \min_{1 \leq i \leq M} \{2[(b_i - a_i) + z_i]d_L\},$$

for every $i = 1, 2, \dots, M$, as an estimate of the difference depending only on the given functions p_i and on d_L .

Estimate 3. If $p_1, p_2, \dots, p_M$ are the projections of a convex body K , (43) is true, and so

$$\lambda_2(K \setminus C^*) \leq \max_{1 \leq j \leq t} \left\{ \int_{a_i}^{b_i} (p_i(u) - [P_i C_j](u)) du \right\} \leq \min_{1 \leq i \leq M} \{2[(b_i - a_i) + z_i]d_L\},$$

where $(C^*, E^*) \in \{(C_j, E_j) | j = 1, 2, \dots, t\}$ is the approximative solution having the property (19).

6. Reconstruction of Convex Bodies

In the reconstruction of convex bodies we can use Algorithm E in the following way: using a more “dense” lattice in the input data, we get approximative solutions (C_j, E_j) , $j = 1, 2, \dots, t$, where $\lambda_2(E_j \setminus C_j)$ becomes smaller (see Theorem 3E), and so one of them is a better approximation of an (exact) solution of the RP.

Let $(M, \mathfrak{P}, \mathbf{p}, L_1)$ be the input data of Algorithm E such that they satisfy the input predicate φ' . Let the approximative solutions corresponding to L_1 be denoted by $(C_j^{(1)}, E_j^{(1)})$, $j = 1, 2, \dots, t_1$. If $C_j^{(1)} = E_j^{(1)}$, for some j , $C_j^{(1)}$ is a solution of the RP (see Corollary 1E(iii)), and we omit it from further process. After this, we have t'_1 approximative solutions ($t_1 \geq t'_1 \geq 0$).

If $t'_1 \geq 1$, then we increase the density of L_1 (i.e., decrease d_L) by adding new lines to L_1 in such a way that for the new lattice L_2 ($L_2 \supset L_1$)

$$d_{L_1} > d_{L_2}.$$

Now, we apply Algorithm E to the input data $(M, \mathfrak{P}, \mathbf{p}, C_1^{(1)}, E_1^{(1)}, L_2)$, where $C_1^{(1)}$ and $E_1^{(1)}$ in the input data mean that instead of $C^{(0)}$ and $E^{(0)}$ constructed according to Section 3(a), we shall use $C_1^{(1)}$ and $E_1^{(1)}$ as the initial core and initial envelope, respectively. Then, we apply Algorithm E to the input data $(M, \mathfrak{P}, \mathbf{p}, C_2^{(1)}, E_2^{(1)}, L_2)$, and so on. That is, Algorithm E will be applied t'_1 times on L_2 giving new approximative solutions $(C_j^{(2)}, E_j^{(2)})$, $j = 1, 2, \dots, t_2$. From Corollary 1E(ii) and from the core and envelope construction in Algorithm E it follows that there is exactly one $(C_j^{(1)}, E_j^{(1)})$ for each $(C_k^{(2)}, E_k^{(2)})$ such that

$$C_j^{(1)} \subseteq C_k^{(2)} \quad \text{and} \quad E_j^{(1)} \supseteq E_k^{(2)}.$$

If $C_j^{(2)} = E_j^{(2)}$, for some $j = 1, 2, \dots, t_2$, $C_j^{(2)}$ is a solution of the RP, and we can omit it from further process. After this selection $t'_2 (\leq t_2)$ approximative solutions remain.

We can continue the reconstruction process on a new lattice $L_3 (\supset L_2)$, where

$$d_{L_3} < d_{L_2}.$$

We use $C_j^{(2)}$ and $E_j^{(2)}$, $j = 1, 2, \dots, t'_2$, as the initial core and envelope sets, respectively. After the repeated application of Algorithm E t'_2 times on L_3 we get the newer approximative solutions $(C_j^{(3)}, E_j^{(3)})$, $j = 1, 2, \dots, t_3$, and so on.

In this way we obtain a double-sequence

$$\{(C_j^{(n)}, E_j^{(n)}) | n = 1, 2, \dots; j = 1, 2, \dots, t_n\}$$

of approximative solutions. This double-sequence can be illustrated by a directed tree (see Fig. 7). The vertices are the approximative solutions. There is an arc from $(C_j^{(n)}, E_j^{(n)})$ to $(C_k^{(n+1)}, E_k^{(n+1)})$ if and only if $C_j^{(n)} \subseteq C_k^{(n+1)}$ and $E_j^{(n)} \supseteq E_k^{(n+1)}$. The initial core and envelope, $(C^{(0)}, E^{(0)}) = (C_1^{(0)}, E_1^{(0)})$, are the roots of the tree.

Let (C_n, E_n) be a subsequence of $(C_j^{(n)}, E_j^{(n)})$ such that

$$C_n \subseteq C_{n+1} \quad \text{and} \quad E_n \supseteq E_{n+1}. \quad (44)$$

That is, (C_n, E_n) , $n = 0, 1, \dots$, are the vertices of a path from the root $(C_1^{(0)}, E_1^{(0)})$. If $d_{L_n} \rightarrow 0$ ($n \rightarrow \infty$) then there are the following possibilities:

- (a) $C_\infty = E_\infty$ on a lattice L_∞ after a finite number of steps. C_∞ (and E_∞) is a convex body and $P_i C_\infty = p_i = P_i E_\infty$, for every $i = 1, 2, \dots, M$ (see Corollary 1E(iii)). That is, C_∞ is a solution of the RP.

Fig. 7. Illustration of the approximative solutions, arranging them by a directed tree according to the relations $C_j^{(n)} \subseteq C_k^{(n+1)}$ and $E_j^{(n)} \supseteq E_k^{(n+1)}$.

- (b) The sequence $\{(C_n, E_n)\}$ has an infinite number of elements. From Theorem 3E we know that

$$\lambda_2(E_n \setminus C_n) \rightarrow 0 \quad (d_{L_n} \rightarrow 0). \quad (45)$$

From (44) and (45) we have

$$\lim_{n \rightarrow \infty} C_n = K = \lim_{n \rightarrow \infty} E_n, \quad (46)$$

where K is a convex body. Since

$$P_i C_n \leq p_i \leq P_i E_n, \quad i = 1, 2, \dots, M,$$

for every n , from (46)

$$\lim_{n \rightarrow \infty} P_i C_n = p_i = \lim_{n \rightarrow \infty} P_i E_n, \quad i = 1, 2, \dots, M,$$

follows. That is,

$$P_i K = p_i, \quad i = 1, 2, \dots, M.$$

So, K is a solution of the RP.

- (c) On a certain lattice L , the selected sequence $\{(C_n, E_n)\}$ is cancelled because there is no approximative solution belonging to $C_{\epsilon-1}$ and $E_{\epsilon-1}$ as the initial core and envelope, respectively.

On the basis of these statements we get the following consequence: if a convex body K is uniquely determined by the given input data, then:

- (a) we have a finite tree and at the end of one of its paths $C_\epsilon = E_\epsilon = K$, or
- (b) there is only one infinite path of the tree and the approximative solutions belonging to this path converge to K .

It would be very useful to know how far an approximative solution is from a solution of the RP. Theorem 3E gives information only about the difference between the corresponding projections, and Estimate 3 can be used only in the case of an approximative solution having the property (19). It is impossible to estimate $\lambda_2(K \setminus C_i)$ only from the value of $\lambda_2(E_i \setminus C_i)$, even in the case of uniquely determined convex bodies, as the following two examples show.

Fig. 8. Illustration of Example 1.

Example 1. Let T_1 be a triangle having sides of equal length, and let T_2 be another triangle such that $\text{conv}(T_1 \cup T_2)$ is a regular hexagon (Fig. 8). It is easy to see that the three directions, ϑ_1 , ϑ_2 , and ϑ_3 , of the sides of the hexagon do not distinguish between the two triangles.

Now let R be one of the vertices of T_1 , and let R_1 be a point near to R , belonging to one of the edges of the hexagon. The convex hull of $T_1 \cup \{R_1\}$ is a quadrangle Q . It is easy to prove, identifying the positions of the four vertices, that Q is uniquely determined by the projections in the directions ϑ_1 , ϑ_2 , and ϑ_3 . It is also clear from the construction that the projections of T_2 and Q are very near and that the distances between the projections tend to zero when R_1 tends to R . On the other hand, $\lambda_2(T_2 \Delta Q)$ tends to $\frac{1}{2}\lambda_2(T_2)$, when R_1 tends to R .

Example 1 shows that it is not possible to determine a $\delta > 0$, depending only on $\varepsilon > 0$ and not only on the body, such that if the projections are at a distance smaller than δ , then the distance between the bodies is smaller than ε .

Example 2. A similar problem arises when uniqueness is assumed for all the convex bodies (e.g., when the condition of Theorem 1 of Gardner and McMullen [3] is satisfied).

Consider two squares Q_1 and Q_2 such that $\text{conv}(Q_1 \cup Q_2)$ is a regular octagon (Fig. 9). The directions ϑ_1 , ϑ_2 , ϑ_3 , and ϑ_4 of the sides of the octagon do not distinguish between Q_1 and Q_2 . If we now take a quadruplet $(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta'_4)$, with ϑ'_4 near to ϑ_4 , and such that the cross-ratio $R(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta'_4)$ is transcendental, then (by Theorem 4 of Gardner and McMullen [3]) Q_1 is uniquely determined

Fig. 9. Illustration of Example 2.

by the corresponding projections. But since the projections with respect to ϑ_4 and ϑ'_4 are very close, we might find, if the approximation is not fine enough, a body K , whose projections approximate quite well those of Q_1 , but is in fact close to Q_2 .

7. Reconstruction of Convex Bodies from a Finite Number of Discrete Projectional Data

In this section we show that Algorithm E, with small modifications, is suitable to reconstruct approximative solutions from discrete projectional data; and the corresponding theorems of the RP (Theorems 1E and 2E) remain true.

Reconstruction Problem for Discrete Projectional Data (RPD). Suppose that a finite number of distinct diretions $\mathfrak{D} = (\vartheta_1, \vartheta_2, \dots, \vartheta_M)$,

$$0 \leq \vartheta_1 < \vartheta_2 < \dots < \vartheta_M < \pi \quad (M \geq 2),$$

the same number of vectors,

$$\begin{aligned} \mathbf{u}_1 &= (u_{11}, u_{12}, \dots, u_{1r_1}), \\ \mathbf{u}_2 &= (u_{21}, u_{22}, \dots, u_{2r_2}), \\ &\vdots \\ \mathbf{u}_M &= (u_{M1}, u_{M2}, \dots, u_{Mr_M}); \end{aligned}$$

and the same number of projectional vectors,

$$\begin{aligned} \mathbf{p}_1 &= (p_{11}, p_{12}, \dots, p_{1r_1}), \\ \mathbf{p}_2 &= (p_{21}, p_{22}, \dots, p_{2r_2}), \\ &\vdots \\ \mathbf{p}_M &= (p_{M1}, p_{M2}, \dots, p_{Mr_M}), \end{aligned}$$

are given. We want to find a convex body K such that

$$p_{ik} = [P_i K](u_{ik})$$

for every $i = 1, 2, \dots, M$ for every $k = 1, 2, \dots, r_i$ (in other words, $\mathbf{p}_i = [P_i K](\mathbf{u}_i)$, $i = 1, 2, \dots, M$). If there is such a set K , we say that K is a *solution of the RPD*.

In the following we use the notations

$$\mathbb{U} = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_M)$$

and

$$\mathbb{P} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_M).$$

For two vectors, $\mathbf{a}$ and $\mathbf{b}$, we write $\mathbf{a} < \mathbf{b}$ if strict inequality holds for all the components.

Suppose that the data $\mathfrak{P}$, $\mathbb{U}$, $\mathbb{P}$ of the RPD and a finite subset H of $\mathbb{R}^2$ are given. We say that (C, E) is an *approximative solution of the RPD* on H if:

- (i) $C \subseteq E \subset \mathbb{R}^2$,
- (ii) $E \setminus C_H = \emptyset$,
- (iii) $[P_i C](\mathbf{u}_i) \leq \mathbf{p}_i \leq [P_i E](\mathbf{u}_i)$, $i = 1, 2, \dots, M$.

Construction of the Initial Envelope and Core in the case of Discrete Projectional Data. Consider the vectors $\mathbf{u}_i$ and $\mathbf{p}_i$. Let

$$[a'_i, b'_i] = \sup \left(\inf_{g \in G(\mathbf{u}_i, \mathbf{p}_i)} \{g(u)\} \right)$$

and

$$[a''_i, b''_i] = \sup \left(\sup_{g \in G(\mathbf{u}_i, \mathbf{p}_i)} \{g(u)\} \right),$$

where $G(\mathbf{u}_i, \mathbf{p}_i) = \{g \mid g(\mathbf{u}_i) = \mathbf{p}_i, g \text{ is a nonnegative concave function on its support}\}$. (If $G(\mathbf{u}_i, \mathbf{p}_i) = \emptyset$ for some i , there is no convex body corresponding to the discrete projectional data $\mathbf{u}_i$ and $\mathbf{p}_i$.) It is clear that

$$[a''_i, b''_i] \supseteq \sup(P_i K), \quad i = 1, 2, \dots, M,$$

for each convex body K satisfying

$$[P_i K](\mathbf{u}_i) = \mathbf{p}_i, \quad i = 1, 2, \dots, M.$$

The strip S''_i is defined as

$$S''_i = \{(u, v) \mid a''_i \leq u \leq b''_i\}, \quad i = 1, 2, \dots, M.$$

Since $S''_i \supseteq K$ for each i , let

$$E^{(0)} = \bigcap_{i=1}^M S''_i.$$

If $M \geq 3$, we can use the following method to get an initial core (if $M = 2$, let $C^{(0)} = \emptyset$): let

$$\begin{aligned} Q_1 &= I(a'_1, \vartheta_1) \cap I(b''_M, \vartheta_M), \\ Q_i &= I(a'_i, \vartheta_i) \cap I(a''_{i-1}, \vartheta_{i-1}), \quad i = 2, 3, \dots, M, \\ Q_{M+1} &= I(b'_1, \vartheta_1) \cap I(a''_M, \vartheta_M), \\ R_i &= I(a'_i, \vartheta_i) \cap I(a''_{i+1}, \vartheta_{i+1}), \quad i = 1, 2, \dots, M-1, \\ R_M &= I(a'_M, \vartheta_M) \cap I(b''_1, \vartheta_1), \\ R_{M+1} &= I(b'_1, \vartheta_1) \cap I(b''_2, \vartheta_2), \\ R_{M+2} &= I(b'_2, \vartheta_2) \cap I(b''_3, \vartheta_3) \end{aligned}$$

Fig. 10. Construction of the point X_i as an element of the initial core in the case of discrete projectional data.

(some of them possibly coincide), and let

$$X_i \in Q_i R_{i+1} \cap Q_{i+1} R_{i+2}, \quad i = 1, 2, \dots, M.$$

(If $Q_i = R_{i+1}$ for some i , let $X_i = Q_i$.) We shall prove that $X_i \in K$ for each i : the intersecting lines $Q_i R_{i+1}$ and $Q_{i+1} R_{i+2}$ divide the polygon $E^{(0)}$ into four disjoint sets: T_1 , T_2 , T_3 , and T_4 (Fig. 10). It is not difficult to see that $T_1 \cap K \neq \emptyset$, $T_2 \cap K \neq \emptyset$, and $T_3 \cap K \neq \emptyset$. If the interior of T_4 contains a point of K then $X_i \in K$ because of the convexity of K . In the other case $Q_{i+1} \in K$ or $R_{i+1} \in K$. From $Q_{i+1} \in K$ it follows that the segment $[Q_{i+1}, R_{i+2}] \subseteq K$. From $R_{i+1} \in K$ it follows that the segment $[Q_i, R_{i+1}] \subseteq K$. In both cases $X_i \in K$.

This idea can be used for each $i = 1, \dots, M$. Furthermore, it can be repeated with the points

$$\begin{aligned} X_i &\in Q_i R_{i+1} \cap Q_{i+1} R_{i+2}, & i = M+1, M+2, \dots, 2M-2, \\ X_{2M-1} &\in Q_{2M-1} R_{2M} \cap Q_{2M} R_1, \\ X_{2M} &\in Q_{2M} R_1 \cap Q_1 R_2, \end{aligned}$$

where

$$\begin{aligned} Q_{M+i} &= l(b'_i, \vartheta_i) \cap l(b''_{i-1}, \vartheta_{i-1}), & i = 2, 3, \dots, M, \\ R_{M+i} &= l(b'_i, \vartheta_i) \cap l(b''_{i+1}, \vartheta_{i+1}), & i = 1, 2, \dots, M-1, \end{aligned}$$

and

$$R_{2M} = l(b'_M, \vartheta_M) \cap l(b''_1, \vartheta_1).$$

Thus, we have $2M$ points of K .

We can take the convex hull of these points as an initial core $C^{(0)}$:

$$C^{(0)} = \text{conv}\left(\bigcup_{i=1}^{2M} X_i\right).$$

We have

$$C^{(0)} \subseteq K \subseteq E^{(0)}$$

for all convex bodies K such that $p'_i \leq P_i K \leq p''_i$, $i = 1, 2, \dots, M$.

Remark. Since in the algorithm we shall assume only $C^{(0)} \subseteq K \subseteq E^{(0)}$ about the initial core and envelope, other $C^{(0)}$ and $E^{(0)}$ can also be used.

Since the projectional values are known only in the points of $\mathbb{U}$, the core and envelope constructions, $C(E, \mathbf{p}_i)$ and $E(C, \mathbf{p}_i)$, are defined on the lines $l(u_{ik}, \vartheta_i)$, $i = 1, 2, \dots, M$, $k = 1, 2, \dots, r_i$.

We give *Algorithm ED* to reconstruct approximative solutions of the RPD.

The *input data* of Algorithm ED are $(M, \mathfrak{D}, \mathbb{U}, \mathbb{P}, H)$ and the *input predicate* of Algorithm ED is $\varphi_D(M, \mathfrak{D}, \mathbb{U}, \mathbb{P}, H)$, where:

- (a) $M \geq 2$ is the number of projections.
- (b) $\mathfrak{D} = (\vartheta_1, \vartheta_2, \dots, \vartheta_M)$, $0 \leq \vartheta_1 < \vartheta_2 < \dots < \vartheta_M < \pi$, are the directions.
- (c) $\mathbb{U} = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_M)$, the positions of the discrete projectional data.
- (d) $\mathbb{P} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_M)$, the discrete projectional data.
- (e) H is a finite subset of $\mathbb{R}^2$.

Algorithm ED is given by Chart 2.

The *output data* of Algorithm ED are the ordered pairs of cores and envelopes, $(C^{(k)}, E^{(k)})$. The *output predicate* of Algorithm ED is $\psi_D(M, \mathfrak{D}, \mathbb{U}, \mathbb{P}, H, \{(C^{(k)}, E^{(k)})\})$, where:

- (a) $C^{(0)} \subseteq C^{(k)} \subseteq E^{(k)} \subseteq E^{(0)}$.
- (b) $\underline{E^{(k)}} \setminus \underline{C^{(k)}}_H = \emptyset$.
- (c) $[P_i C^{(k)}](\mathbf{u}_i) \leq \mathbf{p}_i \leq [P_i E^{(k)}](\mathbf{u}_i)$, $i = 1, 2, \dots, M$.

Statements (a), (b), (c) can be proved similarly as in the case of output predicate ψ of Algorithm ED.

Algorithm ED is totally correct with respect to φ_D and ψ_D , as follows from Lemma 1ED and Theorem 1ED.

Lemma 1ED. *If, in the k th iteration step of Algorithm ED, we have $\underline{E}_H^{(k-1)} \neq \underline{E}_H^{(k)}$ then:*

- (a) $E^{(k+1)} \subseteq E^{(k)}$ and
- (b) $C^{(k+1)} \supseteq C^{(k)}$.

Chart 2. Flowchart of Algorithm ED.

Proof. The same as that of Lemma 1E. □

Theorem 1ED. If, for a given input data, $\varphi_D(M, \mathfrak{S}, U, P, H)$ is true, the computation of Algorithm ED terminates after a finite number of steps.

Proof. The same as that of Theorem 1E. □

Corollary 1ED. The output of Algorithm ED is a finite number of approximative

solutions on the set $H: (C_1, E_1), (C_2, E_2), \dots, (C_t, E_t)$. ($t=0$ is also possible.) They have the following properties:

- (i) $C_j, j = 1, 2, \dots, t$, are convex sets.
- (ii) $C_{jH} \neq C_{nH}$ and $E_{jH} \neq E_{nH}$ if $j \neq n$ ($1 \leq j, n \leq t$).
- (iii) $C_i = E_j$ iff $[P_i C_i](u_i) = p_i = [P_i E_j](u_i)$ for every $i = 1, 2, \dots, M$.

Theorem 2ED. Let K be a convex body. Let $(M, \mathfrak{D}, \mathbb{U}, \mathbb{P}, H)$ be the input data of Algorithm ED satisfying the input predicate φ_D . If

$$p_i = [P_i K](u_i), \quad i = 1, 2, \dots, M,$$

then among the approximative solutions reconstructed by Algorithm ED, there is exactly one (C^*, E^*) , such that

$$C^* \subseteq K \subseteq E^*.$$

Proof. The same as that of Theorem 2E. □

Corollary 2ED.

- (a) If, by application of Algorithm ED, we do not get an approximative solution, there is no convex body to the given input data.
- (b) It is possible to begin Algorithm ED with arbitrary $C^{(0)}$ and $E^{(0)}$ ($C^{(0)} \subseteq E^{(0)}$), which have the property $C^{(0)} \subseteq K \subseteq E^{(0)}$. Theorem 2ED remains true in this case.

In the case of discrete data we can use Estimate 1 directly. Estimate 2 can be used if we know upper bounds to the functions $p_i, i = 1, 2, \dots, M$. Let

$$\bar{p}_i(u) = \sup_{g \in G(u, p_i)} \{g(u)\}, \quad (47)$$

where $G(u, p_i) = \{g \mid g(u_i) = p_i \text{ and } g \text{ is a concave function on its support}\}$. Then

$$p_i = P_i K \leq \bar{p}_i, \quad i = 1, 2, \dots, M.$$

From (22) we have

$$\begin{aligned} \lambda_2(K \setminus C^*) &\leq \max_{1 \leq j \leq t} \left\{ \int_{-\infty}^{\infty} (p_i(u) - [P_i C_j](u)) du \right\} \\ &\leq \max_{1 \leq j \leq t} \left\{ \int_{-\infty}^{\infty} (\bar{p}_i(u) - [P_i C_j](u)) du \right\} \end{aligned}$$

for arbitrary $i = 1, 2, \dots, M$.

To give an *a priori* estimate in the case of discrete data we need the following new input predicate $\varphi'_D(M, \mathfrak{D}, \mathbb{U}, \mathbb{P}, L)$, where:

- (a) $M \geq 2$ is the number of directions.
- (b) $\mathfrak{D} = (\vartheta_1, \vartheta_2, \dots, \vartheta_M)$, $0 \leq \vartheta_1 < \vartheta_2 < \dots < \vartheta_M < \pi$, is the vector of directions.
- (c) $\mathbb{U} = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_M)$, the positions of the discrete projectional data.
- (d) $\mathbb{P} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_M)$, the projectional vectors, where vector $\mathbf{p}_i$ can be considered as the values of a nonnegative function f_i taken at the points of $\mathbf{u}_i$

$$\mathbf{p}_i = f_i(\mathbf{u}_i), \quad i = 1, 2, \dots, M,$$

where f_i is concave on its support.

- (e) Lattice L consisting of the lines $l(u_{ik}, \vartheta_i)$, $k = 1, 2, \dots, r_i$, $i = 1, 2, \dots, M$, where $u_{ik} \in \mathbf{u}_i$.

Theorem 3ED. Let $(M, \mathfrak{D}, \mathbb{U}, \mathbb{P}, L)$ be input data satisfying the input predicate φ'_D . Then for every approximative solution (C_j, E_j) , $j = 1, 2, \dots, t$, reconstructed by Algorithm ED we have

$$\int_{a'_i}^{b'_i} (f_i(u) - [P_i C_j](u)) du \leq 2[(b'_i - a'_i) + z'_i] d_L,$$

where f_i is an arbitrary function satisfying (d) of input predicate φ'_D and $[a'_i, b'_i] = \text{supp}(f_i)$, $z'_i = \max_u \{f_i(u)\}$, $i = 1, 2, \dots, M$, and d_L is the density of lattice L .

The proof is similar to that of Theorem 3E.

As a consequence of Theorem 3ED we get

Estimate 3ED. If $\mathbf{p}_i = [P_i K](\mathbf{u}_i)$, where K is a convex body, and $(C^*, E^*) \in \{(C_j, E_j) | j = 1, 2, \dots, t\}$ is the approximative solution having the property (19), then

$$\begin{aligned} \lambda_2(K \setminus C^*) &= \int ([P_i K](u) - [P_i C^*](u)) du \\ &\leq \max_{1 \leq j \leq t} \left\{ \int ([P_i K](u) - [P_i C_j](u)) du \right\} \\ &\leq \min_{1 \leq i \leq M} \{2[(b_i - a_i) + z_i] d_L\}, \end{aligned} \tag{48}$$

where $[a_i, b_i] = \text{supp}(P_i K)$, $z_i = \max_u \{[P_i K](u)\}$. Since the projections $P_i K$ are known only on a finite number of points, we have to estimate a_i , b_i , and z_i from these discrete data: let

$$[\bar{a}_i, \bar{b}_i] = \text{supp}(\bar{p}_i),$$

and

$$\bar{z}_i = \sup_u \{\bar{p}_i(u)\},$$

where the functions $\bar{p}_i$, $i = 1, 2, \dots, M$, are defined by (47). From (48) we have

$$\lambda_2(K \setminus C^*) \leq \min_{1 \leq i \leq M} \{2[\bar{b}_i - \bar{a}_i] + \bar{z}_i\} d_L.$$

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