Note on Induced Subgraphs of the Unit Distance Graph E^n

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Abstract. The unit distance graph E^n is the graph whose vertices are the points in Euclidean n -space, and two vertices are adjacent if and only if the distance between them is 1. We prove that for any n there is a finite bipartite graph which cannot be embedded in E^n as an induced subgraph and that every finite graph with maximum degree d can be embedded in E^N , $N = (d^3 - d)/2$, as an induced subgraph.

1. Introduction

The *unit distance graph* of Euclidean space E^n is the infinite graph with vertex set E^n and edge set $\{xy : x, y \in E^n, \|x - y\| = 1\}$. This graph is also denoted by E^n . (Whether E^n means the space or the graph will become clear from the context.) All graphs in this paper are finite simple graphs, except the graph E^n .

To estimate the chromatic number $\chi(E^n)$ of E^n is an interesting and difficult problem. For instance, it is well known that $4 \leq \chi(E^2) \leq 7$ (see, e.g., p. 111 of [1]), but the exact value of $\chi(E^2)$ is not yet known. In general, $(1.2 + o(1))^n < \chi(E^n) < (3 + o(1))^n$ is known, see [4].

Another interesting problem is to decide whether a given graph G can be embedded in E^n as an (induced) subgraph. The smallest integer n needed to embed G in E^n as an (induced) subgraph is called the *dimension* of G or the *Euclidean dimension* of G [2], [3], [5]. Here, to avoid any possible confusion, let us state the definition.

Definition. For a graph G , the smallest integer n needed to embed G in E^n as an “induced” subgraph is called the *Euclidean dimension* of G and is denoted by $e(G)$.

Clearly, the complete graph K_p of order p has Euclidean dimension $p - 1$.

2. Euclidean Dimension and Chromatic Number

Theorem A [3]. *Every graph with chromatic number n can be embedded in E^{2n} as a subgraph.*

This is not the case for induced subgraphs as will be seen in Theorem 1.
Recall the following Helly-type theorem for spheres.

Theorem B [6]. *Let F be a family of at least $n+2$ spheres in E^{n-1} such that any n members of F have a common point. Then all members in F have a common point, and there exist n or fewer spheres in F whose intersection coincides with the intersection of all spheres in F .*

Theorem 1. *For any n , there is a bipartite graph G with $e(G) = n$.*

Proof. Let $X = \{1, 2, \dots, n+2\}$ and $Y = \{A \subset X : |A| = n\}$. Let G be the bipartite graph with vertex set $V(G) = X \cup Y$ and edge set $E(G) = \{xA : x \in A\}$. First we show $e(G) \geq n$. Suppose that there is an embedding of G in E^{n-1} as an induced subgraph. Then there is a map $f: V(G) \rightarrow E^{n-1}$ such that $\|f(i) - f(A)\| = 1$ if and only if $i \in A$ ($i \in X, A \in Y$). Let F be the family of $n+2$ unit spheres S_i centered at $f(i)$, $i \in X$. Then any n members of F have a point in common. Hence, by Theorem B, there must be an A in Y such that $f(A)$ is common to all S_i , $i \in X$, which is a contradiction since $\|f(A) - f(i)\| = 1$ for some $i \notin A$. Thus $e(G) \geq n$.

Next we show $e(G) \leq n$ by construction. Let S be a sphere of radius $r < 1$ (say $r = \frac{1}{2}$) in E^n . Take $n+2$ points x_i , $i = 1, \dots, n+1$, on S in such a way that no $n+1$ points lie on a hyperplane of E^n . This is clearly possible. For each $i = 1, \dots, n+2$, let S_i be the unit sphere centered at x_i . Then, for each $A \in Y$, the point set $\{x_i : i \in A\}$ spans a hyperplane of E^n , and hence the n spheres S_i , $i \in A$, have exactly two points in common. On the other hand, any $n+1$ spheres of $\{S_i : i \in X\}$ have no common point because any $n+1$ points of $\{x_i : i \in X\}$ lie on S and span E^n . Now, for every A of Y , take a point y_A in the intersection of S_i , $i \in A$. Then $y_A \in S_j$ if and only if $j \in A$. Define a map $f: V(G) \rightarrow E^n$ by $f(i) = x_i$, $i \in X$, and $f(A) = y_A$, $A \in Y$. Then f is an embedding of G in E^n as an induced subgraph. $\square$

Since the complete bipartite graph $K(m, n)$, $m, n > 2$, has Euclidean dimension 4 [3], [5], we have

Corollary 1. *The graph E^4 contains finite subgraphs of arbitrarily high Euclidean dimension.*

Thus, it is impossible to give an upper bound on $e(G)$ in terms of the chromatic number of G alone.

We now modify a lemma of Reiterman *et al.* [7] which we will use to obtain a bound for $e(G)$.

Lemma 1. *Let G be a graph with maximum degree d . Then there exist two maps $f, g: V(G) \rightarrow E^{d+1}$ such that for every $u, v \in V(G)$,*

$$\begin{aligned} f(u) \cdot f(v) &= 0, & g(u) \cdot g(v) &> 0, \\ f(u) \cdot g(v) &= 0 & \text{if and only if } uv &\in E(G), \end{aligned}$$

where the dot $\cdot$ denotes the inner product.

Proof. Let $V(G) = \{v_1, \dots, v_n\}$ and let c be a positive number. For each i , $i = 1, \dots, n$, let $j_1, \dots, j_r$ be the list of all j with $v_i v_j \in E(G)$. Put

$$p_i(t) = (t - j_1/c) \cdots (t - j_r/c)(t + 1/c)^{d-r}.$$

Then, since $r \leq d$, $p_i(t)$ is a monic polynomial of degree d ,

$$p_i(t) = a_{i0} + a_{i1}t + \cdots + a_{id}t^d, \quad a_{id} = 1.$$

Suppose that c is large enough so that $(a_{ik})^2 < 1/d$ for $k < d$ and $i = 1, \dots, n$. Now, for $i = 1, \dots, n$, put

$$f(v_i) = (a_{i0}, \dots, a_{id}), \quad g(v_i) = ((i/c)^0, \dots, (i/c)^d).$$

Then, clearly, $f(u) \cdot g(v) = 0$ if and only if $uv \in E(G)$, and $g(u) \cdot g(v) > 0$. Since $(a_{ik})^2 < 1/d$ for $k < d$, and $a_{id} = 1$, we also have $f(u) \cdot f(v) > 0$ $\square$

Theorem 2. *If G has chromatic number k and maximum degree d , then*

$$e(G) \leq \binom{k}{2}(d+1).$$

Proof. Let G be a k -partite graph with vertex classes $V_1, \dots, V_k$ and having maximum degree d . Let $f, g: V(G) \rightarrow E^{d+1}$ be the two maps as in Lemma 1. Take a tournament T on k vertices and let M be the vertex-arc incidence matrix of the tournament T . Then M is a $k \times \binom{k}{2}$ matrix, each column of which has one $+1$ and one -1 . Now, for each $v \in V_i$, let $x(v)$ be the vector obtained from the i th row of M by replacing each -1 by $f(v)$, each $+1$ by $g(v)$, and the 0's by 0-vectors of length $d+1$, $i = 1, \dots, k$. Then $x(v) \in E^N$, $N = \binom{k}{2}(d+1)$.

For example, let $k = 4$, $d = 5$. Let T be the tournament with vertices 1, 2, 3, 4 and arcs 12, 23, 34, 41, 13, 24. Then M is

	12,	23,	34,	41,	13,	24
1	+1	0	0	-1	+1	0
2	-1	+1	0	0	0	+1
3	0	-1	+1	0	-1	0
4	0	0	-1	+1	0	-1

and, for $v \in V_i$, $x(v) = (0, f(v), g(v), 0, f(v), 0) \in E^{16}$, where $0 = (0, 0, 0, 0, 0, 0)$.

Observe that if $u, v \in V_i$, then $x(u) \cdot x(v) > 0$; and if $u \in V_i$, $v \in V_j$ ($i \neq j$), then $x(u) \cdot x(v) = f(u) \cdot g(v)$. Hence

$$x(u) \cdot x(v) = 0 \quad \text{if and only if} \quad uv \in E(G).$$

Finally, let $y(v) = x(v)/(2x(v) \cdot x(v))^{1/2}$. Then $\|y(u) - y(v)\| = 1$ if and only if $uv \in E(G)$. Hence G can be embedded in E^∞ as an induced subgraph. $\square$

By Brooks' theorem (see, e.g., p. 91 of [1]) a connected graph G with maximum degree d has chromatic number at most d provided that G is neither a complete graph nor an odd cycle. Hence we have the following.

Corollary 2. *If G has maximum degree d , then*

$$e(G) \leq \binom{d}{2}(d+1) = (d^3 - d)/2.$$

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