Geometric Optimization and D^P -Completeness*

Chanderjit Bajaj¹ and Ming Li^{2†}

¹ Department of Computer Science, Purdue University, West Lafayette, IN 47907, USA

² Aiken Computation Laboratory, Harvard University, Cambridge, MA 02138, USA

Abstract. In this paper we show a number of natural geometric optimization problems in the plane to be *complete* for a class D^P . The class D^P contains both NP and Co-NP and is contained in $\Delta_2^P = P^{NP}$. Completeness in D^P is exhibited under many-one and positive reductions. Further an $OptP(O(\log n))$ result is also obtained for some of these optimization problems.

1. Introduction

The task of classifying the complexity of optimization problems *accurately* in the polynomial hierarchy is one of continuing interest and importance [3], [9], [11], [15]. We show that a large number of geometric optimization problems that arise naturally from the optimal placement of geometric objects are *complete* for a class D^P . The class of D^P was defined in [15] as follows: L is in D^P iff L is an intersection of L_1 and L_2 such that L_1 is in NP and L_2 is in Co-NP. The class D^P contains both NP and Co-NP and is contained in $\Delta_2^P = P^{NP}$. Completeness in D^P is exhibited under many-one and positive reductions. These results also prove the existence of natural geometric optimization problems that are *proper* in $\Delta_2^P = P^{NP}$. A problem Q is *proper* for Δ_2^P if Q is in Δ_2^P and not in NP or Co-NP, assuming $NP \neq Co-NP$.

The geometric optimization problems we consider here are inherently not pure combinatorial. The adaptation of combinatorial analysis methods to these problems thus provides added significance. In the past the recognition versions of various geometric location problems have been shown to be NP-complete [3],

* Work on this paper by the first author was supported in part by NSF Grant No. MIP 85-21356 and ARO Contract No. DAAG29-85-C-0018 under Cornell/MSI. Work by the second author was supported in part by NSF Grant No. DCR 86-06366 at Ohio State and ONR Grant No. N00014-k-0045 at Harvard.

† On leave from Ohio State University.

[6], [12], [13]. However, the results have been few and far between. In Section 2 we list the various geometric optimization problems. In Section 3 we show these problems to be D^P -complete under positive and many-one reductions. In Section 4 we give an $OptP(O(\log n))$ result. In Section 5 we show that all the constructions generalize to arbitrary metrics.

2. Geometric Optimization Problems

Our geometric optimization problems are all natural optimization questions we may ask concerning the size and number of geometric objects when dealing with optimal planar packing and covering. Initially we verse the optimization problems in terms of circles and then generalize to other geometric figures in the succeeding sections.

In the following denote the real Euclidean plane by E^2 and a circle *locatable* anywhere in E^2 to mean that the center of the circle can be any point in the real Euclidean plane. Furthermore, let an R -circle be a circle of radius R . We are given an input set $T = \{(x_i, y_i), i = 1, \dots, n\}$ of n fixed *points* in E^2 and integers k and m . For polynomially bounded input descriptions, we may restrict the coordinates of the input points to the class of algebraic real numbers, which are the roots of some polynomial equation over the rationals [2], [4].

The set of optimization problems then are:

- (PC₁) Is R the *minimum* radius of k equal-sized *circles* locatable anywhere in E^2 to cover the n *points* of T ?
- (PC₂) Is m ($m \leq n$) the *maximum* number of *points* of T that k R -circles locatable anywhere in E^2 can cover?
- (PC₃) Is k the *minimum* number of R -circles locatable anywhere in E^2 to cover the n *points* of T ?

Further consider the case where, besides the set T of n *points* in E^2 , we are also given a finite discrete set of points $S \subset E^2$ and the location of the *circles* are constrained to be from this set. Again a circle *locatable* anywhere in S means the center of the circle is a point of this set S .

- (QC₁) Is R the *minimum* radius of k -equal sized circles locatable anywhere in S to cover the n *points* of T ?
- (QC₂) Is m ($m \leq n$) the *maximum* number of *points* of T that k R -circles locatable anywhere in S can cover?
- (QC₃) Is k the *minimum* number of R -circles locatable anywhere in S to cover the n *points* of T ?

For circles (circular disks) an alternate though similar set of optimization questions as above, may be asked. The *piercing* number for a set of circles is the number of “needles” required to pierce all the circles of the system [8].

Given a set C of n fixed R -circles in the plane:

- (DC_1) Is R the *minimum* radius of R -circles of C that k *needles* can pierce?
- (DC_2) Is m the *maximum* number of R -circles of C that k *needles* can pierce?
- (DC_3) Is k the *minimum* number of *needles* required to pierce all n R -circles of C ?

Note that in these problems we optimally locate “needles” (points), in order to tag each of the given circles. These packing and covering optimization problems also arise from a series of location-allocation problems under a minimax (minimizing the maximum) optimization criterion [3]. In geometric optimization problems with the Euclidean distance norm there is however the problem with radicals [2], [4]. More specifically this occurs under the minsum criterion (minimizing the summation), where at best an exponential procedure is known to verify a comparison involving a sum of radicals. This however does not cause concern with the minimax optimization criterion where a comparison would involve at most two radicals and an easy comparison can be accomplished after simple squarings.

3. D^P -Completeness

We now show all the above geometric optimization problems to be complete for the complexity class D^P . The model of computation could be the multitape Turing machine or any other polynomially related computation model under a logarithmic cost criterion [1]. We begin by first proving the following lemma.

Lemma 1. *The problems DC_1 , DC_2 , and DC_3 are dual problems of PC_1 , PC_2 , and PC_3 (respectively) and are polynomial-time transformable to each other.*

Proof. The duality arises from the fact that if the k R -circles with centers $\{c_i\}$, $i = 1, \dots, k$, have a common intersection point then they can all be pierced by a single *needle*. Furthermore, a single R -circle centered on any point of the common intersection can cover the k centers, $\{c_i\}$, $i = 1, \dots, k$. For the problem DC_2 and similarly for the others, given the set $T = \{n \text{ points in the plane}\}$, obtain the set C of n circles of radius R with centers being the n points of T . Then m is the maximum number of R -circles of C that k needles can pierce iff m is the maximum number of points of T that k R -circles can cover. The proof follows, since, for each subset S of C of R -circles that a *needle* pierces, an R -circle centered on the piercing point of the needle can cover the centers (members of set T) of the R -circles of set S . Conversely, for each subset S of points, $S \subseteq T$, that are coverable by an R -circle, a *needle* pierces the subset of C of R -circles having S as their centers. $\square$

We now show that the problem PC_3 of determining the minimum number of R -circles in E^2 to cover all the n demand points is D^P -complete by reducing (*Sat*, *UnSat*), a known D^P -complete problem [15], to it. To show membership in D^P we use the fact that D^P can be defined as the class of all predicates $R(x)$

that can be expressed as $R(x) = [\exists y, P(x, y)] \wedge [\forall z, Q(x, z)]$ for some polynomially balanced and polynomial-time checkable P and Q .

It is quite well known that, similar to polynomial-time many-one reductions, polynomial-time positive reductions¹ preserve the class of NP [16]. That is if a language L_1 polynomial-time positive reduces (or polynomial-time many-one reduces) to a language L_2 then $L_2 \in \text{NP} \Rightarrow L_1 \in \text{NP}$. A similar fact is true for the class Co-NP. Therefore these positive reductions are *adequate* to separate the class of D^p -complete languages from the classes of NP and Co-NP (assuming $\text{NP} \neq \text{Co-NP}$). Thus in our proofs we illustrate both many-one and positive polynomial-time reductions. It is important to note that polynomial-time Turing reductions which do not preserve the class of NP, are not adequate in separating D^p languages from NP and Co-NP. Thus, for instance, it is possible to polynomial-time Turing reduce the D^p -complete language $(\text{Sat}, \text{Unsat})$ to (Sat) , a known NP-complete problem. Also note that the special case of disjunctive positive reductions which we use here are by far the strongest of the various other positive and truth-table reducibilities known [10]. In turn, any truth-table reduction is stronger than a Turing reduction.

Theorem 2. *The problem PC_3 of determining if k is the minimum number of R -circles in E^2 to cover the given n points, is D^p -complete under polynomial-time positive reductions.*

Proof. The problem is in D^p since it can be rephrased as the conjunction of a predicate in NP and a predicate in Co-NP: $(\exists(p_1, \dots, p_k) \in E^2)[R\text{-circles with centers at } p_1, \dots, p_k \text{ cover } n \text{ points}] \wedge (\forall(q_1, \dots, q_{k-1}) \in E^2)[R\text{-circles with centers at } q_1, \dots, q_{k-1} \text{ cover } < n \text{ points}]$. Both the quantified predicates within square brackets can easily be seen to be verifiable in polynomial time.

To prove the completeness we reduce $(\text{Sat}, \text{Unsat})$ to PC_3 , using polynomial-time positive reductions. Starting from (F_1, F_2) and adapting a polynomial-time construction in [6], we construct two separate sets of points S_1 and S_2 in the plane such that, for $i = 1, 2$, exactly k_i R -circles are required to cover all the n_i points in S_i if F_i is satisfiable. Further, if F_i is not satisfiable, at least $(k_i + 1)$ and at most $(k_i + c_i)$ R -circles are needed, where c_i is the number of clauses in the CNF formula F_i .

Now construct c_2 additional copies of the set of points S_1 . We now have $(c_2 + 1)$ copies of sets of points S_1 and a single set of points S_2 . (The reader may wish to see for himself why $(c_2 + 1)$ copies of S_1 are required.) Let $n = (c_2 + 1)n_1 + n_2$. It is not hard to see that k , the minimum number of R -circles needed to cover all the n points, satisfies $(c_2 + 1)k_1 + k_2 + 1 \leq k \leq (c_2 + 1)k_1 + k_2 + c_2$ iff F_1 is satisfiable and F_2 is not satisfiable. Since this is a disjunction of at most c_2 calls of PC_3 , problem PC_3 is D^p -complete under a polynomial-time positive (disjunctive) reduction from $(\text{Sat}, \text{Unsat})$. $\square$

¹ A function $F(X)$ is a polynomial-time positive reduction from language L_1 to language L_2 if F is polynomial-time computable and is a disjunction of a polynomial number of polynomial-time computable functions $f_i(X)$ with $X \in L_1$ iff there exists some i_0 such that $f_{i_0}(X) \in L_2$.

Corollary 2(a). *The problem PC_2 of determining if m is the maximum number of the given points that k R -circles in E^2 can cover, is D^P -complete under polynomial-time positive reductions.*

Proof. The problem is in D^P since it can be rephrased as before, as the conjunction of a predicate in NP and a predicate in Co-NP: $(\exists(p_1, \dots, p_k) \in E^2)[R\text{-circles with centers at } p_1, \dots, p_k \text{ cover } m \text{ points}] \wedge (\forall(q_1, \dots, q_k) \in E^2)[R\text{-circles with centers at } q_1, \dots, q_k \text{ cover } \leq m \text{ points}]$. Again the predicates are polynomial-time checkable.

To prove the completeness we reduce (Sat, Unsat) to PC_3 , using polynomial-time positive (disjunctive) reductions in a way very similar to Theorem 2. $\square$

Theorem 3. *The problem PC_1 of determining if R is the minimum radius of k equal-sized circles required to cover all the given n points, is D^P -complete under polynomial-time positive reductions.*

Proof. The problem is in D^P , when R is an integer parameter,² since it can be rephrased as before, as the conjunction of a predicate in NP and a predicate in Co-NP: $(\exists(p_1, \dots, p_k) \in E^2)[R\text{-circles with centers at } p_1, \dots, p_k \text{ cover } n \text{ points}] \wedge (\forall(q_1, \dots, q_k) \in E^2)[(R-1)\text{-circles with centers at } q_1, \dots, q_k \text{ cover } < n \text{ points}]$ for polynomial-time checkable predicates.

To prove it complete we show that PC_3 polynomial-time positive reduces to PC_1 . We construct a set S of the radii of all possible circles which minimally cover n points in the plane. Since the minimum enclosing circle for a set of points is defined by exactly two or three of the points, the total size of S is at most $\binom{n}{2} + \binom{n}{3}$ which is $O(n^3)$. We claim that k is the minimum number of R -circles that cover all n points iff, for some $s \in S$, $s \leq R$, s is the minimum radius of k circles to cover all n points and for some $s \in S$, $s > R$, s is the minimum radius of $k-1$ circles to cover all n points. The proof is straightforward and follows from the definitions of the two problems PC_1 and PC_3 . Again since we have a conjunction of two sets of disjunctive calls of PC_1 (at most $O(n^3)$ calls), we have a polynomial-time positive reduction from PC_3 to PC_1 . $\square$

To show D^P -completeness under the stronger many-one reductions is slightly more involved.

Theorem 4. *The problem PC_3 of determining if k is the minimum number of R -circles in E^2 to cover the given n points, is D^P -complete under polynomial-time many-one reductions.*

Proof. We need only describe the many-one reduction from (Sat, Unsat) to PC_3 .

The adapted polynomial-time construction of [6] gives a set of points S_i in

² The problem appears to be D^P -hard when R is an algebraic real-numbered parameter, for then it seems that the set of radii of possible circles is not polynomially bounded.

Fig. 1. (a) Points construction for Theorem 4. (b) Clause configuration.

the plane, such that exactly k_i R -circles are required to cover all the points if a CNF formula F_i is satisfiable. Further, if F_i is not satisfiable, then at least $(k_i + 1)$ and at most $(k_i + c_i)$ R -circles are needed, where c_i is the number of classes in the CNF formula F_i . We then modify this construction by adding a new line of points L at an appropriate distance from each of the previous clause points, see Fig. 1(a). The new line of points could be covered by l R -circles if none of the clause points is covered. Further $(l + 1)$ R -circles would be required to cover even only one of the clause points, see Fig. 1(b). Clearly, $(l + 1)$ R -circles are also sufficient to cover all the clause points. Now if F_i is satisfiable, then the original set of points S_i uses k_i R -circles and with the added line of points L an additional l R -circles, thus in total $(k_i + l)$ R -circles. Further, if F_i is not satisfiable, then S_i uses k_i R -circles while L uses an additional $(l + 1)$ R -circles, thus in total $(k_i + l + 1)$ R -circles.

Then starting from two CNF formulas (F_1, F_2) , we construct two copies of the above sets of points, $S_1^1 \cup L_1^1$ and $S_2^1 \cup L$ corresponding to CNF formula F_1 and one copy of the set of points $S_2 \cup L_2$ corresponding to CNF formula F_2 . Then the minimum number of R -circles needed to cover all the points are $2k_1 + k_2 + 3l + 1$ iff F_1 is satisfiable and F_2 is unsatisfiable. For if F_1 is unsatisfiable and F_2 is satisfiable or unsatisfiable, the number of R -circles needed are $\geq 2k_1 + k_2 + 3l + 2$ and if both F_1 and F_2 are satisfiable the number of R -circles needed are $2k_1 + k_2 + 3l$. $\square$

Corollary 4(a). *The problem PC_2 of determining if m is the maximum number of the given points that k R -circles in E^2 can cover, is D^P -complete under polynomial-time many-one reductions.*

Proof. The polynomial-time many-one reduction is the same as for PC_3 , except that we add an extra isolated point to the construction of Theorem 4 (two copies of the sets of points corresponding to F_1 and one copy corresponding to F_2). Then for $K = 2k_2 + k_1 + 3l + 1$ and $M = \text{number of points in } (S_1^1 \cup S_1^2 \cup S_2 \cup 3^*L)$ we have that K R -circles can cover at most M points iff F_1 is unsatisfiable and F_2 is unsatisfiable. For if F_1 is unsatisfiable and F_2 is satisfiable or unsatisfiable, K R -circles can cover less than M points and if both F_1 and F_2 are satisfiable, then K R -circles can cover $M + 1$ points as the extra R -circle covers the additional isolated point. $\square$

Corollary 4(b). *The problem PC_1 of determining if R is the minimum radius of k equal-sized circles required to cover all the given n points, is D^P -complete under polynomial-time many-one reductions.*

Proof. The polynomial-time many-one reduction is the same as for PC_3 , except that we add two extra isolated points to the construction of Theorem 4. These two points are at a distance $2R$ forcing the minimum radius of the covering circles to be R . Then let $K = 2k_2 + k_1 + 3l + 2$ and these many minimum radius R circles can cover all the points iff F_1 is unsatisfiable and F_2 is unsatisfiable. $\square$

Remark. The dual problems DC_1 , DC_2 , and DC_3 are also D^P -complete under both positive and many-one reductions. This from the earlier lemma and theorems.

Theorem 5. *The finite solution set problem QC_3 is D^P -complete under polynomial-time many-one reductions.*

Proof. The problem is in D^P since it can be rephrased as the conjunction of a predicate in NP and a predicate in Co-NP: $(\exists(p_1, \dots, p_k) \in S)[R\text{-circles with centers at } p_1, \dots, p_k \text{ cover } n \text{ points}] \wedge (\forall(q_1, \dots, q_{k-1}) \in S)[R\text{-circles with centers at } q_1, \dots, q_{k-1} \text{ cover } < n \text{ points}]$ for polynomial-time checkable predicates.

To prove it complete we prove that PC_3 polynomial-time many-one reduces to QC_3 . It suffices to show that for any set T of n destination points in the plane there exists a finite set $S \subset E^2$, such that if a minimum of k R -circles can cover T , then these R -circles can be chosen to have their centers in S . Furthermore, S must be constructible in time polynomial in n .

We claim that the set S can be chosen to be the union of set T with the following set $T_a = \{\text{intersection points of } R\text{-circles centered at the points of } T\}$. For a proof of this claim let F be a (minimal) set of circles of radius R covering T and let circle $C \in F$. If C contains only a single point $p \in T$, replace C by an R -circle centered at $p \in T \subset S$. Otherwise, if C contains more than one point, move C without uncovering any point of T , until two points $p, q \in T$, lie on the boundary of the moved circle C' . Clearly, the center c of C' lies at an intersection of the R -circles centered at p and q . Thus $c \in S$. Finally note that S contains at most $O(n^2)$ points and can be constructed in $O(n^2)$ time. $\square$

Corollary 5(a). *The finite feasible solution set problems QC_1 and QC_2 are D^P -complete under many-one reductions, respectively.*

Proof. The problems are in D^P since they can be rephrased as before, as the conjunction of a predicate in NP and a predicate in Co-NP.

To prove them complete we exhibit a polynomial-time many-one reduction from PC_1 to QC_1 and from PC_2 to QC_2 , similar to the proof of Theorem 5. $\square$

4. $OptP(O(\log n))$ and Metric Reductions

Following [9], we restate some definitions.

- (1) An NP metric Turing machine, N , is a nondeterministic polynomially time-bounded Turing machine such that every branch writes a binary number and accepts; and for $x \in \Sigma^*$, we write $opt^N(x)$ for the largest value on any branch of N on input of x .
- (2) A function $f: \Sigma^* \rightarrow N$ is in $OptP$ if there is an NP metric Turing machine N such that $f(x) = opt^N(x)$ for all $x \in \Sigma^*$. We say that f is in $OptP[z(n)]$ if $f \in OptP$ and the length of $f(x)$ in binary is bounded by $z(|x|)$ for all $x \in \Sigma^*$.
- (3) Let $f, g: \Sigma^* \rightarrow N$. A metric reduction from f to g is a pair of polynomial-time computable functions (T_1, T_2) where $T_1: \Sigma^* \rightarrow \Sigma^*$ and $T_2: \Sigma^* \times N \rightarrow N$ such that for all $x \in \Sigma^*$ we have $f(x) = T_2(x, g(T_1(x)))$.

Krentel has proved that there are natural complete problems (under metric reduction in $OptP[O(\log n)]$). In particular, he showed that the following problem is $OptP[O(\log n)]$ -complete:

- Maximum Satisfiability (MAX-SAT):
Instance: CNF formula φ .
Output: the maximum number of simultaneously satisfiable clauses.

In what follows we show

Theorem 6. PC_2 is $OptP[O(\log n)]$ complete.

Proof. We first show that MAX-3-SAT, which is defined below, is $OptP[O(\log n)]$ -complete.

- Maximum Satisfiability for 3-CNF (MAX-3-SAT):
Instance: CNF formula φ with three variables in each clause.
Output: the maximum number of simultaneously satisfiable clauses.

Lemma 6(a). MAX-3-SAT is $OptP[O(\log n)]$ -complete.

Proof. Use standard reduction given, say, in [7, pp. 48–49]. Then the minimum number of unsatisfiable clauses in the transformed 3-CNF formula equals the

minimum number of unsatisfiable clauses in the original CNF. From this, we can derive the maximum number of satisfiable clauses in the original CNF formula. This is a metric reduction from MAX-SAT to MAX-3-SAT. $\square$

We now reduce MAX-3-SAT to PC_2 via a metric reduction. Given φ , a 3-CNF formula with c clauses and n variables, in the construction of [6], replace each nonclause point by a “superpoint.” A superpoint contains $c + 1$ points which are clustered together in a small enough area. The purpose of the superpoints are to insist that all the nonclause points must be covered. Then it follows that the maximum number of clause points that can be covered is equivalent to the maximum number of clauses in φ that can be satisfied. We omit the remaining details and refer the reader to [6]. $\square$

5. The $(l_p, 1 \leq p \leq \infty)$ and Arbitrary Metrics

Between two points $p = (x_1, y_1)$ and $q = (x_2, y_2)$ in the plane the general l_p distance, for $1 \leq p \leq \infty$, is $[|x_1 - x_2|^p + |y_1 - y_2|^p]^{1/p}$; some of the more common distance metrics used are *Rectilinear* (l_1), *Euclidean* (l_2), and *Infinity* (l_∞). For the general l_p metrics the *unit disks* given by $|x|^p + |y|^p = 1$, correspond to planar geometrical figures of fixed orientation. For the *rectilinear* l_1 distance metric the geometric figures involved in our above optimization problems P 's and Q 's are equal-sized *diamonds* (squares rotated by 45°) of half-diagonal length R instead of R -circles. For the *infinity* l_∞ distance metric the optimization problems reduce to placement optimization problems of equal-sized squares of half-edge length R , having sides parallel to the respective coordinate axes. We note that, for our optimization problems, locating a diamond (square) anywhere in S or E^2 translates to locating the intersection point of the diagonals of the diamond (square) in the finite discrete set S or the Euclidean plane, respectively.

For arbitrary metrics we consider the following. A point set U is *symmetric* with respect to the origin if for each point (x, y) belonging to U , the point $(-x, -y)$ also belongs to U . A point set is *convex* if, for each pair of points in U , the line segment joining these points is entirely in U . If U is a *bounded* point set containing the origin in its interior and U is *convex* and *symmetric* with respect to the origin, then there exists a distance function d for which U is the *unit disk* [5]. For a *unit disk* U which is both *convex* and *symmetric*, define the distance from the origin o to z to be $d(oz) = oz/o\hat{z}$ where $\hat{z}$ is the intersection of the line oz with the boundary of U . The distance $d(oz)$ is *less* than 1 if z is an interior point of U , *equal* to 1 if z is a boundary point of U , and *greater* than 1 if z is an exterior point of U . In locating these unit disks in S or E^2 we specify a certain fixed orientation and again this translates to locating the origin (the point of symmetry) in S or E^2 , respectively. Note then that each of the earlier constructions in earlier theorems can be modified in a direct fashion for these arbitrary planar geometric figures with fixed orientations. Both the convexity and symmetric properties of these figures prove essential in the reduction of the infinite solution set problems P 's to the finite solution set problems Q 's, as in Theorem 5. The duality used in these theorems apply to all convex figures having a point of symmetry. Hence,

Theorem 7. *The above geometric optimization problems for the geometric figures corresponding to l_p and arbitrary metrics are also D^p -complete.*

6. Conclusion and Open Issues

We have shown a large number of natural geometric optimization problems concerning various types of geometric figures, to be *complete* for a class D^p . We also note that positive reductions are *adequate* to separate the class of D^p -complete languages from the classes of NP and Co-NP. These results also prove the existence of natural geometric optimization problems that are *proper* in $\Delta_2^p = P^{NP}$. Further, all of the above problems are *strongly D^p -complete* analogous to the similar concept for NP-complete languages, since all the above constructions hold even when the largest number occurring in any instance of the problems, that is parameter R and the coordinate points in set T , are restricted to be of size bounded by a polynomial in n .

There are a number of interesting open issues. First, is there a natural geometric optimization problem which is Δ_3^p *proper*? This has been a persistent open problem in the context of combinatorial optimization [11], [16]. A problem Q is *proper* for Δ_3^p if Q is in Δ_3^p and not in Σ_2^p or Π_2^p , assuming $\Sigma_2^p \neq \Pi_2^p$. Another open question is the precise complexity of the UNIQUE versions of the above geometric optimization problems [14]. It is not too difficult to note that the optimization solution of the above problems is not unique. Finally, the structure of the class D^p as well as the existence of similar classes higher up in the polynomial hierarchy, are yet to be fully explored.

Acknowledgments

We thank Mark Krentel whose suggestions strengthened this paper.

References

1. Aho, A., Hopcroft, J., and Ullman, J., *The Design and Analysis of Computer Algorithms*, Addison-Wesley, Reading, MA, 1974.
2. Bajaj, C., Limitations to Algorithmic Solvability: Galois Methods and Models of Computation, *Proc. ACM Symposium on Symbolic and Algebraic Computation*, SYMSAC 86, 1986, pp. 71-76.
3. Bajaj, C., Geometric Optimization and the Polynomial Hierarchy, *Theoretical Computer Science*, **54**, 1987, 87-102.
4. Bajaj, C., The Algebraic Degree of Geometric Optimization Problems, *Discrete and Computational Geometry*, **3**, 1988, 177-191.
5. Beckenbach, E., and Bellman, R., *An Introduction to Inequalities*, Random House, New York, 1961.
6. Fowler, R., Paterson, M., and Tanimoto, S., Optimal Packing and Covering in the Plane are NP-complete, *Information Processing Letters*, **12**, 1981, 133-137.
7. Garey, M., and Johnson, D., *Computers and Intractability: A Guide to NP-completeness*, Freeman, San Francisco, 1979.
8. Hadwiger, H., and Debrunner, H., *Combinatorial Geometry in the Plane* (translated by Klee, V.), Holt, Rhinehart & Winston, New York, 1964.

9. Krentel, M., The Complexity of Optimization Problems, *Proc. Eighteenth Symposium on Theory of Computing*, 1986, pp. 69–76.
10. Ladner, R., Lynch, N., and Selman, A., Comparison of Polynomial-Time Reducibilities, *Theoretical Computer Science*, **1**, 1975, 103–123.
11. Legget, E. W., and Moore, D. J., Optimization Problems and the Polynomial Hierarchy, *Theoretical Computer Science*, **15**, 1981, 279–289.
12. Megiddo, N., and Supowit, K. J., On the Complexity of Some Common Geometric Location Problems, *SIAM Journal of Computing*, **13**, 1984, 182–196.
13. Papadimitriou, C., Worst-case and Probabilistic Analysis of a Geometric Location Problem, *SIAM Journal of Computing*, **10**, 1981, 542–557.
14. Papadimitriou, C., On the Complexity of Unique Solutions, *Journal of the Association for Computing Machinery*, **31**, 1984, 392–400.
15. Papadimitriou, C., and Yannakakis, M., The Complexity of Facets (and Some Facets of Complexity), *Journal of Computer and System Sciences*, **28**, 1984, 244–259.
16. Selman, A. L., Analogues of semirecursive sets and effective reducibilities to the study of NP completeness, *Information and Control*, **52**, 1982, 36–51.

Received January 6, 1986, and in revised form August 25, 1987, and in final form March 4, 1988.