Densest Packing of Translates of the Union of Two Circles

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Abstract. Let u be the union of two unit circles whose centers have a distance at most 2. Motivated by more general problems it is proved that the density of a packing of translates of u never exceeds the density of the densest lattice-packing.

In the Euclidean plane let w be a domain. Let $d(w)$ be the density of the densest packing of translates of w . Let $\bar{d}(w)$ be the density of the densest lattice-packing of translates of w . It is known [1], [2], [3] that if w is convex then

$$d(w) = \bar{d}(w). \quad (1)$$

An interesting field of research arises by trying to extend (1) to more general domains [4]. From some wider families of domains, which presumably share property (1) with the convex domains, we emphasize

Conjecture 1. *Let u be the union of two convex domains having a point in common. Then $d(u) = \bar{d}(u)$.*

As a modest step in this direction we shall prove the following:

Theorem. *Let u be the union of two unit circles centered at a distance at most 2 from one another. Then $d(u) = \bar{d}(u)$.*

Let u_1 and u_2 be translates of u such that u and u_1 have three points of contact, and u_2 has one point of contact with both u and u_1 . The densest lattice-packing of translates of u is generated by the translations $u \rightarrow u_1$ and $u \rightarrow u_2$. Let $2t$ be the distance of the centers of the two unit circles contained in u . Then $\bar{d}(u)$ is given in terms of t by the function

$$f(t) = \frac{2(\arcsin t + t\sqrt{1-t^2}) + \pi}{2t\sqrt{4-t^2} + \sqrt{12}}.$$

For $0 < t < 1$ we have $f(t) > f(0) = f(1) = \pi/\sqrt{12}$. The maximum of $f(t)$ is attained at a value $t_0 \approx 0.582$. Since translates of the union of two disjoint circles cannot be packed with a density greater than $\pi/\sqrt{12}$ the theorem implies that the density of a packing of translates of the union of two equal circles is at most $f(t_0) = 0.936599 \dots$

What can be said about the supremum of $d(u)$ extended over the set S_n of all unions u of n unit circles? We shall return to this question after the proof of the theorem.

The proof of the theorem is based on an idea used in a proof of Thue's well-known theorem which claims that for a circle c we have $d(c) = \bar{d}(c) = \pi/\sqrt{12}$. First we will reproduce this proof [5] in a slightly modified form. We shall denote the area of a domain w by $|w|$. By a circle we shall mean an open circular disc.

Let the unit circles $c_1, c_2, \dots$ with centers $C_1, C_2, \dots$ form a packing. Let z_i be the Dirichlet cell of c_i defined as the set of those points which are nearer to C_i than to the center of any other circle. We shall show that the density $|c_i|/|z_i|$ of any circle c_i in its Dirichlet cell is at most $\pi/\sqrt{12}$. Writing $c_i = c$ and $z_i = z$, this means that $|z| \geq |h|$ where h is a regular hexagon circumscribed about c .

We shall prove a sharper inequality: If k is the circumcircle of h then

$$|z \cap k| \geq |h|.$$

Let $s_1, \dots, s_n$ be the segments cut off from k by the sides of z . Let $F_1, \dots, F_n$ be the orthogonal projections of C to the respective sides of z . Since the distance between any two of the C_i 's is at least 2, the distance between the F_i 's is at least 1. It follows that the s_i 's do not overlap. Therefore

$$|z \cap k| = |k| - |s_1| - \dots - |s_n| \geq |k| - n|s|,$$

where s is a segment cut off from k by a tangent of c . For $n \leq 6$ this implies the desired inequality

$$|z \cap k| \geq |k| - 6|s| = |h|.$$

The case when $n = 7$ can be settled by a very rough estimate which automatically rules out the possibility that $n > 7$. Since the length

$$\frac{4}{\sqrt{3}} \sin \frac{\pi}{7} = 1.00201 \dots$$

of a side of a regular heptagon inscribed into k is close to 1, it is clear that the points $F_1, \dots, F_7$ must lie close to the boundary $\text{bd } k$ of k , suggesting that the s_i 's are very small. Though this is sufficiently convincing, we present the details.

The distance CF_7 attains its minimum $m = 1.1383 \dots$ if $CF_1 = \dots = CF_6 = 2/\sqrt{3} = 1.1547 \dots$, and all sides of the heptagon $F_1 \dots F_7$ are equal to 1. It follows that

$$|s_i| \leq \frac{1}{2} \left(\frac{2}{\sqrt{3}} \right)^2 (2\omega - \sin 2\omega) = 0.00424 \dots, \quad i = 1, \dots, 7,$$

where $\omega = \arccos \sqrt{3} m/w$. Hence

$$|s_1| + \dots + |s_7| < 0.0297 < 6|s| = 0.724 \dots$$

Turning to the proof of the theorem, we consider a rectangular coordinate system (x, y) in which the centers of the two unit circles contained in u coincide with $(-t, 0)$ and $(t, 0)$. Let c^- and c^+ be the left- and right-sided circles, and C^- and C^+ their centers. Let the translates $u_1, u_2, \dots$ of u form a packing. Let $c_1^-, c_1^+, c_2^-, c_2^+, \dots$ be the set of pertaining circles. In this set let z_i^- and z_i^+ be the Dirichlet cells of c_i^- and c_i^+ . We define the Dirichlet cell of u_i by $z_i = z_i^- \cup z_i^+$. In order to simplify the notations we suppose that u is an arbitrary member of the packing, and denote its Dirichlet cell by z .

We shall prove the theorem by showing that

$$|u|/|z| \leq \bar{d}(u).$$

We shall prove a sharper inequality: If k^- and k^+ are circles concentric with c^- and c^+ of radius $2/\sqrt{3}$ and $k = k^- \cup k^+$ then

$$|z \cap k| \geq |u|/\bar{d}(u) = |\bar{z}|, \quad (2)$$

where $\bar{z}$ is a Dirichlet cell in the densest lattice-packing.

Let $u_1, \dots, u_n$ be those domains other than u whose Dirichlet cells intersect k . Let the respective intersections be $s_1, \dots, s_n$. Since the Dirichlet cells do not overlap neither do the s_i 's. Therefore

$$|z \cap k| = |k| - |s_1| - \dots - |s_n|. \quad (3)$$

We claim that among $s_1, \dots, s_n$ there are at most two whose areas are greater than $|s|$. We shall prove this by showing that, with the exception of at most two, the sets $s_1, \dots, s_n$ are circular segments.

Let u_0 be a translate of u not necessarily disjoint of u . Let C_0^- and C_0^+ be the corresponding translates of C^- and C^+ . In the arrangement of translates of u , consisting only of u and u_0 , let z_0 be the Dirichlet cell of u_0 . We say that the intersection $s_0 = k \cap z_0$ is exceptional if it is not empty and not a circular segment. Let p be the set of points defined by the following property: If either $C^- \in p$ or $C^+ \in p$ then s_0 is exceptional.

Let a and b be circles of radius $2/\sqrt{3}$ centered at the points of intersection of $bd\ k^-$ and $bd\ k^+$ (Fig. 1). The difference $(2k^- \cap 2k^+) \setminus (a \cup b)$ consists of four arc-sided triangles. Let v and w be the triangles bisected by the y -axis. We claim that $p = a \cup b \cup v \cup w$. This can easily be seen by the following alternative definition of p : The set p consists of the centers of those circles of radius $2/\sqrt{3}$ whose boundaries intersect both arcs constituting $bd\ k$.

Presently we are only interested in the part of p which lies in the ring r defined by $r = (2k^- \cup 2k^+) \setminus (2c^- \cup 2c^+)$. This part consists of two disjoint components whose diameter is less than or equal to 2 with equality only in the limiting case as $t \rightarrow 0$. Since the distance between centers C_i^- or C_i^+ belonging to different u_i 's is at least 2, the assertion is proved.

Fig. 1

Now we make use of the easily seen fact that under the sole condition that u and u_0 are disjoint $|s_0|$ attains its maximum if u_0 and u have three points of contact. Denoting s_0 in this position by $\hat{s}$, and assuming that $n \leq 6$, we have, by (3), in accordance with (2),

$$|z \cap k| \geq |k| - 4|s| - 2|\hat{s}| = |\bar{z}|.$$

Now we consider seven pairs of centers $(C_1^-, C_1^+), \dots, (C_7^-, C_7^+)$ such that of each pair at least one center is in r . Let (C_i^-, C_i^+) be a pair of centers such that C_i^- lies in the half-plane $x \leq 0$ and C_i^+ lies in the half-plane $x \geq 0$. We assume that among the seven pairs of centers there are (i) two such pairs, or (ii) one such pair, or (iii) none.

Case (i). Let $C_7^+, C_1^-, C_2^-, C_3^+$ be in $x \geq 0$, and let $C_3^-, C_4^+, C_5^+, C_6^+, C_7^-$ be in $x \leq 0$. Translating C_4^+, C_5^+ , and C_6^+ through the vector C^-C^+ we obtain, along with C_7^+, C_1^-, C_2^- , and C_3^+ seven points in the ring $2k^+ \setminus 2c^+$ with a distance at least 2 from one another. Referring to the above proof of Thue's theorem we see that these points must be close to $\text{bd } 2k^+$. Therefore the original points must be close to $\text{bd } (2k^- \cup 2k^+)$ so that $|z \cap k|$ is, by far, greater than $|\bar{z}|$.

Case (ii). We can translate the domains $u_1, \dots, u_7$ continuously "outwards" without overlapping each other until, of each pair of centers, one center lies on $\text{bd}(2k^- \cup 2k^+)$, and the other lies either on $\text{bd}(2k^- \cup 2k^+)$ or outside $2k^- \cup 2k^+$. We number the centers so that $C_1^-, C_2^-, C_3^-, C_4^+, C_5^+, C_6^+, C_7^-$, and C_7^+ lie, in

Fig. 2

this cyclic order, on $\text{bd}(2k^- \cup 2k^+)$ (Fig. 2). Translate C_5^+ through the vector C^-C^+ and C_2^- through C^+C^- obtaining the points U and V . Let $\text{bd } 2k^-$ and $\text{bd } 2k^+$ intersect at the points F and G , G being in the same half-plane $y > 0$ or $y < 0$ as C_7^- and C_7^+ . Let Σ and σ be the length of the arcs on $\text{bd } 2k^+$ subtended by the chords of length 2 and $2t$, respectively. Using an obvious notation for the length of an arc on $\text{bd } 2k^-$ or $\text{bd } 2k^+$, we have

$$\begin{aligned} 8\pi/\sqrt{3} + 2\sigma &= \widehat{C_7^-G} + \widehat{GC_7^+} + \widehat{C_7^+C_2^-} + \widehat{C_2^-F} + \widehat{FC_5^+} + \widehat{C_5^+C_7^-} \\ &= \sigma + \widehat{C_7^+C_2^-} + \widehat{VF} + \sigma + \widehat{FU} + \sigma + \widehat{C_5^+C_7^-} \geq 4\Sigma + 3\sigma + \widehat{UF} + \widehat{FV}, \end{aligned}$$

whence

$$\Lambda = \widehat{UF} + \widehat{FV} + \sigma + 4\Sigma - 8\pi/\sqrt{3} \leq 0.$$

For a given value of t , $\widehat{UF} + \widehat{FV}$ attains its minimum if, in the inequalities $C_4^+C_3^- \geq 2$, $C_4^+U = C_4^-C_5^+ \geq 2$, $C_3^-V = C_3^+C_2^- \geq 2$, equality holds. Geometrical considerations suggest, and numerical computations, support the conjecture that in the extremal position either C_4^+ or C_3^- coincides with F . From the computed values of Λ belonging to different values of t and different positions of C_4^+ and C_3^- we present, for a few values of t , only the smallest one which belong to the

suggested position:

t	0	0.065	0.066	0.1	0.2	0.3	0.4
Λ	-0.0313	-0.0002	0.0004	0.0156	0.0572	0.0944	0.1280
t	0.5	0.6	0.7	0.8	0.9	1	
Λ	0.1588	0.1954	0.2140	0.2390	0.2626	0.2852	

This table shows that there is a value $\bar{t} = 0.065 \dots$ such that for $t > \bar{t}$ case (ii) cannot occur.

The same argument shows that case (ii) (for $t \leq \bar{t}$) cannot occur either if, from all seven pairs of centers, at least one center lies in $\lambda c^- \cup \lambda c^+$ where $\lambda = 1/\sin(\pi/7)$ is the circumradius of a regular heptagon of side-length 2. But if for some $j \leq 7$ both c_j^- and c_j^+ are outside $\lambda c^- \cup \lambda c^+$ then $|s_j|$ is very small, more exactly less than $3|w| = 0.00067 \dots$ where w is a circular segment cut off from k^+ by a tangent of $(\lambda/2)c^+$. On the other hand, u_j prevents the rest of the centers under consideration from getting close to $\text{bd } 2k^- \cup 2k^+$, so that inequality (2) is again amply fulfilled.

Case (iii) can be settled in a similar manner to (ii). Without going into details we mention that there is a constant $\bar{t} = 0.034 \dots$ such that for $t > \bar{t}$ case (iii) cannot occur.

This ends the proof of the theorem.

We still make some remarks about $d_n = \sup_{u \in S_n} d(u)$, where S_n is the set of all possible unions of n unit circles.

Let u be the union of n unit circles with centers on a line equally spaced at a distance $2t \leq 2$. Then $\bar{d}(u)$ is given in terms of t by the function

$$f_n(t) = \frac{2(n-1)(\arcsin t + t\sqrt{1-t^2}) + \pi}{2(n-1)t\sqrt{4-t^2} + \sqrt{12}}.$$

As we have seen, we have $d_2 = \max_{0 \leq t \leq 1} f_2(t)$. Now we can phrase

Conjecture 2. For $n = 3, 4$, and 5 we have $d_n = \max_{0 \leq t \leq 1} f_n(t)$.

However, passing from $n = 5$ to $n = 6$ the extremal configuration seems to change drastically. Let v be the union of six unit circles centered at the vertices of a regular hexagon of side-length 1. Then we have

$$\bar{d}(v) = \frac{3\sqrt{3} + 2\pi}{(\sqrt{3}/2)(7 + 3\sqrt{5})} = 0.96695 \dots,$$

whilst $\max_{0 \leq t \leq 1} f_6(t) = 0.96686 \dots$. This phenomenon is similar to the "sausage catastrophe" observed by Wills [6].

Improving a construction suggested by J. Pach we now shall show that

$$\lim_{n \rightarrow \infty} (1 - d_n) n \leq \frac{4\sqrt{3} - 2\pi}{3\sqrt{3}}. \quad (4)$$

Let $k > 1$ and $l \geq 0$ be integers. In the thinnest lattice-covering of the plane with unit circles the centers of the circles form a lattice generated by the vertices of a regular triangle of side-length $\sqrt{3}$. Let H be a closed regular hexagon of side-length $\sqrt{3} k$ containing $3k(k+1)+1$ lattice-points. On $\text{bd } H$ we put, on each segment determined by consecutive lattice-points l , new points so as to divide the segment into $l+1$ parts of length $2t = \sqrt{3}/(l+1)$. Now we have in H altogether

$$n = 1 + 3k(k+1) + 6kl$$

points. Let U be the union of the unit circles centered at these points. We have

$$\begin{aligned} |U| &= |\text{conv } U| - |\text{conv } U \setminus U| \\ &= \frac{9\sqrt{3}}{2} k^2 + 6\sqrt{3} k + \pi - 6k(l+1)(2t - \arcsin t - t\sqrt{1-t^2}), \end{aligned}$$

where $\text{conv } U$ denotes the convex hull of U .

In the densest lattice-packing of translates of U each domain is touched by six others so that adjacent domains have $2kl-1$ points of contact. A unit cell C of this lattice is given by joining to H along three consecutive sides parallelograms of side-length 2 and $\sqrt{3} k$ including an angle $\arcsin(\frac{1}{2}\sqrt{4-t^2})$ and two equilateral triangles of side-length 2 between the parallelograms. Thus we have

$$|C| = \frac{9\sqrt{3}}{2} k^2 + 3\sqrt{3} k\sqrt{4-t^2} + 2\sqrt{3}$$

and

$$|C| - |U| = 3\sqrt{3} k \left(\sqrt{4-t^2} - \frac{1}{t} \arcsin t - \sqrt{1-t^2} \right) + 2\sqrt{3} - \pi.$$

Since

$$\lim_{t \rightarrow 0} \frac{1}{t^2} \left(\sqrt{4-t^2} - \frac{1}{t} \arcsin t - \sqrt{1-t^2} \right) = \frac{1}{12}$$

there is a positive constant c such that for $0 < t \leq \sqrt{3}/2$ we have

$$\sqrt{4-t^2} - \frac{1}{t} \arcsin t - \sqrt{1-t^2} < ct^2.$$

It follows that

$$1 - d_n \leq \frac{|C| - |U|}{|C|} < \frac{3\sqrt{3} \, ckt^2 + 2\sqrt{3} - \pi}{\frac{9\sqrt{3}}{2} k^2}.$$

Let p be a positive number less than $1/8$. Letting n tend to infinity and choosing l so that $n^p < l \leq m^{1/2-p}$ we have $\lim_{n \rightarrow \infty} n/k^2 = 3$ and $\lim_{n \rightarrow \infty} k/l^2 = 0$. Therefore multiplying the last inequality by n and going over to the limiting value as $n \rightarrow \infty$ we obtain inequality (4).

It is conjectured that in (4) the sign of equality holds. We formulate the following sharper

Conjecture 3. *Let $D(u)$ be the density of the densest packing of congruent copies of a domain u . Then*

$$\sup_{u \in S_n} D(u) \sim 1 - \frac{4\sqrt{3} - 2\pi}{3\sqrt{3} \, n}.$$

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