CORRECTION AND REMARK ON A FORMER PAPER „A METHOD FOR FINDING PRECISE ERROR BOUNDS OF NUMERICAL INTEGRATION FORMULAS IN HIGHER DIMENSIONS”

By

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(Presented by P. TURÁN)

The recent work of I. M. SOBOL concerning precise estimation of errors of multidimensional quadrature formulas for function classes $\tilde{W}_1$ and $\tilde{H}_1$ (see *Journal of Computational Mathematics and Mathematical Physics* (SSSR), **1**(2) (1961), pp. 208—216) calls my attention to an error in my former paper mentioned in the title (see *Acta Math. Acad. Sci. Hung.* **11** (1960), pp. 163—171).

The boundedness conditions $|f^{(m,n)}(x, y)| \leq M$, $|f^{(\mu,n)}(x, y)| \leq M_\mu$, $|f^{(m,v)}(x, y)| \leq N_v$ ($\mu=0, 1, \dots, m-1$; $v=0, 1, \dots, n-1$) contained in (1) of my paper (loc. cit., p. 164) should be replaced by the following

$$(1)^* \quad |f^{(m,n)}(x, y)| \leq M, \quad |f^{(\mu,n)}(a, y)| \leq M_\mu, \quad |f^{(m,v)}(x, b)| \leq N_v.$$

Accordingly we get a function class $\tilde{W}^{(m,n)}(M, M_\mu, N_v; Q)$ which is larger than the original class $W^{(m,n)}(M, M_\mu, N_v; Q)$ considered in former paper. Employing the original constructive proof of Lemma 2 and analyzing in the same way as that of SOBOL, we easily find that Lemma 2 is valid for the function class $\tilde{W}^{(m,n)}$ but not generally true for the smaller class $W^{(m,n)}$ (as may easily be observed by estimating the both sides of (11)), in which the functions $F(x, y)$, $G_v(x)$, $H_\mu(y)$ are defined by (19) (see p. 168). Consequently, our main theorem is true with respect to the class $\tilde{W}^{(m,n)}$. Actually the error bound $\varepsilon(L)$ given by (20) is attainable by a function of $\tilde{W}^{(m,n)}$ (see loc. cit., pp. 168—169).

Evidently it is not difficult to generalize our main result to the case where the number of variables is greater than 2. Moreover, a remark should be added here. Both the results of SOBOL and that of mine are, in fact, closely related to those of I. A. EZROHI (see *Mat. Sbornik* (SSSR), **38** (80) (1956), pp. 389—416) and of A. SARD (see *Numerical Properties of Functions of More Than One Independent Variable*, *New York Acad. Sci.*, **86** (3) (1960), pp. 700—757).

Finally, let us sketch how to evaluate $F(t_1, t_2)$, $G_v(t)$, $H_\mu(t)$ of (19) by means of line integrals. Let C be the boundary of D , and let $C(x \geq t)$ denote the part of C which lies on the right-hand side of the line $x=t$ on the (x, y) -plane. Similarly, $C(x \geq t_1, y \geq t_2)$ denotes the part of C lying on the right-upper side of the lines $x=t_1$ and $y=t_2$. Then, recalling

$$K_r(u) = \begin{cases} \frac{1}{(r-1)!} u^{r-1} & \text{for } u \geq 0, \\ 0 & \text{for } u < 0, \end{cases}$$

and noticing $K_r(u)du = K_{r+1}(u) + \text{constant}$, we easily make use of GREEN's theorem to obtain (cf. (19), loc. cit.)

$$\left\{ \begin{aligned} F(t_1, t_2) &= - \int_{C(x \geq t_1, y \geq t_2)} \frac{(x-t_1)^{m-1} \cdot (y-t_2)^n}{(m-1)! n!} dx - L(K_m(x-t_1) K_n(y-t_2)), \\ G_v(t) &= - \int_{C(x \geq t)} \frac{(x-t)^{m-1} \cdot (y-b)^{v+1}}{(m-1)! (v+1)!} dx - L\left(K_m(x-t) \frac{(y-b)^v}{v!}\right), \\ H_\mu(t) &= - \int_{C(y \geq t)} \frac{(x-a)^\mu \cdot (y-t)^n}{\mu! n!} dx - L\left(K_n(y-t) \frac{(x-a)^\mu}{\mu!}\right). \end{aligned} \right.$$

Clearly, these line integrals involved can be directly evaluated in case C is a polygonal line. In the general case we may approximate the curve C by an inscribed polygon so that $F(t_1, t_2)$, $G_v(t)$, $H_\mu(t)$ can be approximately calculated. This final remark may be useful for the numerical evaluation of the precise error bound given by our main result.

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