Extension of Some Results on Forced Nonlinear Oscillations (*) (**).

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A GAETANO VILLARI, mio padre, per i suoi sessanta anni

Sunto. — Si considera l'equazione

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = e(t) = e(t + T).$$

Si danno due teoremi per l'esistenza di soluzioni di periodo T basandosi su una applicazione del teorema di punto fisso di Brouwer.

Introduction.

In the present paper we consider generalized Liénard's differential equation

$$(1) \quad \ddot{x} + f(x, \dot{x})\dot{x} + g(x) = e(t) = e(t + T)$$

We give sufficient conditions for the existence of T -periodic solutions.

Results of this kind have been obtained by N. LEVINSON [1], C. LANGENHOP [2], Z. OPIAL [3], R. REISSIG [4] and GAETANO VILLARI [5].

The technique used in the demonstration is analogous to that used in the papers mentioned above, and is based on an application of Brouwer's fixed point theorem.

When equation (1) is written out in the form of the system

$$(2) \quad \begin{cases} \dot{x} = y \\ \dot{y} = -f(x, y)y - g(x) + e(t) \end{cases}$$

in the plane $t = \text{constant}$ we may locate a finite simply connected region V , defined by one or more simple line arcs. These lines are for the most part trajectories of autonomous systems or projections on the plane $t = \text{constant}$ of solutions of non-autonomous systems which, for convenience, we shall also continue to refer to as trajectories.

Let ω be the transformation which associates the projection on the plane $t = \text{constant}$ of the point obtained by following for time T the integral curve of

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system (2) which passes through P , with the generic point P of the plane $t = \text{constant}$. One may prove that the region V has the property of being transformed by ω into one of its parts.

The Brouwer theorem, therefore, assumes the existence of a fixed point for the transformation ω , and hence, for the periodicity of $e(t)$, one has the existence of a periodic solution for system (2) and thus for equation (1).

In this paper we give two theorems for the existence of periodic solutions based on two different ways of defining a suitable region V . The type of construction used in Theorem 2 may also be applied relatively to the autonomous equation

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0$$

obtaining additionally in this case a theorem for the existence of periodic solutions.

If in equation (1) we consider the function f dependent only on x , we obtain

$$(3) \quad \ddot{x} + f(x)\dot{x} + g(x) = e(t) = e(t + T).$$

This type of equation has been the object of study in the last few years, because in this case one may use a different technique.

J. MAWHIN [6] generalizing a result by C. LAZER [7], has demonstrated the existence of a periodic solution for equation (3) with only the hypothesis of continuity relative to $f(x)$. The demonstration is based on Schauder's fixed point theorem. This result has been expanded on by R. REISSIG [8], [9], M. MARTELLI [10] and many other authors.

It does not, however, seem possible, as Mawhin and Martelli assert in the above-mentioned papers, to extend these results to equation (1) without adding further hypotheses on $f(x, \dot{x})$ besides that on continuity; this does not seem convenient considering that the hypothesis on $g(x)$ are heavier with respect to the works already considered.

In the final part of the paper, we show that the results obtained are more general than their predecessors, and compare the differences between the hypotheses relevant to a study of equation (3).

1. - THEOREM 1. - Let $f: \mathbf{R}^2 \rightarrow \mathbf{R}$, $y: \mathbf{R} \rightarrow \mathbf{R}$ and $e: \mathbf{R} \rightarrow \mathbf{R}$ be such that:

- (1) f and g are continuous and locally Lipschitz continuous functions;
- (2) $e(t)$ is continuous and periodic, $e(t) = e(t + T) \quad \forall t, e \leq e(t) \leq E$;
- (3) there exist $a, b, M > 0$ with $b > a$ such that

$$\begin{aligned} f(x, y) > -M & \quad \text{if } |x| < a, \quad |y| < +\infty \\ & \quad x > b, \quad y < -a \\ & \quad x < -a, \quad y > x \end{aligned}$$

$$(4) \quad \begin{aligned} f(x, y)y + g(x) &\geq E && \text{if } x > a \quad y \geq 0 \\ f(x, y)y + g(x) &\leq e && \text{if } x < -a \quad y \leq 0 \end{aligned}$$

$$(5) \quad \begin{aligned} f(x, y) + f(x, z) &\geq k > 0 && \text{if } x > b, \quad y \geq z > a \\ &&& x < -a, \quad y \leq z < -a \end{aligned}$$

(6) there exists $\Phi(x)$ such that

$$f(x, y) \geq \Phi(x) \quad \text{if } a \leq x \leq b \quad y < -a_1 \leq -a$$

and

$$\int_a^b \Phi(x) dx > 4aN + 1$$

where $N = M + H$, $H > 0$ is such that

$$g(x) - e(t) \geq -H \quad \text{if } |x| < a \quad 0 \leq t \leq T$$

and $a_1 > a$ is such that

$$0 < \frac{E - e}{a_1} \leq \frac{k}{3}.$$

Then there exists a T -periodic solution of equation (1).

PROOF. — We write the equation

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = e(t)$$

in the form of the system

$$(1.1) \quad \begin{cases} \dot{x} = y \\ \dot{y} = -f(x, y)y - g(x) + e(t). \end{cases}$$

We assume $E > e$; if indeed $E = e$, then $e(t)$ is a constant, system (1.1) is autonomous and hypotheses (4) guarantee the existence of a stationary point and thus of a periodic solution.

In the plane-region bounded by $x \geq b$, $y \geq 0$ we consider the autonomous system

$$(1.2) \quad \begin{cases} \dot{x} = y \\ \dot{y} = -f(x, y)y - g(x) + E \end{cases}$$

It follows from assumption (4) that for such a system it is

$$\dot{y} \leq 0.$$

If we assume, as in [5], $\dot{y} \leq -k/3 < 0$, considering a point $P_0(x_0, y_0)$, having coordinates x_0 and y_0 with $x_0 \geq b$ and $y_0 > 0$, the positive half-trajectory of system (1.2) which passes through the point P_0 crosses in a finite time the x -axis at a point $P_1(x_1, 0)$ and it is $x_1 - x_0 \leq \frac{3}{2}(y_0^2/k)$.

Now we consider points such that for system (1.2)

$$(1.3) \quad -\frac{k}{3} \leq \dot{y} \leq 0$$

and assume, for convenience, that (1.3) holds for every point of the region bounded by $x \geq b$, $y \geq 0$.

Let $E - e > 0$ and let

$$\delta = \frac{E - e}{4}.$$

We consider the function $e(t)$ in the interval $[0, T]$. Let t_0 be a point of the open interval $(0, T)$ such that $e(t_0) = e + 2\delta$. Recalling the uniform continuity of $e(t)$ which is due to periodicity one may choose $h > 0$ such that from $|t - t_0| \leq h$ it follows that $|e(t) - e(t_0)| < \delta$.

Divide the interval $[0, T]$ into two sets T_1 and T_2 :

$$[0, T] = T_1 \cup T_2$$

$$T_1 = (t_0 - h, t_0 + h)$$

$$T_2 = [0, t_0 - h] \cup [t_0 + h, T].$$

Let $\varphi: \mathbf{R} \rightarrow \mathbf{R}$, $\Psi: \mathbf{R} \rightarrow \mathbf{R}$ and $v: \mathbf{R} \rightarrow \mathbf{R}$ be such that φ , Ψ and v are T -periodic functions and

$$\varphi(t) = \begin{cases} E - \delta & \text{if } t \in T_1 \\ E & \text{if } t \in T_2 \end{cases}$$

$$\Psi(t) = \begin{cases} e + \delta & \text{if } t \in T_1 \\ e & \text{if } t \in T_2 \end{cases}$$

$$v(t) = \begin{cases} -\delta & \text{if } t \in T_1 \\ 0 & \text{if } t \in T_2 \end{cases}$$

We have $\Psi(t) < \varphi(t)$

$$\Psi(t) \leq e(t) \leq \varphi(t) \quad \forall t$$

One may choose $a_1 > a$ such that

$$(1.4) \quad 0 < \frac{E - e}{a_1} \leq \frac{k}{3}.$$

Now we may consider the system

$$(1.5) \quad \begin{cases} \dot{x} = y & x \geq b \\ \dot{y} = v(t) & y > 0 \end{cases}$$

LEMMA 1. — *When the time $t_0 - h$ and the point $P_0(x_0, y_0)$ are fixed, the positive half-trajectory of system (1.5) which passes through the point P_0 at the time $t_0 - h$, crosses in a finite time the x -axis at a point $P_1(x_1, 0)$ and it is possible to give a bound for the distance $x_1 - x_0$ which does not depend on x_0 .*

The proof of this Lemma and the proofs of Lemma 2 and Lemma 3 which follow are in the Appendix at the end of this paper.

REMARK 1.1. — We consider the system

$$(1.6) \quad \begin{cases} \dot{x} = y & x \geq b \\ \dot{y} = -f(x, y)y - g(x) + \varphi(t) & y \geq 0 \end{cases}$$

From a comparison with the system (1.5) it follows that the positive half-trajectory of system (1.6) which passes through the point P_0 at the time $t_0 - h$ has an upper boundary which is the trajectory of system (1.5) which passes through the point P_0 at the same time. So the considered trajectory of system (1.6) crosses in a finite time the x -axis at a point $\hat{P}(\hat{x}, 0)$. Clearly $\hat{x} - x_0 \leq x_1 - x_0$.

We consider the point $B(x_B, a_1)$, with $x_B > x_0$.

Based on Lemma 1 one may see that the positive half-trajectory of system (1.5) which passes through the point B at the time $t_0 - h$, meets the point $C(x_C, a)$, with $x_C > x_B$, and then crosses at the time $\hat{t}$ the x -axis at a point $D(x_D, 0)$ with $x_D > x_C$.

In the half-plane $y \leq 0$ we consider the autonomous system

$$(1.7) \quad \begin{cases} \dot{x} = y \\ \dot{y} = -f(x, y)y - g(x) + e \end{cases}$$

Recalling that $f(x, y)$ has a lower bound in the plane region bounded by $x > a$ and $y < -a$, it is easy to prove that the positive half-trajectory of system (1.7) which passes through the point $F(x_D, -a_1)$ crosses in a finite time the axis $x = x_0$ in $y < 0$.

Fig. 1.

Let $\hat{y}(x)$ be the symmetric line, with respect to the x -axis, of a trajectory of system (1.7).

Such a line satisfies the equation

$$(1.8) \quad \hat{y}' = f(x_1, -\hat{y}) - \frac{g(x) - e}{\hat{y}}$$

which comes from

$$y' = -f(x, y) - \frac{g(x) - e}{y}$$

a consequence of system (1.7), where one makes the substitution

$$\hat{y} = -y.$$

We consider the point $S_0(x_D, a_1)$, which is the symmetric point of F .

The symmetric line of the trajectory of system (1.7) which passes through F , passes through D_0 and meets the point $S_1(x_C, y)$ and then the point $S_2(x_B, y)$.

LEMMA 2. — *The length of the segment S_2B has an upper bound which does not depend on x_0 .*

A consequence of Lemma 2 is the following

LEMMA 3. — *One may choose $y_0 > a_1$ such that*

- 1) *The positive half-trajectory of system (1.5) which passes through the point $P_0(b, y_0)$ at the time $\hat{t}$ crosses in a finite time the x -axis at a point $\hat{P}(\hat{x}, 0)$.*
- 2) *The positive half-trajectory of system (1.7) which passes through the point $\hat{Q}(\hat{x}, a_1)$ crosses in a finite time the axis $x = b$ at a point $Q_0(b, -y_Q)$ and*

$$0 < y_Q < y_0$$

- 3) *These assumptions hold for every $y > y_0$, and the choice of the time $\hat{t}$ has no influence.*

REMARK 1.2. — The result of Lemma 3 still holds if we consider the axis $x = -a$ instead of the axis $x = b$, the system

$$\begin{aligned} \dot{x} &= y & x &\leq -a \\ \dot{y} &= -f(x, y)y - g(x) + \Psi(t) & y &< 0 \end{aligned}$$

instead of the system (1.6) and the system

$$(1.9) \quad \begin{cases} \dot{x} = y & x \leq -a \\ \dot{y} = -v(t) & y < 0 \end{cases}$$

instead of the system (1.5).

We assume that for every point of the region bounded by $x \leq -a$, $y < 0$ it is

$$0 \leq \dot{y} \leq \frac{k}{3}.$$

Such an assumption is made for convenience and is analogous with (1.3).

In the half-plane $y \geq 0$ we consider the autonomous system (1.2) instead of the system (1.7).

Clearly we consider the symmetric lines, with respect to the x -axis, of the trajectories of system (1.2).

The result still holds for every $y < -y_0$ with $-y_0 < -a_1$.

Again, the choice of time has no influence.

It is possible to choose a constant $H > 0$ such that

$$g(x) - e(t) \geq -H \quad \text{if } |x| < a \\ 0 \leq t \leq T.$$

Let $N = M + H$.

It is trivial to see that if the positive half-trajectory of system (1.1) which passes through a point $R(-a, \bar{y})$ of the axis $x = -a$, with $\bar{y} > 1$, at the time t , crosses in a finite time the axis $x = a$ it does it at a point $\hat{R}(a, y)$ with $y < \bar{y} + 2aN$. Such a part of the trajectory is bounded on the upper side by the segment whose extremes are R and $\bar{R}(a, \bar{y} + 2aN)$.

The same result holds if we consider a trajectory of system (1.7) and the points $R_1(a, -\bar{y})$ and $\bar{R}_1(-a, -(\bar{y} + 2aN))$.

Based on hypothesis (6) one may choose $Z > 0$ such that

$$(1.10) \quad -\int_a^b \Phi(x) dx + \frac{G}{Z}(b-a) < -4aN$$

where

$$G = \max \{g(x) - e, a \leq x \leq b\}$$

Let

$$\hat{\Phi} = \max \{\Phi(x), \quad a \leq x \leq b\},$$

one may choose $\hat{Z}$ such that

$$(1.11) \quad \hat{Z} > Z + (b-a) \left[\hat{\Phi} + \frac{G}{Z} \right].$$

Based on (1.10), (1.11) and hypothesis (6) one can see that the positive half-trajectory of system (1.7) which passes through a point $P_1(b, y)$ of the axis $x = b$, with $y \leq -\hat{Z}$, crosses in a finite time the axis $x = a$ at a point $P_2(a, y)$, with y such that

$$(1.12) \quad -(\hat{Z} - 4N) < y < -Z.$$

Finally we may locate in the phase-plane a suitable region V for which it is possible to apply the method which is described in the introduction.

With reference to the figure, one may choose $\hat{Z}$ such that it satisfies the assumption (1.11), where the value Z satisfies the assumption (1.10) and is greater than $y_0 > a_1$ with y_0 as in Lemma 3. Here we assume that y_0 satisfies Lemma 3 also for the axis

Fig. 2.

$x = -a$, which is clearly not restrictive on a possible enlargement of the above choice of y_0 .

The point P_1 has coordinates $x = b$ and $y = -\hat{Z}$. The point P_2 has coordinates $x = a$ and $y = -\bar{Z}$.

Based on (1.12) it is $-(\hat{Z} - 4aN) < -\bar{Z} < -Z < -y_0$.

The point P_3 has coordinates $x = -a$ and $y = -\bar{Z} - 2aN$, P_4 and P_5 have the same abscissa. P_6 has coordinates $x = a$ and $y < \bar{Z} + 2aN$. P_7 has coordinates $x = -a, y = \bar{Z} + 2aN$. P_8 has coordinates $x = a$ and $y = \bar{Z} + 4aN$. P_0 has coordinates $x = b$ and $y = \bar{Z} + 4aN$. $\hat{P}$ and $\hat{Q}$ have the same abscissa.

At last, Q_0 has coordinates $x = b$ and $y > -\bar{Z} - 4aN > -\hat{Z}$. The line $P_1 - Q_0$ is formed in this way:

$$\overline{P_2 P_3}, \quad \overline{P_4 P_5}, \quad \overline{P_7 P_8}, \quad \overline{P_8 P_0} \text{ and } \overline{\hat{P} \hat{Q}} \text{ are segments.}$$

$\widehat{QQ_0}$ and $\widehat{P_1 P_2}$ are trajectories of the autonomous system (1.7). $\widehat{P_5 P_6}$ is a trajectory of the autonomous system (1.2). $\widehat{P_0 \hat{P}}$ is a trajectory of system (1.5) with the initial condition $t_0 + h$. $\widehat{P_3 P_4}$ is a trajectory of system (1.9) with the initial condition $t_0 + h$.

Notice that the lines $P_0 Q_0$ and $P_3 P_7$ are based on Lemma 3, while the line $P_1 P_2$ is based on hypothesis (6). One may verify that it is possible to apply the Brouwer's fixed point theorem to the region V which is bounded by the line $P_1 - Q_0$.

We project the region V on the plane $t = t_0 - h$. Let ω be the transformation which associates the projection on the plane $t = t_0 - h$ of the point obtained by following for time T the positive half-trajectory of system (1.1) which passes through P at the time $t_0 - h$, with the generic point P of the plane $t = t_0 - h$; one may prove that the region V has the property of being transformed by ω in one of its parts.

The result is trivial for the parts of the region V which are bounded by segments and by arcs of trajectories of system (1.2) and (1.7).

In fact the trajectories of system (1.7) are in the half-plane $y < 0$, while the trajectory of system (1.2) is in the half-plane $y \geq 0$.

With respect to the parts of V which are bounded by the lines $P_0 \hat{P}$ and $P_3 P_4$, a direct proof is necessary. For simplicity we examine in detail only the region bounded by the line $P_0 \hat{P}$, the axis $x = b$ and the axis $y = 0$. It is necessary to prove that a generic point P of this region is transformed by ω into a point P' which is in the same region if it is in the half-plane $y \geq 0$. We may examine only the points of the trajectory $P_0 \hat{P}$, because two trajectories of system (1.5) which pass at the same time, through two points with the same abscissa cannot cut each other, having the same behaviour of autonomous trajectories.

A comparison between system (1.1) and system (1.5) shows that the positive half-trajectory of (1.1) which passes through a point of the considered region is bounded by the corresponding trajectory of system (1.5) which passes at the same time through the same point.

We suppose, for simplicity, that the positive half-trajectory of system (1.5) which passes through the point $P_0(\bar{Z} + 4aN, b)$ at the time $t_0 + h$, crosses in a time $2T$ the x -axis at a point $\hat{P}$. This situation is described in the figure.

Fig. 3.

Consider only the line $P_0\hat{P}_2$. If we choose a positive half-trajectory of system (1.5) which passes through any point of the line $P_0\hat{P}_2$ at the time $t_0 - h$, such a trajectory does not cross the line $P_0\hat{P}$ during the time T . Notice that the result is true only for the initial time $t_0 - h$. This result holds for the whole line $P_0\hat{P}$ and the proof is analogous for the line P_3P_4 .

Thus the region V has the property of being transformed by ω into one of its parts and this completes the proof of the Theorem.

REMARK 1.3. — Assumption (1.3) is not restrictive. In the general case the line $P_0\hat{P}$ may be construct with trajectories of the autonomous system (1.2) and trajectories of system (1.5) with the initial condition $t_0 + h$.

One may then construct the line P_3P_4 in the same way, using trajectories of the autonomous system (1.7) and system (1.9) with the initial condition $t_0 + h$.

REMARK 1.4. — We just note that in the hypothesis (6) the assumptions

$$f(x, y) \geq \Phi(x) \quad \text{if } a \leq x \leq b \quad y < -a_1 \leq -a$$

$$\int_a^b \Phi(x) dx > 4aN + 1$$

may be replaced by the assumptions

$$f(x, y) \geq \Phi(x) \quad \text{if } -b \leq x \leq -a, \quad y > a_1 \geq a$$

$$\int_{-b}^{-a} \Phi(x) dx > 4aN + 1$$

provided that one makes the corresponding changes in the hypotheses (3) and (5), namely

$$x > a \quad \text{instead of } x > b$$

$$x < -b \quad \text{instead of } x < -a$$

REMARK 1.5. — Recalling the construction of the plane-region V it is clear that Theorem 1 may be invoked to assert that

Every solution of the equation

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = e(t)$$

is bounded for $t > 0$.

This result is still true if the function $e(t)$ is not a periodic function.

If $e(t)$ is a constant, or if

$$\lim_{t \rightarrow +\infty} e(t) = E$$

the hypothesis (4) is replaced by

$$(4^*) \quad \begin{aligned} f(x, y)y + g(x) &\geq E + k && \text{if } x \geq b, \quad y \geq 0 \\ f(x, y)y + g(x) &\leq E - k && \text{if } x \leq -a, \quad y \leq 0 \end{aligned}$$

where $k > 0$ is a constant.

2. — With the same technique, and using the previous results, one may state the following

THEOREM 2. — Let $f: \mathbf{R}^2 \rightarrow \mathbf{R}$, $g: \mathbf{R} \rightarrow \mathbf{R}$ and $e: \mathbf{R} \rightarrow \mathbf{R}$ such that:

- (1) f and g are continuous and locally Lipschitz continuous functions;
- (2) $e(t)$ is continuous and periodic, $e(t) = e(t + T) \quad \forall t, \quad e \leq e(t) \leq E$;
- (3') there exist $a, M > 0$ such that

$$f(x, y) > -M \quad \text{if } x < a, \quad y > a$$

$$x > -a, \quad y < -a$$

$$(4) \quad \begin{aligned} f(x, y)y + g(x) &\geq E && \text{if } x > a, \quad y \geq 0 \\ f(x, y)y + g(x) &\leq e && \text{if } x < -a, \quad y \leq 0; \end{aligned}$$

$$(5') \quad \lim_{x \rightarrow -\infty} f(x, E - g(x)) > g'(x) + 1.$$

Then there exists a T -periodic solution of equation (1).

PROOF. — Based on hypothesis (5'), there exists $\bar{x} < -a$ such that

$$(2.1) \quad f(x, E - g(x)) > g'(x) + 1$$

if $x \leq \bar{x}$.

We consider the point P_0 having coordinates $\bar{x}$ and $E - g(\bar{x})$. Clearly $E - g(\bar{x}) > 0$. Based on hypothesis (3'), the positive half-trajectory of system (1.2) which passes through the point P_0 crosses in a finite time the axis $x = -a$.

Later on such a trajectory either crosses the x -axis, or crosses the axis $x = a$ at a point P_1 having coordinates a and $y_1 > 0$. In the latter case we consider the positive half-trajectory of system (1.5) which passes through the point P_1 at the time $t_0 + h$. Recalling Lemma 1 this trajectory crosses in a finite time the x -axis at a point P_2 . Now we consider the positive half-trajectory of system (1.7) which passes through the point P_2 . Again based on hypothesis (3') this trajectory crosses in a finite time the axis $x = a$. Later on such a trajectory either crosses the x -axis, or crosses the axis $x = -a$ at a point P_3 having coordinates $-a$ and $y_3 < 0$. In the latter case we consider the positive half-trajectory of system (1.12) which passes through the point P_3 at the time $t_0 + h$. Again recalling Lemma 1 this trajectory crosses in a finite time the x -axis at a point P_4 having abscissa x_4 .

If $x_4 \geq \bar{x}$ we consider on the x -axis the point P_5 having abscissa $\bar{x}$. Then we join P_4 with P_5 and P_5 with P_0 : If $x_4 < \bar{x}$ we consider on the axis $x = x_4$ the point P_5 having ordinate $E - g(x_4)$. Then we join P_4 with P_5 and we consider the line P_5P_0 having coordinates x and $E - g(x)$, with $x_4 \leq x \leq \bar{x}$.

One may verify that it is possible to apply the Brouwer's fixed point theorem to the region V which is bounded by the line $P_0 - P_5$.

Indeed if $x_4 \geq \bar{x}$ one may prove, with the same arguments of theorem 1, that the region V has the property of being transformed by ω , which is defined as in Theorem 1, into one of its parts. In the second case, when $x_4 < \bar{x}$, a direct proof is necessary with respect to the part of V which is bounded by the line P_5P_0 . For the points of the line P_5P_0 we have

$$y(x) = E - g(x)$$

which implies

$$y'(x) = -g'(x).$$

One may write system (1.2) for these points and obtain

$$(2.2) \quad \begin{cases} \dot{x} = E - g(x) \\ \dot{y} = -f(x, E - g(x))(E - g(x)) + E - g(x) . \end{cases}$$

It follows immediately from assumption (2.2) that

$$y'(x) = -f(x, E - g(x)) + 1 .$$

It is necessary that any trajectory of system (1.2) which passes through a generic Point P of the region V cannot cross the line P_5P_0 .

This implies that

$$\begin{aligned} -f(x, E - g(x)) + 1 &< -g'(x) \\ f(x, E - g(x)) &> g'(x) + 1 \end{aligned}$$

which is precisely assumption (2.1).

The Theorem is proved, because now it is possible to argue in a manner analogous to that of Theorem 1.

REMARK 2.1. - We suppose that $g(x)$ is differentiable, but this is not restrictive. Indeed if $g(x)$ is not differentiable we consider a differentiable function $\hat{g}(x)$ such that

$$\lim_{x \rightarrow -\infty} \frac{g(x)}{\hat{g}(x)} = 1$$

and we write assumption (5') in the form

$$\lim_{x \rightarrow -\infty} f(x, E - \hat{g}(x)) > \hat{g}'(x) + 1 .$$

Another way of writing assumption (5') is the following:

$$(2.3) \quad \text{let } \lim_{x \rightarrow -\infty} \frac{g(x)}{x^n} = \alpha \neq \infty .$$

Define the function $y(x) = (-1)^n x^n$.

We consider the system (1.2) for the points having coordinates x and $(-1)^n x^n$.

Arguing as in the previous theorem, based on assumption (2.3), it follows that

$$-f(x, (-1)^n x^n) - (-1)^n \alpha < (-1)^n n x^{n-1} .$$

Hence the hypothesis (5') may be written in the following way

$$\liminf_{x \rightarrow -\infty} f(x, (-1)'x^n) > (-1)^{n+1}(\alpha + nx^{n-1})$$

$$\lim_{x \rightarrow -\infty} \frac{g(x)}{x^n} = \alpha \neq \infty.$$

The particular case given by $\lim_{x \rightarrow -\infty} (g(x)/x) = 0$ is interesting. Indeed we have, for $\delta > 0$

$$\lim_{x \rightarrow -\infty} f\left(x, -\frac{\delta}{2}x\right) > \delta.$$

Then the assumption

$$f(x, y) > \delta, \quad \text{if } x < -a, \quad y > a$$

implies the hypothesis (5'). This result is known [5] when $g(x)$ is a bounded function for x negative.

REMARK 2.2. - We just note that the hypothesis (5') may be replaced by the hypothesis

$$(5'') \quad \lim_{x \rightarrow +\infty} f(x, e - g(x)) > g'(x) + 1.$$

Also in this case one may consider the observation of previous Remark 2.1.

REMARK 2.3. We consider the following autonomous equation

$$(2.4) \quad \ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0.$$

For this equation we may give a result analogous with Theorem 2. More precisely one may prove that the autonomous system

$$(2.5) \quad \begin{cases} \dot{x} = y \\ \dot{y} = -f(x, y)y - g(x) \end{cases}$$

has (in the phase-plane) at least one stable limit cycle. This is equivalent to the existence of a nontrivial periodic solution of equation (2.4).

THEOREM 3. - Let $f: \mathbf{R}^2 \rightarrow \mathbf{R}$, $g: \mathbf{R} \rightarrow \mathbf{R}$ such that

- (1) f and g are continuous and locally Lipschitz continuous functions;
- (2) $xg(x) > 0$ if $x \neq 0$

(3) $f(0, 0) < 0$ and there exist $a, M > 0$ such that

$$f(x, y) > -M \quad \text{if } x < a, y > a \\ x > -a, y < -a.$$

There exists $\varepsilon > 0$ such that

(4) $f(x, y)|y| + |g(x)| \geq \varepsilon$ if $|x| > a$

(5) $\lim_{x \rightarrow -\infty} f(x, -g(x)) > g'(x) + 1$

(or (5') $\lim_{x \rightarrow +\infty} f(x, -g(x)) > g'(x) + 1$).

Then there exists a non trivial periodic solution of equation (2.1).

PROOF. — The proof of this theorem follows immediately based on the previous observations. We note that

- a) The origin is the only stationary point of system (2.5) and it is a repulsive stationary point.
- b) Arguing as in Theorem 2 it is possible to produce a bounded trajectory for system (2.5).

The Poincaré-Bendixson theorem now implies that system (2.5) has at least one stable limit cycle.

Thus theorem 3 is proved.

We observe that the assumption

$$f(x, y)|y| + |g(x)| > 0 \quad |x| > a$$

cannot replace the hypothesis (4), because in this case a trajectory of system (2.5) may have an horizontal asymptote.

3. — Let us briefly examine the results we have obtained and let us compare them with the results obtained by Gaetano VILLARI [5] as his are the most general among the works considered. Theorem 1 extends theorem 1 of the above-mentioned work: indeed the hypotheses (4) improve on the hypotheses

$$f(x, y)y + g(x) \geq E + \delta \quad \text{if } x > a, y \geq 0 \\ f(x, y)y + g(x) \leq e - \delta \quad \text{if } x < -a, y \leq 0.$$

This improvement is in line with the classical results.

LEVINSON [1], indeed, has put forward the following hypotheses on the function $g(x)$

$$\begin{aligned}\lim_{|x| \rightarrow +\infty} |g(x)| &= +\infty \\ xg(x) &> 0 \quad |x| > a \\ \lim_{x \rightarrow \infty} \frac{g(x)}{\int_0^x g(S) dS} &= 0.\end{aligned}$$

Subsequent papers have little by little improved on such hypotheses: those at (4) of theorem 1 imply the hypotheses

$$(3.1) \quad \begin{cases} g(x) \geq E & \text{if } x > a \\ g(x) \leq e & \text{if } x < -a. \end{cases}$$

Which improves on the results already noted, and it does not seem possible to improve on them any further with the same technique since otherwise it will be impossible to avoid the possible existence of a horizontal asymptote in the construction of region V .

One may use an analogous argument under hypotheses (3) and (5) which are an improvement on the hypotheses suggested in [5]

$$\begin{aligned}f(x, y) &\geq \delta > 0 && \text{if } x > a, \quad y < -a \\ &&& x < -a, \quad y > a \\ f(x, y) &\geq -M && \text{if } |x| < a, \quad |y| < +\infty \\ &&& x > a, \quad 0 < y < a \\ &&& x < -a, \quad -a < y < 0.\end{aligned}$$

Even a result of this kind concords with the classical results because Levinson supposes

$$\begin{aligned}f(x, y) &\geq \delta > 0 && \text{if } |x| > a, \quad |y| > a \\ f(x, y) &> -M && |x| < a, \quad |y| < +\infty\end{aligned}$$

and such a hypothesis has been gradually improved on.

Theorem 2 may be regarded as an extension of a result due to Gaetano VILLARI [5], the applications of which are interesting when $g(x)$ is bounded, but which gives few indications otherwise.

If, finally, we consider the equation

$$\ddot{x} + f(x)\dot{x} + g(x) = e(t)$$

the results of theorems 1 and 2 permit both assumption (3.1) and the hypothesis relative to $f(x)$

$$(3.2) \quad f(x) \geq \delta > 0 \quad \text{if } |x| > a.$$

Let us compare such hypotheses with the results obtained by MAWHIN [6], REISSIG [8], [9] and MARTELLI [10].

In these papers other than continuity, no hypotheses on $f(x)$ are required.

Other hypotheses are

$$(3.3) \quad \begin{cases} \int_{\delta}^{\pi} e(S) dS = 0 \\ xg(x) \geq 0 \end{cases} \quad \text{if } |x| > a.$$

We may observe that the assumptions (3.3) are for the most part equivalent to those at (3.1).

Indeed if $e(t)$ does not have mean value zero we put

$$\int_0^{\pi} e(S) dS = I.$$

One should consider the function

$$\hat{e}(t) = e(t) - I/T$$

which turns out to be T -periodic and with mean value zero. It follows that

$$\begin{aligned} g(x) &\geq I/T & \text{if } x > a \\ g(x) &\leq I/T & \text{if } x < -a. \end{aligned}$$

On the other hand, it is not possible to rule out the hypothesis that

$$\int_0^{\pi} e(S) dS = 0 \quad \text{maintaining the hypothesis that } xg(x) > 0$$

if $|x| > a$, as may be verified by the following example:

EXAMPLE 3.1. – Let $g: \mathbf{R} \rightarrow \mathbf{R}$ and $e: \mathbf{R} \rightarrow \mathbf{R}$ be such that

$$\begin{aligned} g(x) &= x & \text{if } |x| \leq 1 \\ &\frac{1}{x} & \text{if } |x| \geq 1 \\ e(t) &= \sin t + 3. \end{aligned}$$

Let us consider the equation $\ddot{x} + f(x)\dot{x} + g(x) = e(t)$. It may be verified that no matter how one takes $f(x)$ continuous, no periodic solution exists.

The result is immediate if one examines the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -f(x)y - g(x) + e(t). \end{aligned}$$

With respect to the function $g(x)$ one may observe that the hypotheses are more restrictive because

$$\begin{aligned} \limsup_{|x| \rightarrow \infty} \frac{g(x)}{x} &< 4\pi^2 \\ \left(\lim_{|x| \rightarrow \infty} \frac{g(x)}{x} = 0 \text{ in [6]} \right) \end{aligned}$$

is required.

In conclusion with respect to the function $f(x)$ the hypothesis (3.2) is not required, but the behaviour of $g(x)$ as $x \rightarrow \infty$ is constrained by that of x .

Finally we observe that theorems 1 and 2 provide information on boundedness of solutions for $t > 0$ and give a limitation for the periodic solutions, because they must obviously be situated in the region V , but the method used in [6], [8], [9], [10] is valid even in the case of functions from $\mathbf{R}^n$ to $\mathbf{R}^n$ and in a more general situation.

Appendix.

PROOF OF LEMMA 1. – We suppose that the positive half-trajectory of system (1.5) which passes through the point P_0 at the time $t_0 - h$ does not cross the x -axis in a time less than $2h$ otherwise the lemma is trivial.

At the time $t_0 + h$ such a trajectory passes through the point having coordinates $x_0 + 2h(y_0 - \delta h)$ and $y_0 - 2\delta h$.

At the time $t_0 - h + T$, based on the definition of function $v(t)$, the trajectory passes through the point having coordinates $x_0 + 2h^2\delta + T(y_0 - 2\delta h)$ and $y_0 - 2\delta h$.

Fig. 4.

The function $v(t)$ is T -periodic; hence for every period T the ordinate of the trajectory of system (1.5) decreases by $2\delta h$. Let n be the smallest integer such that

$$y_0 < n2\delta h$$

$$n = \left[\frac{y_0}{2\delta h} + 1 \right]$$

Thus we have

$$x_1 - x_0 < \frac{1}{2} \frac{y_0^2}{\delta} + \left[\frac{y_0}{2\delta h} + 1 \right] \cdot (T - 2h)(y_0 - 2\delta h)$$

now we replace δ , namely

$$x_1 - x_0 < \frac{2y_0^2}{E - e} + \left[\frac{2y_0}{h(E - e)} + 1 \right] \cdot (T - 2h) \left(y_0 - \frac{1}{2} h(E - e) \right) \quad \text{Q.E.D.}$$

PROOF OF LEMMA 2. — We consider the negative half-trajectory of system (1.6) which passes through the point D at the time $\hat{t}$.

This trajectory crosses the axis $x = x_c$ at a point $\hat{C}$ and the axis $x = x_B$ at a point $\hat{B}$.

The point $\hat{C}$ has ordinate $\hat{a}$, while the point $\hat{B}$ has ordinate $\hat{a}_1$ and it is $\hat{a} > a$, $\hat{a}_1 > a_1$.

Recalling (1.3), if we compare systems (1.5) and (1.6) it follows that $(\hat{a}_1 - a)$ and $(\hat{a}_1 - a_1)$ depend on $(x_D - x_c)$ and $(x_D - x_B)$ respectively, but they do not depend on x_c or x_B .

A consequence of system (1.6) is

$$(A.1) \quad y' = -f(x, y) - \frac{g(x) - \varphi(t)}{y}$$

We have

$$y' = -f(x, y) - \frac{g(x) - \varphi(t)}{y} \leq -f(x, y) - \frac{g(x) - E}{y}$$

If we suppose $f(x, y) > -M$ in the region bounded by $x > x_c$, $0 \leq y \leq \hat{a}$ from

$$\int_{x_0}^x y' dx = \hat{a}$$

it follows

$$\int_{s_0 s_1} \hat{y}' dx < \hat{a} + 2M(x_D - x_c) + \frac{|e|}{a_1}(x_D - x_c).$$

We note that with respect to the function $f(x, y)$ the hypothesis of boundedness in the region above mentioned is made for simplicity. In fact if $f(x, y)$ has no lower bound in this region, we may replace $f(x, y)$ with $\hat{f}(x, y) > -M$.

In this case the assumption,

$$f(x, y)y + g(x) \geq E \quad \text{if } x > a \quad y \geq 0$$

which is in hypothesis (4), is still valid.

Consider the system

$$(A.2) \quad \begin{cases} \dot{x} = y \\ \dot{y} = -\hat{f}(x, y)y - g(x) + \varphi(t) \end{cases}$$

From a comparison with the system (1.6) it follows that the positive half-trajectory of system (A.2) which passes through the point $\hat{C}$ in the meantime, crosses the x -axis at a point $\hat{D}$ having coordinates $x_{\hat{D}}$ and 0, with $x_c < x_{\hat{D}} < x_D$. Since $(x_{\hat{D}} - x_c) < (x_D - x_c)$, arguing in the same way we have the same statement.

Now we prove that

$$(A.3) \quad \hat{y}' - y' \geq \frac{2}{3}k$$

if $x > a$, $y \geq z > a$, $\hat{y} > a$.

This follows from hypothesis (5) and assumption (1.4).

In fact, from

$$\hat{y} > a_1 > y > a$$

we have

$$\begin{aligned} \hat{y}' - y' &= f(x - \hat{y}) + f(x, y) - \frac{g(x)}{\hat{y}} + \frac{e}{\hat{y}} + \frac{g(x)}{y} - \frac{\varphi(t)}{y} > \\ &\geq k - \frac{g(x)}{\hat{y}} + \frac{e}{\hat{y}} + \frac{g(x) - E}{y} \geq k - \frac{g(x)}{\hat{y}} + \frac{e}{\hat{y}} + \frac{g(x)}{\hat{y}} - \frac{E}{\hat{y}} = k - \frac{E - e}{\hat{y}} \geq \frac{2}{3} k. \end{aligned}$$

From (A.3) we have

$$\int_{S_1 S_2} \hat{y}' dx < \hat{a}_1 - \hat{a} - \frac{2}{3} k(x_C - x_B).$$

Thus $\overline{S_2 B} = \int_{S_0 S_2} \hat{y} dx < \hat{a}_1 + (2M + |e|/a_1)(x_D - x_C)$. Since $(x_D - x_C) < (x_D - x_B)$ and, recalling Lemma 1 and assumption (1.3), $(x_D - x_B)$ and $\hat{a}_1$ do not depend on x_0 , Lemma 2 is proved. Q.E.D.

PROOF OF LEMMA 3. — We consider the negative half-trajectory of system (1.5) which passes through the point B at the time $t_0 - h$. This trajectory crosses at the time $\bar{t} < t_0 - h$ the axis $x = b$ at a point A having coordinates b and y_A , with $y_A > a_1$.

The negative half-trajectory of system (1.6) which passes through the point B at the time $t_0 - h$ crosses the axis $x = b$ at a point $\hat{A}$ having coordinates b and $y_{\hat{A}}$, with $y_{\hat{A}} > y_A$.

Assume $a_1 > 1$, which clearly is not restrictive.

Based on (1.3) we have

$$(A.4) \quad y_A \leq \hat{y}_{\hat{A}} < y_A + \frac{k}{3}(x_B - b).$$

The line $\hat{y}(x)$ which passes through the point S_0 crosses the axis $x = b$ in a point S_3 .

By a possible integration of assumption (A.3) in the interval $[b, x_B]$, based on assumptions (1.8) and (A.1), we have

$$\hat{y}(x_B) - a_1 - |\hat{y}(b) - y_{\hat{A}}| \geq \frac{2}{3} k(x_B - b)$$

recalling (A.4) it follows

$$\hat{y}(x_B) - a_1 - |\hat{y}(b) - y_A| > \frac{k}{3}(x_B - b).$$

Lemma 2 may be invoked to state that

$$\lim_{x_B \rightarrow +\infty} (y_A - \hat{y}(b)) = +\infty.$$

This means that when x_B is big enough the line S_2S_3 crosses the arc of trajectory BA .
Therefore, recalling assumptions (A.3) and (1.3),

$$y_A > \hat{y}(b).$$

In the light of the meaning of the line S_0S_3 Lemma 3 is proved. Q.E.D.

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