The Plancherel Measure for Symmetric Graphs (*).

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Summary. — Let G be the free product of r copies of the cyclic group $\mathbf{Z}_k$. We obtain the Plancherel formula for the commutative C^* -algebra of radial convolution operators on $l^2(G)$. The Plancherel measure is expressed in terms of the c -function appearing in the expansion of spherical functions on G as linear combinations of exponentials.

1. — Introduction.

A function on the free group $\mathbf{F}_r$ with r generators is called radial if it depends only on the length of the words. The C^* -algebra of radial convolution operators on $l^2(\mathbf{F}_r)$ is commutative, and its spectrum σ supports a measure m such that the l^2 -norm of a radial function f is the norm of its Gelfand transform $\hat{f}$ in $L^2(\sigma, m)$. This measure, called the *Plancherel measure*, has been computed in [2, 10, 3, 9, 5].

The group $\mathbf{F}_r$ acts isometrically and simply transitively on the homogeneous tree T_d with $d = 2r$. Every group acting in this way on T_d is isomorphic to the free product $\left(\bigstar_{j=1}^p \mathbf{Z} \right) * \left(\bigstar_{j=1}^s \mathbf{Z}_2 \right)$, with $2p + s = d$. The Plancherel theorem has been extended to these groups in [1]. The Plancherel measure can be expressed in terms of the asymptotic behaviour of spherical functions on trees. Following [5], we shall denote the spherical functions by φ_z , $z \in \mathbf{C}$: then the support σ of the Plancherel measure is the segment $\{\operatorname{Re} z = \frac{1}{2}, 0 \leq \operatorname{Im} z \leq \pi/\ln q\}$ where $q = d - 1$, and d is the homogeneity degree. There exists a meromorphic function c (depending on the degree), such that, except at its poles, $\varphi_z = c(z)h_z + c(1-z)h_{1-z}$ where $h_z(x) = q^{-|x|}$ (here $|x|$ denotes the length of the vertex x). Then it turns out that $dm(\frac{1}{2} + it)$ is a multiple of $|c(\frac{1}{2} + it)|^{-2}$ [5, 1]. Similar relations had long been known for semisimple Lie groups.

The theory of spherical functions has been recently extended to the free products $\bigstar_{j=1}^r \mathbf{Z}_k(r, k, \geq 2)$, which act on «symmetric graphs» [7], and on semihomogeneous trees.

A Plancherel formula for these groups has been proved in [8], with no reference to the c -function. In this note we give a short and simple computation of the Plancherel measure, for groups acting on trees or on symmetric graphs (semihomogeneous

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trees), as described above, by a method which sheds light upon the role played by the c -function. For rank-one semisimple Lie groups, a similar approach is in [6].

For uniformity of notations, we shall always consider groups of the type $G = \bigstar_{j=1}^r \mathbf{Z}_k$.

In the case $k = 2$, when the associated graph is a homogeneous tree, only trivial modifications are needed to adapt our results to free groups and to every other group acting on the tree.

Unexplained terminology and notations are as in [7]. We were informed by W. BETORI and M. PAGLIACCI of an alternative deduction of the Plancherel formula for free groups, which makes use of Radon transforms.

An extension of the results of [8] to more abstract settings was communicated to us by J. M. COHEN and A. TRENHOLME.

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2. - The Plancherel measure.

Let $G = \bigstar_{j=1}^r \mathbf{Z}_k$, and denote by μ_1 the probability measure equidistributed on words of length one. We shall consider the self-adjoint operator T , on the Hilbert space $l^2_\#$ of square-integrable radial functions on G , defined by $Tf = \mu_1 * f$. It has been shown in [7, thm. 3] that the spectrum of T is the set $E \cup D$, where E is the segment

$$\{\lambda \in \mathbf{R}: \lambda = \gamma(\tfrac{1}{2} + it), 0 \leq t \leq \pi/\log q\},$$

and

$$D = \left\{ \lambda_0 = \frac{1}{1-k} \right\} \quad \text{for } k > r, \quad D = \emptyset \quad \text{for } k \leq r.$$

(Here, as in [7],

$$q = (k-1)(r-1) \quad \text{and} \quad \gamma(z) = \frac{1}{(k-1)r} (q^z + q^{1-z} + k-2).$$

Let $z_0 = (\log(k-1) + i\pi)/\log q$, so that $\gamma(z_0) = \lambda_0$. The Plancherel measure m is the positive measure on the spectrum of T such that, for $f \in l^1_\#$,

$$f(e) = \int f dm$$

where $f(z) = \sum_{x \in G} f(x) \varphi_z(x)$ and φ_z is the spherical function.

THEOREM. — For every $f \in l_{\#}^1$,

$$f(e) = \frac{1}{2\pi} \frac{q \log q}{(k-1)r} \int_0^{\pi/\log q} \left| f\left(\frac{1}{2} + it\right) \right| \left| c\left(\frac{1}{2} + it\right) \right|^{-2} dt + \frac{q \log q}{(k-1)r} \operatorname{Res}_{z_0} \left[\frac{1}{c(z)c(1-z)} \right] f(z_0).$$

(The second term on the right hand side is omitted if $k \leq r$.)

PROOF. — Recall the following explicit expression for the c -function [7, thm. 2]:

$$(1) \quad c(z) = \frac{q^{1-z} - (k-1)q^{z-1} + k-2}{(k-1)r(q^{-z} - q^{z-1})}.$$

The resolvent $R_{\lambda} = (\lambda I - T)^{-1}$ of the operator T , computed in [7, thm. 3], can be expressed in terms of the c -function: For $\lambda = \gamma(z)$, $\operatorname{Re} z > \frac{1}{2}$, $f \in l_{\#}^2$,

$$R_{\lambda} f = r_{\lambda} * f$$

with

$$r_{\lambda}(x) = \frac{1}{c(1-z)} \frac{1}{q^{z-1} - q^{-z}} q^{-z|x|}.$$

The resolvent is analytic with respect to λ in the complement of the spectrum of T , and has a simple pole at $\lambda_0 = (1-k)^{-1}$ for $k > r$. Carleman's formula [4, p. 920] yields a relation between R_{λ} and the spectral resolution $E(d\lambda)$ of T : for any $f \in l_{\#}^2$ and any continuous function φ on $\mathbf{R}$ with compact support,

$$\int_{-\infty}^{\infty} \varphi(\lambda) (E(d\lambda)f, f) = -\operatorname{p} \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \varphi(\lambda) \left(\frac{1}{2i} [R_{\lambda+i\varepsilon} - R_{\lambda-i\varepsilon}] f, f \right) d\lambda.$$

The measure $(E(d\lambda)f, f)$ is absolutely continuous with respect to Lebesgue measure on E , and has an atom at $\lambda_0 = 1/(1-k)$ for $k > r$.

We now compute its continuous part. For $\lambda = \gamma(\frac{1}{2} + it)$, $0 < t < \pi/\log q$,

$$\begin{aligned} \frac{1}{2i} [r_{\lambda+i0}(x) - r_{\lambda-i0}(x)] d\lambda &= \frac{1}{2i} \left[\frac{1}{c(\frac{1}{2} - it)} \frac{q^{-(\frac{1}{2}+it)|x|}}{q^{-\frac{1}{2}+it} - q^{-\frac{1}{2}-it}} - \frac{1}{c(\frac{1}{2} + it)} \frac{q^{-(\frac{1}{2}-it)|x|}}{q^{-\frac{1}{2}-it} - q^{-\frac{1}{2}+it}} \right] \\ &\quad \cdot \frac{\log q}{(k-1)r} |q^{\frac{1}{2}+it} - q^{\frac{1}{2}-it}| dt = \\ &= -\frac{1}{2} \frac{q \log q}{(k-1)r} \left[\frac{1}{c(\frac{1}{2} - it)} q^{-(\frac{1}{2}+it)|x|} + \frac{1}{c(\frac{1}{2} + it)} q^{-(\frac{1}{2}-it)|x|} \right] dt = \\ &= -\frac{1}{2} \frac{q \log q}{(k-1)r} \frac{1}{|c(\frac{1}{2} + it)|^2} \varphi_{\frac{1}{2}+it}(x) dt. \end{aligned}$$

Observing that $(\varphi_z * f, f) = |\hat{f}(z)|^2$ for $z = \frac{1}{2} + it$, we have:

$$\frac{1}{2i} ([R_{\lambda+i0} - R_{\lambda-i0}]f, f) d\lambda = -\frac{1}{2} \frac{q \log q}{(k-1)r} \left| \hat{f}\left(\frac{1}{2} + it\right) \right|^2 \frac{dt}{|c(\frac{1}{2} + it)|^2}.$$

Let us assume $k > r$ and consider the singular part of m . We have $c(1 - z_0) = 0$, $c(z_0) = 1$; hence

$$\varphi_{z_0}(x) = q^{-z_0|x|} = (1 - k)^{-|x|}.$$

The orthogonal projector onto the eigenspace corresponding to the eigenvalue $\lambda_0 = \gamma(z_0) = (1 - k)^{-1}$ is given by the residue of the resolvent R_λ at λ_0 :

$$(E(\{\lambda_0\})f, f) = \text{Res}_{\lambda_0} [(R_\lambda f, f)].$$

We have

$$\text{Res}_{\lambda_0} [r_\lambda] = \gamma'(z_0) \text{Res}_{z_0} \left[\frac{1}{c(1-z)} \frac{1}{q^{z-1} - q^{-z}} q^{-z|x|} \right] = \frac{q \log q}{(k-1)r} \text{Res}_{z_0} \left[\frac{1}{c(1-z)} \right] q^{-z_0|x|}.$$

Therefore

$$(E(\{\lambda_0\})f, f) = \frac{q \log q}{(k-1)r} \text{Res}_{z_0} \left[\frac{1}{c(1-z)} \right] |\hat{f}(z_0)|^2.$$

Since $\int_{-\infty}^{\infty} (E(d\lambda)f, f) = \|f\|_2^2$, we obtain the Plancherel formula

$$\begin{aligned} \sum_{x \in G} |f(x)|^2 &= \frac{1}{2\pi} \frac{q \log q}{(k-1)r} \int_0^{x/\log q} \left| \hat{f}\left(\frac{1}{2} + it\right) \right|^2 \left| c\left(\frac{1}{2} + it\right) \right|^{-2} dt + \\ &\quad + \frac{q \log q}{(k-1)r} \text{Res}_{z_0} \left[\frac{1}{c(z)c(1-z)} \right] |\hat{f}(z_0)|^2, \end{aligned}$$

where the second term has to be omitted for $k \leq r$.

REMARK. - The residue which yields the singular part of m can be easily computed by means of (1). One obtains

$$\frac{q \log q}{(k-1)r} \text{Res}_{z_0} \left[\frac{1}{c(z)c(1-z)} \right] = \frac{k-r}{k}.$$

One can also check that the expression of the Plancherel measure given in the Theorem agrees with the formulas of [8, thm. 2] (in our notations, the parameter x which appears in these formulas becomes

$$x = q^z + q^{1-z} + k - 2 = (k-1)r\gamma(z).$$

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