Riemannian Metrics on Tangent Bundles (*).

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Summary. – *Some « natural » metrics on the tangent and on the sphere tangent bundle of Riemannian manifold are constructed and studied via the moving frame method.*

1. – Introduction.

The tangent bundle TM of a Riemannian manifold (M, g) admits a natural Riemannian metric: the *Sasaki metric* g_s.

In order to define g_s we consider two vectors X and Y tangent to TM at the point (p, v). Suppose that X and Y are tangent at the time $t = 0$ to the curves $\tilde{\alpha}(t) = (\alpha(t), V(t))$ and $\tilde{\beta}(t) = (\beta(t), W(t))$ respectively. Denote with DV/dt and DW/dt the covariant derivatives of the vector fields $V(t)$ and $W(t)$ along $\alpha(t)$ and $\beta(t)$, then g_s is defined by:

$$g_s|_{(p,v)}(X, Y) = g_p(\dot{\alpha}(0), \dot{\beta}(0)) + g_p\left(\frac{DV}{dt}\bigg|_0, \frac{DW}{dt}\bigg|_0\right). \tag{1.1}$$

g_s is perhaps the most natural metric on TM depending only on the Riemannian structure on M, but it is extremely rigid. For instance, g_s has constant scalar curvature if and only if g is flat. Therefore, the Sasaki metric is locally homogeneous, or locally symmetric, or Einstein only if it is flat (see n. 3). But, if we consider TM as a vector bundle associated with OM we may easily construct other interesting metrics on TM.

In section 4 we discuss this general construction and we shall prove that the Sasaki metric can be obtained in this way. We also give an explicit expression of a complete metric g_{CG} introduced by Cheeger and Gromoll in [CG].

If (M, g) is the standard n-sphere, the metric g_{CG} has non negative curvature and S^n is the soul of (TS^n, g_{CG}).

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In section 5 we study the spherical tangent bundle $T_1M = \{(p, v) \in TM/\|v\| = 1\}$ endowed with the induced Sasaki metric g_s'. It is interesting to observe that (T_1M, g_s') is an homogeneous Riemannian space if (M, g) is a rank one symmetric space (see n. 5).

T_1M can be regarded as an hypersurface of TM, thus the Levi Civita connection and the curvature tensor of g_s' could be computed using Gauss equation. Instead, we prefer to identify T_1M with a quotient of OM and make use of the moving frame method (see n. 6). In section 6 we study the spherical tangent bundle T_1S^n of the standard n-sphere generalising the results obtained in [KS].

Section 7 deals with deformations of the metric g_s'. We prove that the Einstein metric defined on T_1S^n by S. KOBAYASHI ([KO$_1$], [Je]) can be obtained deforming g_s' along the direction of the canonical contact form on T_1S^n.

We are indebted to O. KOWALSKI for the remark 4.3 and several useful discussions.

2. – The Sasaki metric.

Let (M, g) be an n-dimensional Riemannian manifold with tangent bundle TM and natural projection $\pi: TM \to M$.

A curve $\bar{\gamma}: I \to TM$, $t \to (\gamma(t), V(t))$ is *horizontal* if the vector field $V(t)$ is parallel along $\gamma = \pi \circ \bar{\gamma}$. A vector on TM is *horizontal* if it is tangent to an horizontal curve, or *vertical* if is tangent to a fiber.

Let $\gamma: I \to M$, $t \to \gamma(t)$ be a curve through the point $p = \gamma(0)$.

For each tangent vector $v \in T_pM$ there exists a unique horizontal curve $\gamma^H: I \to \to TM$ through (p, v) which projects onto γ. This curve is defined by:

$$\gamma^H(t) = (\gamma(t), V(t)),$$

where $V(t)$ is the parallel vector field along γ with $V(0) = v$. The curve γ^H is called an *horizzontal lift of* γ.

The horizontal lift of a vector field X on M is the unique vector field X^H on TM which is horizontal and which projects onto X.

Let $(e_1, \ldots, e_n)$ be an orthonormal frame field defined on the open set $U \subset M$, and let $(x^1, \ldots, x^n)$ be a local coordinate system on U. We define a local coordinate system $(x^1, \ldots, x^n, v^1, \ldots, v^n)$ on $\pi^{-1}(U)$ as follows:

$$x^i(p, v) = x^i(p), \qquad v^i(p, v) = v^i, \qquad (p, v) \in \pi^{-1}(U), \tag{2.1}$$

where $v = \sum_i v^i e_i(p)$. We denote with Γ_j^i the local 1-forms defined by:

$$\nabla_X e_j = \sum \Gamma_j^i(X) e_i. \tag{2.2}$$

It is easy to verify that the horizontal lift X^H of a vector field X on M is given, in terms of the local coordinate system above, as follows:

$$X^H = X - \sum_{ij} \Gamma^i_j(X) v^j \frac{\partial}{\partial v^i} . \tag{2.3}$$

The *vertical lift* X^V is defined by:

$$X^V = \sum_i X^i \frac{\partial}{\partial v^i} . \tag{2.4}$$

Horizontal and vertical vectors generate two complementary distributions on TM: the *horizontal distribution* and the *vertical distribution.* Those two distributions are orthogonal with respect to g_s.

From (1.1) we obtain:

$$\begin{cases} g_s(X^H, Y^H) = g_s(X^V, Y^V) = g(X, Y)\circ\pi , \\ g_s(X^H, Y^V) = 0 , \end{cases} \tag{2.5}$$

for each pair of vector fields X and Y on M.

Clearly (2.5) uniquely determines the Sasaki metric. Then, according to (2.5) we have that $(e_1^H, \dots, e_n^H, e_1^V, \dots, e_n^V)$ is an orthonormal frame field on $\pi^{-1}(U)$ and its dual coframe is given by:

$$\pi^* e^1, \dots, \pi^* e^n , \qquad Dv^1, \dots, Dv^n , \tag{2.6}$$

where e^i denotes the 1-form defined by $e^i(e_k) = \delta^i_k$, and Dv^i is given by

$$Dv^i = dv^i + \sum_j v^j \pi^*(\Gamma^i_j \cdot) . \tag{2.7}$$

From (2.6) and (2.7) we have the following

PROPOSITION 2.1. – *The Sasaki metric g_s can be written as follows:*

$$g_s = \sum_i \pi^*(e^i)^2 + \sum_i (Dv^i)^2 . \tag{2.8}$$

REMARK 2.1. – Observe that the metric induced on the fiber $\pi^{-1}(p)$ is the Euclidean metric. In fact (2.6) and (2.7) imply that the restriction of g_s on $\pi^{-1}(p)$ is given by the quadratic form $\sum_i (dv^i)^2$. Hence g_s is the only metric on TM satisfying the following conditions:

a) horizontal and vertical distributions are orthogonal;

b) the metric induced on the fibers is Euclidean;

c) the projection π is a Riemannian submersion.

The fibers are also *totally geodesic* ([BE], p. 47).

REMARK 2.2. – Let $(e_1', \ldots, e_n')$ be an orthonormal frame defined on the open set $V \subset M$, and suppose that

$$e_i' = \sum_j a_i^j e_j \quad \text{on } U \cap V. \tag{2.9}$$

Then we have:

$$e_i'^H = \sum_j (a_i^j \circ \pi) e_j^H, \quad e_i'^V = \sum_j (a_i^j \circ \pi) e_j^V \quad \text{on } \pi^{-1}(U \cap V). \tag{2.10}$$

This implies that TM admits a natural $O(n) \times O(n)$ structure. Since $O(n) \times O(n)$ is a closed subgroup of $U(2n)$, we can also deduce that TM admits an almost complex structure J compatible with g_s. (For more details on the almost Hermitian manifold (TM, g_s, J) see [DO], [YI] and [BE], pp. 46-48).

3. – The curvature of the Sasaki metric.

The curvature of g_s has been computed by several authors with different methods (see [KW], [YI]). Proposition (2.1) permits the use of the moving frame method and of the structure equations of E. CARTAN.

First we put

$$\varphi^i = \pi^* e^i, \quad \varphi^{n+i} = Dv^i, \quad i = i \ldots n, \tag{3.1}$$

and we observe that $(\varphi^i, \ldots, \varphi^{2n})$ is an orthonormal coframe field. The local 1-forms φ_B^A of the Levi Civita connection of g_s are given by:

$$\begin{cases} d\varphi^A = -\sum_B \varphi_B^A \wedge \varphi^B, \\ \varphi_B^A + \varphi_A^B = 0. \end{cases} \tag{3.2}$$

The curvature forms Φ_B^A can be computed by using the formula:

$$\Phi_B^A = d\varphi_B^A + \sum_C \varphi_C^A \wedge \varphi_B^C. \tag{3.3}$$

From (3.2) and (3.3) we find:

$$\begin{cases} 2\varphi_j^i = 2\pi^* \Gamma_j^i + \sum_{l,m} v^m R_{ijml} \varphi^{n+l}, \\ 2\varphi_{n+j}^i = -2\varphi_i^{n+j} = \sum_{m,l} v^m R_{ilmj} \varphi^l, \\ \varphi_{n+j}^{n+i} = \pi^* \Gamma_i^i. \end{cases} \tag{3.4}$$

$$
(3.5)\quad \begin{cases}
4\Phi^i_j = \sum\limits_{rs}\Big(2R_{ijrs} - \sum\limits_{mlq} v^m v^l R_{ijmq}R_{rslq} - \sum\limits_{mlq} v^m v^l R_{irmq}R_{jslq}\Big)\varphi^r\wedge\varphi^s + \\
\qquad + 2\sum\limits_{rsm} v^m(\nabla_r R)_{ijms}\varphi^r\wedge\varphi^{n+s} + \sum\limits_{rs}\Big(2R_{ijrs} - \sum\limits_{ml} v^m v^l R_{iqmr}R_{jqls}\Big)\varphi^{n+r}\wedge\varphi^{n+l}, \\
4\Phi^i_{n+j} = -4\Phi^{n+j}_i = 2\sum\limits_{r} v^m(\nabla_r R)_{ismj}\varphi^r\wedge\varphi^s + \\
\qquad\qquad + \sum\limits_{rs}\Big(2R_{irjs} - \sum\limits_{lmq} v^m v^l R_{iqms}R_{qrlj}\Big)\varphi^r\wedge\varphi^{n+s}, \\
4\Phi^{n+i}_{n+j} = \sum\limits_{rs}\Big(2R_{ijrs} - \sum\limits_{lmq} v^m v^l R_{qrmi}R_{qslj}\Big)\varphi^r\wedge\varphi^s .
\end{cases}
$$

In the formulae (3.4) and (3.5) we have written R_{ijlm}, $(\nabla_r R)_{ijlm}\ldots$ instead of $R_{ijlm}\circ\pi$, $(\nabla_r R)_{ijlm}\circ\pi\ldots$, and R_{ijlm}, $(\nabla_r R)_{ijlm}\ldots$ denote the components of the curvature tensor R and its covariant derivative ∇R with respect the local frame $(e_1, \ldots, e_n)$.

Now we may state the following lemma:

LEMMA 3.1. – *Let $\bar\tau$ be the scalar curvature of g_s then:*

$$
(3.6)\qquad \bar\tau = \tau\circ\pi - \tfrac{1}{4}\sum R_{ijmq}R_{ijlq}v^m v^l .
$$

where τ is the scalar curvature of g.

PROOF. – Let $\bar R$ denote the curvature tensor of g_s. Then its components with respect to the local frame $(E_1, \ldots, E_{2n}) = (e_1^H, \ldots, e_n^H, e_1^V, \ldots, e_n^V)$ are given by:

$$
(3.7)\qquad \bar R_{ABCD} = 2\Phi^A_B(E_C, E_D) .
$$

Using (3.5) we find:

$$
\bar R_{ijij} = R_{ijij} - \tfrac{3}{4}\sum_{lmq} R_{ijmq}R_{ijlq}v^m v^l ,
$$

$$
\bar R_{i\,n+j\,i\,n+j} = \tfrac{1}{4}\sum_{lmq} R_{iqmj}R_{iqlj}v^m v^l ,
$$

$$
\bar R_{n+i\,n+j\,n+i\,n+j} = 0 .
$$

Then (3.6) follows.

The next proposition is an immediate consequence of (3.6).

PROPOSITION 3.2. – *(TM, g_s) has constant scalar curvature if and only if (M, g) is locally Euclidean.*

COROLLARY 3.3. – *(TM, g_s) is locally homogeneous if and only if g_s is locally Euclidean.*

In particular (see [KW]) (TM, g_s) is locally symmetric if and only if g_s is locally Euclidean.

Corollary 3.3 is still true assuming (TM, g_s) *curvature homogeneous* (see [SI]). In fact this assumption implies that the scalar curvature is constant.

COROLLARY 3.4. – *The Sasaki metric is Einstein if and only if it is locally Euclidean.*

4. – Other metrics on tangent bundles.

Let $\psi\colon OM\times\mathbb{R}^n\to TM$ be the map defined by:

$$\psi\colon (u,\xi)\to\Big(q,\sum_i \xi^i u_i\Big)\,, \tag{4.1}$$

where $u=(q,u_1,\dots,u_n)$ and $\xi=(\xi^1,\dots,\xi^n)$. ψ defines a submersion whose fiber are diffeomorphic to $O(n)$. This map is the canonical projection onto TM regarded as the vector bundle with standard fiber $\mathbb{R}^n$ associated to $O(M)$. Therefore, TM is identified with $OM\times\mathbb{R}^n/O(n)$, where the orthogonal group $O(n)$ acts on the right on OM as follows:

$$(u,\xi)\,a=(ua,a^{-1}\xi)=\Big(q,\sum_i a_1^i u_i,\dots,\sum_i a_n^i\,u_i,\sum_i a_1^i\xi^i,\dots,\sum_i a_n^i\,\xi^i\Big)\,. \tag{4.2}$$

Let now Q be a symmetric, semi-positive defined, tensor field of type $(2,0)$ and rank $2n$ on $OM\times\mathbb{R}^n$. Moreover, we assume that Q is *basic* for ψ. This means that Q is $O(n)$-invariant, and $Q(X,Y)=0$ if X is tangent to a fiber of ψ.

Under these assumptions, there is a unique Riemannian metric g_Q on TM such that $\psi^*(g_Q)=Q$. This metric is determined by the formula

$$g_u|_{(p,v)}(X,Y)=Q|_{(u,\xi)}(X',Y')\,,$$

where (u,ξ) belongs to the fiber $\psi^{-1}(p,v)$, X and Y are elements of $T_{(q,v)}(TM)$, X', Y' are tangent vectors of $OM\times\mathbb{R}^n$ at (u,ξ) with $d\psi(X')=X$ and $d\psi(Y')=Y$.

We observe that is easier to assign Q than to define directly g_Q on TM since $OM\times\mathbb{R}^n$ is parallelizable.

Let $\theta=(\theta^1,\dots,\theta^n)$ denote the *canonical* 1-*form* on OM, and let p be the natural projection $OM\xrightarrow{p}M$.

Then, according to the definition we get:

$$dp_u(X)=\sum_i\theta^i(X)\,u_i\,,\qquad u=(q,u_1,\dots,u_n)\,. \tag{4.4}$$

If we denote with $\omega=(\omega_j^i)$ the $so(n)$-valued differential form defined by the Levi Civita connection of g, then we find that:

$$\theta^i,\ i=1,\dots;n\,;\qquad \omega_k^h,\ 1\leqslant h\leqslant k\leqslant n\,;\qquad d\xi^i,\ i=1,\dots,n\,,$$

is an *absolute parallelism* on $OM\times\mathbb{R}^n$.

We recall two facts:

$$R_a^*(\theta^i) = \sum_h (a^{-1})_h^i \theta^h , \tag{4.5}$$

$$R_a^*(\omega_j^i) = \sum_{hk} (a^{-1})_h^i \omega_k^h a_j^k , \tag{4.6}$$

for each $a \in O(n)$. Moreover, the forms ω_j^i are related to the local 1-forms Γ_j^i defined in (2.2) as follows:

$$\omega_j^i = \sum_h (\psi_U^{-1})_h^i d(\psi_U)_j^h + \sum_{hk} (\psi_U^{-1})_h^i (p^* \Gamma_k^h)(\psi_U)_j^k . \tag{4.7}$$

ψ_U denotes the $O(n)$-valued function on $p^{-1}(U)$ given by:

$$(\psi_U)_j^i(u) = g\big(e_i|_{p(u)}, u_j\big) . \tag{4.8}$$

(4.7) can also be written in matrix form as follows:

$$\omega = \psi_U^{-1} d\psi_U + \psi_U^{-1}(p^* \Gamma) \psi_U . \tag{4.9}$$

Finally, it will be useful later on to note that:

$$p^* e^i = \sum_j (\psi_U)_j^i \theta^j . \tag{4.10}$$

LEMMA 4.1. – *The vertical distribution of ψ is defined by*:

$$\begin{cases} \theta^i = 0 , \\ D\xi^i = d\xi^i + \sum_j \xi^j \omega_j^i = 0 . \end{cases}$$

PROOF. – Let X be a vertical vector of ψ, then X is tangent at $t = 0$ to a curve of the form:

$$\alpha(t) = (ue^{tA}, e^{-tA}\xi) , \qquad A \in so(n) .$$

Then $X = \dot{\alpha}(0) = A^*|_u - A\xi$, A^* is the fundamental vector field on OM generated by A. It follows that

$$\theta^i(X) = 0 ,$$

and

$$D\xi^i(X) = -d\xi^i(A\xi) + \sum_j \xi^j \omega_j^i(A_u^*) = -\sum_j A_j^i \xi^j + \sum_j A_j^i \xi^j = 0 .$$

The converse is obvious.

In particular we see that any basic symmetric form Q on $OM \times \mathbb{R}^n$ is a second order polynomial in θ^i and $D\xi^i$ whose coefficient yield Q invariant under the $O(n)$-action.

For instance, consider

$$Q_s = \sum_i (\theta^i)^2 + \sum_i (D\xi^i)^2 . \tag{4.11}$$

From (4.2) and (4.6) we find:

$$R_a^*(D\xi^i) = \sum_h (a^{-1})^i_h D\xi^h . \tag{4.12}$$

Hence Q_s is basic (see (4.5)).

PROPOSITION 4.2. – *The metric induced by the quadratic form Q_s is the Sasaki metric.*

PROOF. – First, we observe that the diagram

$$\begin{tikzcd} OM \times \mathbb{R}^n \arrow[r, "\psi"] \arrow[d] & TM \arrow[d, "\pi"] \\ OM \arrow[r, "p"] & M \end{tikzcd}$$

commutes. Then, using (4.7) we get:

$$\psi^*(Dv^i) = \sum_j (\psi_U)^i_j D\xi^j . \tag{4.13}$$

Because (4.10) and because ψ_U is $O(n)$-valued, we see that $\psi^*(g_s) = Q_s$.

Consider now the quadratic form

$$g_{OM} = \sum_j (\theta^j)^2 + \sum_{h<k} (\omega^h_k)^2 + \sum_j (d\xi^j)^2 . \tag{4.14}$$

g_{OM} is a metric defined on $OM \times \mathbb{R}^n$; moreover, from (4.5), (4.6) and (4.2), we find that $O(n)$ is a group of isometries for g_{OM}.

Hence the metric g_{OM} projects onto a unique metric g_{CG} on TM so that the projection ψ becames a Riemannian submersion. *g_{CG} is the metric of Cheeger and Gromoll* (see [CG]).

If we assume that g is complete, we have that $\sum_i (\theta^i)^2 + \sum_{h<k} (\omega^h_k)^2$ is a complete Riemannian metric on OM (indeed p is a Riemannian submersion). Then it follows that g_{OM} and g_{CG} are both complete Riemannian metrics.

REMARK 4.1. – If (M, g) is the standard sphere (S^n, can), then $OM = O(n+1)$ and $\sum_i (\theta^i)^2 + \sum_{h<k} (\omega^h_k)^2$ is a biinvariant metric (see n. 6). Since g_{OM} is the product

of two Riemannian metrics with non-negative sectional curvatures, then g_{OM} itself has non-negative sectional curvature. Now, from O'Neill's formula it follows that the Cheeeger Gromoll metric g_{CG} has non-negative curvature (see [CG]).

Proposition 4.3. – The metric g_{CG} is induced by the tensor field

$$Q_{CG} = \sum_i (\theta^i)^2 + \frac{1}{1+\|\xi\|^2}\Big(\sum_i (D\xi^i)^2 + \Big(\sum_m \xi^m D\xi^m\Big)^2\Big). \tag{4.15}$$

Proof. – Let V_ψ be the vertical distribution of ψ, then its orthogonal complement H_ψ is defined by the equation:

$$\omega^h_k = \xi^k d\xi^h - \xi^h d\xi^k. \tag{4.16}$$

Thus, the restriction of g_{OM} on $H_\psi \times H_\psi$ agrees with the restriction of the tensor field given by

$$S = \sum_i (\theta^i)^2 + \sum_{h<k} (\xi^k d\xi^h - d\xi^h \xi^k)^2 + \sum_j (d\xi^j)^2. \tag{4.17}$$

We observe that:

$$\sum_i \xi^i d\xi^i = \sum_i \xi^i D\xi^i. \tag{4.18}$$

From (4.16) and (4.18) we find that:

$$d\xi^i|_{H_\psi} = (1+\|\xi\|^2)^{-1}\Big(D\xi^i + \xi^i \sum_m \xi^m D\xi^m\Big)\Big|_{H_\psi}. \tag{4.19}$$

Replacing (4.19) in (4.17), we get:

$$S|_{H_\psi\times H_\psi} = Q_{CG}|_{H_\psi \times H_\psi}.$$

Since ψ is a Riemannian submersion, we have: $\psi^* g_{CG} = g_{OM}|_{H_\psi\times H_\psi}$. We thus proved (4.15).

Remark 4.2. – From (4.15) we have a local expression of g_{CG}, namely:

$$g_{CG} = g + \frac{1}{1+\|v\|^2}\Big\{\sum_m (Dv^m)^2 + \Big(\sum_m v^m dv^m\Big)^2\Big\}. \tag{4.20}$$

Therefore, the metric induced on the fiber is

$$g_F = \frac{1}{1+\|v\|^2}\Big\{\sum_m (dv^m)^2 + \Big(\sum_m v^m dv^m\Big)^2\Big\} \tag{4.21}$$

which is not flat.

If $\dim M = 2$, we get

$$g_F = dr^2 + (1 + r^2)^{-1} r^2 d\alpha^2 , \tag{4.22}$$

where $v^1 = r\cos\alpha$, $v^2 = r\sin\alpha$ (see [CE], p. 146).

REMARK 4.3. – The Cheeger-Gromoll metric on TM is uniquely determined at the point (p, u) by the following conditions:

$$\begin{aligned} &g_{CG}(X^H, Y^H) = g_p(X, Y) , \\ &g_{CG}(X^V, Y^V) = (1 + \|u\|^2)^{-1}\big(g_p(X, Y) + g_p(X, u)\, g_p(Y, u)\big) , \\ &g_{CG}(X^H, Y^V) = 0 , \end{aligned} \tag{4.23}$$

where $X, Y \in T_p M$, and X^H, X^V are the horizontal and the vertical lifts of X (see n. 2). It is a «natural metric» on TM in the sense of [KWS].

5. – The group of isometries of g_s.

Let $G = I(M, g)$ denote the group of isometries of (M, g). There are two natural left actions of G on TM and on OM defined by:

$$L_a(q, v) = a(q, v) = \big(aq, dL_a(v)\big) , \quad a \in G,\ (q, v) \in TM , \tag{5.1}$$

$$L_a(u) = au = \big(aq, dL_a(u_1), \ldots, dL_a(u_n)\big) , \quad a \in G,\ u \in OM , \tag{5.2}$$

where dL_a denotes the differential of the map $L_a\colon M \to M$, $q \to aq$.

PROPOSITION 5.1. – *g_s is a G-invariant metric on TM.*

PROOF. – First we extend the action (5.2) on $OM \times \mathbb{R}^n$ by setting:

$$L_a(u, \xi) = a(u, \xi) = (au, \xi) . \tag{5.3}$$

The canonical 1-form θ and the Levi Civita connection form ω are G-invariant, i.e.

$$L_a^*(\theta^i) = \theta^i , \tag{5.4}$$

$$L_a^*(\omega_j^i) = \omega_j^i . \tag{5.5}$$

Since G acts trivially on $\mathbb{R}^n$, the differential forms $D\xi^i$ are G-invariant. The projection $\psi\colon OM \to TM$ commutes with the actions (5.1) and (5.2):

$$\psi \circ L_a = L_a \circ \psi , \quad a \in G .$$

From (4.11) we have that $\psi^*(g_s)$ is a G-invariant quadratic form. But:

$$\psi^*(L_a^* g_s) = L_a^*(\psi^* g_s) = \psi^* g_s .$$

Since ψ is a submersion, we obtain for each a in G

$$L_a^*(g_s) = g_s .$$

This proves the proposition.

Remark 5.1. – The same arguments hold if we exchange g_s with g_{CG}, hence the Cheeger-Gromoll metric g_{CG} is G-invariant.

Let u_0 be an orthonormal frame, and let σ_{u_0} be the map defined by

$$\sigma_{u_0}\colon G \to OM, \; a \to au_0 . \tag{5.6}$$

Since the action (5.2) is free, the map σ_{u_0} is an imbedding (see [KN], Vol. I, p. 4). Therefore $\dim G \leqslant \dim OM = \frac{1}{2}n(n+1)$, and the equality holds if and only if (M, g) is isometric to one of the following spaces of constant curvature:

i) the n-dimensional Euclidean space $\mathbb{R}^n$;

ii) the n-dimensional sphere S^n;

iii) the n-dimensional real projective space $\mathbb{R}P^n$;

iv) the n-dimensional hyperbolic space $\mathbb{H}^n$

(see [KO_2], p. 46). In these cases the group G is transitive on T_1M.

This property is characteristic of two-point homogeneous spaces, in fact we have

Proposition 5.2 ([WO], p. 289). – *G is transitive on the spherical tangent bundle if and only if (M, g) is a two-point homogeneous space.*

From prop. (5.1) we see that (T_1M, g_s') is an homogeneous Riemannian space if (M, g) is two-point homogeneous, although (TM, g_s) is never homogeneous unless g_s is flat.

Remark 5.2. – The orbits of the action (5.1) are M, regarded as the zero section of TM, and the spherical tangents bundles T_rM of radius r, $r > 0$. Then (TM, g_s) is a Riemannian space of *cohomogenity one*. The spherical bundles T_rM are the principal orbits of the action, moreover T_rM is a submanifold with *constant mean curvature* [SA], whereas M is totally geodesic.

Remark 5.3. – The tangent bundle TG of every Lie group G has a natural Lie group structure. Under the identification $TG = G \times \boldsymbol{g}$ the product is defined by:

$$(a, A)(b, B) = \big(ab, B + Ad(b^{-1})A\big) , \qquad a, b \in G;\; A, B \in \boldsymbol{g} . \tag{5.7}$$

If G acts on M, TG acts on the tangent bundle TM as follows:

$$L_{(a,A)}(q, v) = \big(aq, dL_a(v) + (Ad(a)A)^*|_{aq}\big) \tag{5.8}$$

where, for each $X \in \mathfrak{g}$, X^* denotes the induced fundamental vector-field on M. If G is transitive on M, then TG is transitive on TM. Clearly this does not mean that if g is a G-invariant metric on M, then g_s is TG-invariant. In fact, $L_{(e,A)}$ is an isometry of (TM, g_s) if and only if $\nabla_X A^* = 0$ for each vector field X on M.

6. – Tangent sphere bundle.

In section 5 we have already noted that the tangent sphere bundle of a two point homogeneous space endowed with the metric g_s' is a Riemannian homogeneous space. In this section we will investigate the tangent sphere bundle T_1S^n of the standard n-sphere, equipped with the induced Sasaki metric, and we will generalize some of the results proven in [KS].

First we need some general facts concerning the tangent sphere bundle of a Riemannian manifold.

Consider the map $\psi_n\colon OM \to T_1M$ defined by:

$$\psi_n\colon (q, u_1, \ldots, u_n) \to (q, u_n) \tag{6.1}$$

(see [CH$_1$], p. 36). This map is a submersion whose fibers are diffeomorphic with $O(n-1)$, identified to the subgroup of $O(n)$ of the matrices $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $a \in O(n-1)$.

Then T_1M can be regarded as the quotient space $OM/O(n-1)$ and ψ_n is the natural projection. Now we shall prove the following proposition:

PROPOSITION 6.1. – *Let g_s' be the induced Sasaki metric on T_1M, then we have*

$$\psi_n^*(g_s') = \sum_i (\theta^i)^2 + \sum_i (\omega_n^i)^2 .$$

PROOF. – First we observe that the following diagram

$$\begin{tikzcd} OM \arrow[r, "\psi_n"] \arrow[dr, "p"'] & T_1M \arrow[r, hook, "i"] & TM \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

commutes. Hence, using (4.9) and (4.10) we get:

$$(i \circ \psi_n)^*(\pi^* e^i) = p^*(e^i) = \sum_j (\psi_U)^i_j \theta^j , \tag{6.2}$$

$$(i \circ \psi_n)^*(\pi^* \Gamma) = p^*(\Gamma) = \psi_U \omega \psi_U^{-1} - d\psi_U \psi_U^{-1} . \tag{6.3}$$

Because $(v^i \circ \psi_n)(u) = v^i(u_n) = (\psi_U)^i_n(u)$, we obtain:

$$(i \circ \psi_n)^* (Dv^i) = \sum_j (\psi_U)^i_j \omega^j_n \,. \tag{6.4}$$

Since $\psi_n^*(g'_s) = \psi_n^*(i^* g_s) = (i \circ \psi_n)^* g_s$ and ψ_U is an $O(n)$-valued function, the formulae (6.2), (6.3) and (6.4) imply proposition (6.1).

REMARK 6.1. – It is an elementary matter to check that the quadratic form $Q = \sum_i (\theta^i)^2 + \sum_i (\omega^i_n)^2$ is $O(n-1)$-invariant and $Q(X, Y) = 0$ if one of X and Y is vertical for ψ_n. Hence Q is *basic*. As for n. 4, we may characterize g'_s as the only metric on T_1M satisfying $\psi_n^*(g'_s) = Q$.

Moreover, from proposition 4.3 it follows that the metric g'_{CG} induced on T_1M by the Cheeger-Gromoll metric is uniquely characterized by the following condition:

$$\psi_n^*(g'_{CG}) = \sum (\theta^i)^2 + \tfrac{1}{2} \sum_i (\omega^i_n)^2 \,.$$

PROPOSITION 6.2. – (T_1S^n, g'_s) is isometric to the *Stiefel manifold* $SO(n+1)/SO(n-1)$ equipped with a metric induced from a biivariant one on $SO(n+1)$.

PROOF. – Let $O_+(S^n)$ denote the bundle of positive orthonormal frames on S^n. Let u_0 be a point of $O_+(S^n)$ and let p be the canonical projection from $SO(n+1)$ onto S^n, then the diagram

$$\begin{tikzcd}
SO(n+1) \arrow[r, "\sigma_{u_0}"] \arrow[dr, "\hat{p}"'] & O_+(S^n) \arrow[r, "\psi_n"] \arrow[d, "p"] & T_1S^n \arrow[r, hook, "i"] \arrow[dl, "\pi"] & TS^n \\
 & S^n = SO(n+1)/SO(n) & &
\end{tikzcd}$$

commutes. Hence T_1S^n is diffeomorphic to the Stiefel manifold $SO(n+1)/SO(n-1)$.

Indeed, a point of T_1S^n is given by a pair of orthonormal vectors in $\mathbb{R}^{n+1}$. To conclude the proof we must check a few facts.

First consider the 1-forms on $SO(n+1)$ defined by:

$$\bar{\theta}^i = \sigma^*_{u_0} \theta^i \,, \qquad i = 1, \ldots, n \,, \tag{6.5}$$

$$\bar{\omega}^i_j = \sigma^*_{u_0}(\omega^i_j) \,, \quad 1 \leqslant i < j \leqslant n \,. \tag{6.6}$$

FACT 1. – $\{\bar{\theta}^i, \bar{\omega}^h_j\}$, $1 \leqslant h < j \leqslant n$ are left-invariant and linearly independent 1-forms, therefore they are Maurer-Cartan forms of $SO(n+1)$.

Proof. – Let $SO(n+1)$ act on $O_+(M)$ (see (5.2)). Then, for each $a, b \in SO(n+1)$, we have:

$$\sigma_{u_0} \circ L_a = L_a \circ \sigma_{u_0}, \tag{6.7}$$

where in the left hand side of (6.7) L_a denotes the left translation of $SO(n+1)$.

Then we get

$$L_a^*(\theta)^i = (\sigma_{u_0} \circ L_a)^* \theta^i = \sigma_{u_0}^* L_a^* \theta^i .$$

In the same way,

$$L_a^*(\omega_j^i) = \sigma_{u_0}^* L_a^*(\omega_j^i) .$$

Since $SO(n+1)$ acts as an isometry group on S^n, the canonical 1-form and the connection form ω are $SO(n+1)$-invariant. This implies Fact 1.

Taking the structure equations of S^n and making use of $\Omega_j^i = \theta^i \wedge \theta^j$ one may see that $\bar{\theta}^i$ and $\bar{\omega}_j^i$ satisfy the following equations:

$$\begin{cases} d\bar{\theta}^i = -\sum_h \bar{\omega}_h^i \wedge \bar{\theta} , \\ d\bar{\omega}_j^i = \bar{\theta}^i \wedge \bar{\theta}^j - \sum_h \bar{\omega}_h^i \wedge \omega_j^h . \end{cases} \tag{6.8}$$

Fact 2. – Let $\bar{g}$ denote the quadratic-form $\sum_i (\bar{\theta}^i)^2 + \sum_{i<j} (\bar{\omega}_j^i)^2$, then $2n\bar{g} = -B$, where B is the Killing form of $SO(n+1)$. Therefore $\bar{g}$ is a biinvariant metric on $SO(n+1)$.

Proof. – $so(n+1)$ is isomorphic with the Lie algebra of left invariant vector fields on $SO(n+1)$. An explicit isomorphism can be defined taking the dual basis $\{\bar{E}_k, \bar{E}_i^j\}$ of $\{\bar{\theta}^k, \bar{\omega}_i^j\}$, hence any left invariant vector field X can be written in the form:

$$X = \sum_i \bar{\theta}^i(X) \bar{E}_i + \sum_{i<j} \omega_j^i(X) \bar{E}_i^j . \tag{6.9}$$

The isomorphism is given by the mapping:

$$X \to (A, \xi) ,$$

where

$$(A, \xi) = \begin{pmatrix} A & \xi \\ -{}^T\xi & 0 \end{pmatrix}, \quad A = (\bar{\omega}_j^i(X)) , \quad {}^T\xi = (\bar{\theta}^1(X), \ldots, \bar{\theta}^n(X)) .$$

Then the Killing form is is given by.

$$B(X, X') = -n\,\mathrm{tr}\,(A, \xi)\circ(A', \xi') = -2n\Big\{\sum_{i,j} \bar{\omega}^i_j(X)\,\bar{\omega}^i_j(X') + \sum_i \bar{\theta}^i(X)\bar{\theta}^i(X')\Big\}, \qquad \text{q.e.d.}$$

Proposition (6.2) follows from Facts 1 and 2, and from proposition (6.1).

REMARK 6.2. - The above proposition implies that T_1S^n with the metric g'_s is a *normal* Riemannian homogeneous space, therefore (T_1S^n, g'_s) is *naturally reductive.* A naturally reductive decomposition of $so(n+1)$ is given by

$$so(n+1) = \boldsymbol{m} \oplus \boldsymbol{k}\,, \tag{6.10}$$

where

$$\boldsymbol{m} = \left\{\begin{pmatrix} 0 & \eta & \xi \\ -{}^T\eta & 0 & \alpha \\ -{}^T\xi & -\alpha & 0 \end{pmatrix} = (\eta, \xi, \alpha)/{}^T\xi,\ {}^T\eta \in \mathbb{R}^n,\ \alpha \in \mathbb{R}\right\}, \tag{6.11}$$

and

$$\boldsymbol{k} = \left\{\begin{pmatrix} B & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} /B \in so(n-1)\right\}. \tag{6.12}$$

The geodesics of (T_1S^n, g'_s) through the origin $(q_0, u_0) = \psi_n\sigma_{u_0}(e)$ are orbits of 1-parameter subgroups generated by elements X belonging to $\boldsymbol{m}$. In the two-dimensional case all geodesics are closed (see [KS]). Of course, for $n \geqslant 3$ this property is no longer true. For instance, the curve

$$\gamma(t) = \exp(tX)$$

where $X = (\eta, \xi, 0)$, ${}^T\eta = (0, \ldots, a, 0)$, ${}^T\xi = (0, \ldots, 0, b)$ and a/b irrational, is a geodesic, dense in a torus contained in T_1S^n, which is not closed.

7. - Deformation of the metric g'_s.

In section 6 we noted that on the spherical tangent bundle T_1M we may define Riemannian metrics in analogy with section 4 (see remark 6.1). For instance, since θ^n is an invariant differential form under the $O(n-1)$ action on OM, and since θ^n vanishes on the vertical vectors of the fibration ψ_n, we may deduce that the quadratic form

$$Q_t = \sum_{i=1}^{n-1} (\theta^i)^2 + t^2(\theta^n)^2 + \sum_i (\omega^i_n)^2\,, \qquad t \neq 0\,, \tag{7.1}$$

induces a metric tensor g_t on T_1M which is uniquely determined by the condition $\psi_n^*(g_t) = Q_t$.

In this section we will show that if (M, g) is a space of constant sectional curvature $\sigma = 1$, then the metric g_t is Einstein when $t^2 = 2/n(n-1)$. Hence $g'_s = g_1$ can be an Einstein metric only if $\dim M = 2$. Indeed (see [KS]), the Riemannian manifold (T_1S^2, g'_s) is isometric to the real projective space $\mathbb{R}P^3$ endowed with a metric of constant positive curvature $\frac{1}{4}$.

The Einstein metrics g_t, $t^2 = 2/n(n-1)$, on T_1S^n was defined by S. KOBAYASHI in [KO$_1$] as a particular case of Einstein metrics on S^1-principal bundles. This construction was generalized by G. JENSEN ([Je]) to other homogeneous spaces, for instance if M is a Stiefel manifold.

Since ψ_n^* is injective, the Riemann and the Ricci curvature tensors are uniquely determined by their pull-backs on OM. Therefore we will work on OM, leaving the computational details to the reader.

First we put:

$$\varphi^i = \theta^i\,, \quad \varphi^n = t\theta^n\,, \quad \varphi^{n+i} = \omega_n^i\,, \quad i = 1, \dots, n-1\,. \tag{7.2}$$

From now on we shall employ the following ranges of indices:

$$i, j, k, h, \ldots = 1, \ldots, n-1\,; \quad A, B, C \ldots = 1, \ldots, 2n-1\,; \quad a, b, c \ldots = 1, \ldots, n\,.$$

We consider the following equations:

$$\begin{cases} d\varphi^A = -\sum\limits_B (\varphi_t)_B^A \wedge \varphi^B\,, \\ (\varphi_t)_B^A + (\varphi_t)_A^B = 0\,. \end{cases} \tag{7.3}$$

To compute the differentials $d\varphi^A$ we make use of (7.2) and of the structure equations for the metric g on M. Recall that the structure equations of g can be written as follows:

$$\begin{cases} d\theta^a = -\sum\limits_a \omega_b^a \wedge \theta^b \\ \Omega_b^a = d\omega_b^a + \sum\limits_c \omega_c^a \wedge \omega_b^c\,. \end{cases} \tag{7.4}$$

The 1-forms $(\theta^a)_{a=1,\dots,n}$, $(\omega_b^a)_{a,b=1,\dots,n}$ are respectively $\mathbb{R}^n$-valued and $so(n)$-valued differential forms globally defined on OM. Moreover:

$$2\Omega_b^a = \sum_{ad} \bar{R}_{abcd}\,\theta^c \wedge \theta^d\,, \tag{7.5}$$

where $\bar{R}_{abcd}$ must be regarded as real-valued functions on OM related to the Riemann curvature tensor R by the formula:

$$\bar{R}_{abcd}(u) = R|_{p(u)}(u_a, u_b, u_c, u_d)\,, \quad u \in OM\,. \tag{7.6}$$

Now, using Cartan's lemma we see that (7.3) uniquely characterize the forms $(\varphi_t)^A_B$: It is a routine matter to deduce the following expression:

$$
(7.7)\qquad \begin{cases}
(\varphi_t)^i_j = \omega^i_j + \frac{1}{2}\sum_h \bar{R}_{ijnh}\varphi^{n+h}\,, \\
(\varphi_t)^i_n = -(\varphi_t)^n_i = \dfrac{t^2+1}{2t}\varphi^{n+i} - \dfrac{1}{2t}\sum_h \bar{R}_{ninh}\varphi^{n+h}\,, \\
(\varphi)^i_{n+j} = -(\varphi_t)^{n+j}_i = -\frac{1}{2}\sum_h \bar{R}_{hinj}\varphi^h + \dfrac{t^2-1}{2t}\delta^i_j\varphi^n - \dfrac{1}{2t}\bar{R}_{ninj}\varphi^n\,, \\
(\varphi_t)^n_{n+j} = -(\varphi_t)^{n+j}_u = \dfrac{t^2-1}{2t}\varphi^j + \dfrac{1}{2t}\sum_h \bar{R}_{nhnj}\varphi^h\,, \\
(\varphi_t)^{n+i}_{n+j} = \omega^i_j\,.
\end{cases}
$$

Let $(\Phi_t)^A_B$ be the forms given by the equation:

$$
(7.8)\qquad (\Phi_t)^A_B = d(\varphi_t)^A_B + \sum_C (\varphi_t)^A_C \wedge (\varphi_t)^C_B\,.
$$

Then

$$
(7.9)\qquad R_t = 2\sum (\Phi_t)^A_B \otimes (\varphi^A \wedge \varphi^B)
$$

is the pull-back of the Riemann curvature tensor of g_t. The pull-back of the Ricci tensor is given by

$$
(7.10)\qquad \varrho_t = \sum_{A,B} (\varrho_t)_{AB}\varphi^A \otimes \varphi^B\,,
$$

where the components $(\varrho_t)_{AB}$ are given by the formula:

$$
(7.11)\qquad \begin{cases}
(\varrho_t)_{ij} = \bar{R}_{ij} - \dfrac{t^4-1}{2t^2}\delta_{ij} - \dfrac{1}{2t^2}\sum_k \bar{R}_{nink}\bar{R}_{njnk} - \frac{1}{2}\sum_{kr}\bar{R}_{irnk}\bar{R}_{jrnk}\,, \\
(\varrho_t)_{in} = \dfrac{4t-(t-1)^2(t+1)}{4t^2}\bar{R}_{in} - \dfrac{1}{2t}\sum_{kr}\bar{R}_{irnk}\bar{R}_{nrnk}\,, \\
(\varrho_t)_{in+j} = \dfrac{1}{2t}(\nabla_n\bar{R})_{ninj} - \frac{1}{2}\sum_r (\nabla_r\bar{R})_{irnj}\,, \\
(\varrho_t)_{nn} = \dfrac{t^4-1}{2t^2}(n-1) + \dfrac{1}{t^2}\bar{R}_{nn} - \dfrac{1}{2t^2}\sum_{rs}\bar{R}_{nrns}\bar{R}_{nrns}\,, \\
(\varrho_t)_{n\,n+j} = -\frac{1}{2}\sum_r (\nabla_r\bar{R})_{nrnj}\,, \\
(\varrho_t)_{n+i\,n+j} = \left\{(n-2) - \dfrac{(t^2-1)^2}{2t^2}\right\}\delta_{ij} + \dfrac{1}{2t^2}\sum_r \bar{R}_{nrni}\bar{R}_{nrnj} + \frac{1}{4}\sum_{rs}\bar{R}_{rsni}\bar{R}_{rsnj}\,,
\end{cases}
$$

with

$$
(7.12)\qquad d(\bar{R}_{abcd}) = \sum_{m=1}^{n}\{(\nabla_m\bar{R})_{abcd}\theta^m + \bar{R}_{mbcd}\omega^m_a + \bar{R}_{amcd}\omega^m_b + \bar{R}_{abmd}\omega^m_c + \bar{R}_{abcm}\omega^m_d\}\,.
$$

If (M, g) is a space of constant curvature λ, (7.11) may be re-written as follows:

$$\left\{\begin{aligned}(\varrho_t)_{ij} &= -\frac{t^4-2\lambda(n-1)t^2+\lambda^2-1}{2t^2}\delta_{ij}\,,\\ (\varrho_t)_{nn} &= \frac{t^4-(\lambda-1)^2}{2t^2}(n-1)\,,\\ (\varrho_t)_{n+j\,n+j} &= -\frac{t^4-2(n-1)t^2-\lambda^2+1}{2t^2}\delta_{ij}\,.\end{aligned}\right. \tag{7.13}$$

where the components that do not appear in (7.13) vanish identically on OM. From (7.13) we obtain

PROPOSITION 7.1. – *Let (M, g) be a space of constant curvature λ, then (T_1M, g_t) is an Einstein space if either $\lambda = 1$, and $t^2 = 2((n-1)/n)$ or else $\lambda = 0$ and $n = 1$, $t^2 = 1$.*

Naturally, the latter case is trivial.

REMARK 7.1. – Equation (7.1) may be written in the form

$$Q_t = \sum_a (\theta^a)^2 + \sum_i (\omega_n^i)^2 + (t^2-1)(\theta^n)^2\,. \tag{7.14}$$

Therefore,

$$g_t = g'_s + (t^2-1)\gamma\,, \tag{7.15}$$

where γ is the 1-form induced on T_1M by θ^n. Since $\psi_n^*(\gamma) = \theta^n$, we get

$$\gamma|_{(a,v)}(X) = g_a(d\pi(X), v)\,, \qquad X \in T_{(a,v)}(T_1M)\,. \tag{7.16}$$

Then γ is the restriction to T_1M of the *Liouville form* of TM (see [BE], p. 21), i.e. γ is the *canonical contact form* on T_1M (see [CH_2] or [BL]).

Thus *the Einstein metric on T_1S^n defined by Kobayashi can be obtained by deforming the induced Sasaki metric g'_s along the direction of the canonical contact form of T_1S^n.*

Clearly the projection π: $T_1S^n \to S^n$ is no longer a Riemannian submersion. This is the price to be paid for an Einstein metric. In context, see CALABI [CA] where the construction of Kähler metrics on holomorphic vector bundles is discussed.

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