Corrigendum

Inverse mean value property of harmonic functions

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Let λ denote Lebesgue measure on $\mathbb{R}^d$, $d \geq 2$, and for every Lebesgue measurable subset A of $\mathbb{R}^d$ such that $0 < \lambda(A) < \infty$ let λ_A denote the normalized Lebesgue measure on A , i.e., $\lambda_A = (\lambda(A))^{-1}1_A$. Moreover define

$$G(z) = \begin{cases} -\log|z| & \text{when } d = 2, \\ |z|^{2-d} & \text{when } d > 2. \end{cases}$$

Our main result in [1] is the following:

Theorem. *Let A be a Lebesgue measurable subset of $\mathbb{R}^d$, $d \geq 2$, such that $0 < \lambda(A) < \infty$ and let B denote the open ball of centre 0 such that $\lambda(B) = \lambda(A)$. Then $\lambda_A(h) = h(0)$ holds for every bounded $h = G^{1_C} - G^{1_D}$, C and D compact sets contained in $\mathbb{C}A$, if and only if $\lambda(B \setminus A) = 0$.*

The proof is based on Lemma 2.1 where we claimed the existence of a constant $c > 0$ such that the following holds: For every unit vector $\theta \in \mathbb{R}^d$, all $a, b \in \mathbb{R}$ satisfying $0 < a < b < \sqrt{2}a$ and every compact subset K of $\mathbb{C}B_b$, there exist neighborhoods V and W of $x = a\theta$ and $y = b\theta$, respectively, such that for all probability measures μ on V and ν on W the function

$$h = \begin{cases} \frac{1}{b/a - 1} (G^\nu - G^\nu(0) - (G^\mu - G^\mu(0))) & \text{when } d = 2 \\ \frac{1}{b/a - 1} (G^\nu/G^\nu(0) - (G^\mu/G^\mu(0))) & \text{when } d > 2 \end{cases}$$

satisfies $h < c$ on $\overline{B_a}$, $h(z) > (1/c)(a/\|z\|)^{d-2}$ for every $z \in K$ and

$$\int_{\{h < 0\} \setminus B_a} |h| d\lambda < ca^d \sqrt{b/a - 1}.$$

We are indebted to D. Singman for having pointed out that, contrary to our statement, the functions h have no upper bound on $\bar{B}_a$ if $d > 2$. Using continuity properties it was sufficient to consider the special case $\mu = \varepsilon_x$ and $\nu = \varepsilon_y$. Unfortunately, the inequality $g(z) \leq g(-x)$ at the beginning of the proof of the lemma is erroneous.

If $d = 2$ we know, however, that

$$h := \frac{1}{b/a - 1} ((G^{\varepsilon_y} - G^{\varepsilon_y}(0)) - (G^{\varepsilon_x} - G^{\varepsilon_x}(0))) < 1$$

on $\bar{B}_a$ since obviously $G^{\varepsilon_y} - G^{\varepsilon_x} < 0$ on $\bar{B}_a$ and

$$\frac{G^{\varepsilon_x}(0) - G^{\varepsilon_y}(0)}{b/a - 1} = \frac{\ln(b/a)}{b/a - 1} < 1.$$

So for $d = 2$ the proof of the theorem itself needs no modification.

If $d > 2$ we have

$$\begin{aligned} h &:= \frac{1}{b/a - 1} (G^{\varepsilon_y}/G^{\varepsilon_y}(0) - G^{\varepsilon_x}/G^{\varepsilon_x}(0)) \\ &= \frac{1}{b/a - 1} \left[\frac{G^{\varepsilon_y} - G^{\varepsilon_x}}{G^{\varepsilon_x}(0)} + \left(\frac{1}{G^{\varepsilon_y}(0)} - \frac{1}{G^{\varepsilon_x}(0)} \right) G^{\varepsilon_y} \right] \end{aligned}$$

where again $G^{\varepsilon_y} \leq G^{\varepsilon_x}$ on $\bar{B}_a$. Moreover,

$$\begin{aligned} \frac{1}{b/a - 1} \left(\frac{1}{G^{\varepsilon_y}(0)} - \frac{1}{G^{\varepsilon_x}(0)} \right) &= \frac{b^{d-2} - a^{d-2}}{b/a - 1} \\ &= a^{d-2} \frac{(b/a)^{d-2} - 1}{b/a - 1} \leq (d-2)(\sqrt{2})^{d-3} a^{d-2}. \end{aligned}$$

So in Lemma 2.1 we should have written that $h < ca^{d-2}G^{\varepsilon_x}$ on $\bar{B}_a$.

It is easy to get the necessary modification in the proof of our theorem for $d > 2$: Assume that $\lambda(B \setminus A) > 0$, but $\lambda_A(h) = h(0)$ for every $h = G^{\lambda_C}$ with C being a compact subset of $\bar{C}A$. As in [1] define

$$r := \sup\{\rho \geq 0: \lambda(B_\rho \setminus A) = 0\}.$$

Let us note first that $r > 0$. Indeed, suppose the contrary and fix a point z in the boundary of B . Obviously,

$$\int_{A \setminus B} G^{\varepsilon_0} d\lambda < \lambda(A \setminus B)G^{\varepsilon_0}(z) = \lambda(B \setminus A)G^{\varepsilon_0}(z) < \int_{B \setminus A} G^{\varepsilon_0} d\lambda.$$

Adding $\int_{A \cap B} G^{\varepsilon_0} d\lambda$ we obtain that $\int_A G^{\varepsilon_0} d\lambda < \int_B G^{\varepsilon_0} d\lambda$, hence $G^{\lambda_A}(0) < G^{\lambda_B}(0)$ (recall that $\lambda(A) = \lambda(B)$). By continuity there exists $\delta > 0$ such that $G^{\lambda_A} < G^{\lambda_B}$ on B_δ . Having assumed that $r = 0$ there exists a compact subset C of $B_\delta \setminus A$ such that $\lambda(C) > 0$. Taking $f = G^{\lambda_C}$ we then know that $\lambda_A(f) = \lambda_C(G^{\lambda_A}) < \lambda_C(G^{\lambda_B}) = \lambda_B(f)$. On the other hand $\lambda_A(f) = f(0) \geq \lambda_B(f)$ since f is superharmonic. This contradiction shows that $r > 0$.

We now choose $K, \eta, \varepsilon, a, b, \mu, \nu$ as in [1]. Considering $h := (b/a - 1)(G^\nu / G^\mu(0) - G^\mu / G^\mu(0))$ we obtain that

$$0 = \int_A h d\lambda - \int_{B_a} h d\lambda = \int_{A \setminus B_a} h d\lambda - \int_{B_a \setminus A} h d\lambda.$$

Since $h < ca^{d-2}G^{\varepsilon_x}$ on $\overline{B_a}$ we have

$$\begin{aligned} \int_{B_a \setminus A} h d\lambda &\leq \int_{B_a \setminus B_r} h^+ d\lambda \\ &\leq ca^{d-2} \int_{B_a \setminus B_r} G^{\varepsilon_x} d\lambda \leq c(r + \varepsilon)^{d-2} \int_{B_{r+\varepsilon} \setminus B_r} G^{\varepsilon_x} d\lambda. \end{aligned}$$

Note that $\int_{B_{r+\varepsilon} \setminus B_r} G^{\varepsilon_x} d\lambda$ is as small as we want since the functions $1_{B_{r+1}} G^{\varepsilon_y}, y \in \mathbb{R}^d$, are uniformly integrable. Furthermore

$$\int_{A \setminus B_a} h d\lambda \geq \int_K h d\lambda - \int_{\{h < 0\} \setminus B_a} |h| d\lambda \geq \eta r^{d-2} - c(r + \varepsilon)^d \sqrt{\varepsilon}.$$

So we conclude that $\int_{B_a \setminus A} h d\lambda < \int_{A \setminus B_a} h d\lambda$ if ε is sufficiently small. This contradiction shows that $\lambda(B \setminus A) = 0$ finishing the proof.

Reference

1. Hansen, W., Netuka, I.: Inverse mean value property of harmonic functions, Math. Ann. **297** (1993), 147–156