Erratum

The Radon Transform for Forms of Type (1, 1) on Complex Projective Space

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Propositions 1 and 2 are true only for $n=2$. If $n \geq 3$ and γ is the element $(q+1)\lambda_0 + \lambda_1 - \lambda_{n-1} - (q+1)\lambda_n$ of $\hat{G}$, with $q \geq 0$, the multiplicity of $C_\gamma^\infty(T^{1,1})$ is equal to one (see [8, Proposition 3.4]). Consequently, Theorem 1 holds only when $n=2$, and is trivial in this case. The proof of Theorem 2 remains valid when $n=2$. Based on joint work with Jacques Gasqui, we now proceed to give a proof of Theorem 2 for all dimensions $n \geq 2$, and to generalize this result to forms of type (p, p) .

Let X be a compact Kähler manifold of dimension n . Let T' be the bundle of holomorphic tangent vectors. Let $\mathcal{A}^p$ be the space of complex-valued forms of degree p on X . We write $\mathcal{A}^{p,q} = C^\infty(T^{p,q})$ and let $\pi_{p,q}$ be the natural projection of $\mathcal{A}^{p+q}$ onto its subspace $\mathcal{A}^{p,q}$. Let $\mathbf{H}^{p,q}$ denote the space of harmonic forms of type (p, q) .

Standard Hodge theory gives us the closed orthogonal decomposition

$$\begin{aligned} \mathcal{A}^{p,q} = \mathbf{H}^{p,q} \oplus \partial\bar{\partial}\mathcal{A}^{p-1,q-1} \oplus \partial\bar{\partial}^*\mathcal{A}^{p-1,q+1} \\ \oplus \bar{\partial}\partial^*\mathcal{A}^{p+1,q-1} \oplus \partial^*\bar{\partial}^*\mathcal{A}^{p+1,q+1}, \end{aligned} \quad (1)$$

from which we immediately deduce

Lemma A. *If $\alpha \in \mathcal{A}^{p,q}$ satisfies $\partial\bar{\partial}\alpha = 0$, then there exist $\varphi \in \mathbf{H}^{p,q}$, $\beta_1 \in \mathcal{A}^{p-1,q}$, $\beta_2 \in \mathcal{A}^{p,q-1}$ such that*

$$\alpha = \varphi + \partial\beta_1 + \bar{\partial}\beta_2.$$

Let ω^p denote the p^{th} exterior power of ω and $\{\omega^p\}$ the sub-bundle of $T^{p,p}$ generated by its section ω^p . If $A: T^{p,q} \rightarrow T^{p-1,q-1}$ is the trace morphism induced by interior multiplication by ω , from the standard formulas

$$\bar{\partial}^* = i(A\partial - \bar{\partial}A), \quad \partial^* = -i(A\bar{\partial} - \partial A),$$

there are constants c_1, c_2 such that

$$\bar{\partial}^*\partial^*(f\omega^p) = c_1(Af)\omega^p + c_2\partial\bar{\partial}f \wedge \omega^{p-1},$$

for all $f \in C^\infty(X)$. Since $T^{n,n}$ is generated by its section ω^n , we obtain the inclusion

$$\partial^*\bar{\partial}^*\mathcal{A}^{n,n} \subset C^\infty(\{\omega^{n-1}\}) + \partial\bar{\partial}C^\infty(\{\omega^{n-2}\}). \quad (2)$$

If $\mathbf{H}_0^{p,p}$ denotes the subspace of $\mathbf{H}^{p,p}$ of harmonic forms of type (p, p) whose trace vanishes, (1) and (2) give us:

Lemma B. *We have the decomposition*

$$\mathcal{A}^{n-1, n-1} = (\mathbf{H}_0^{n-1, n-1} \oplus \partial\bar{\partial}\mathcal{A}^{n-2, n-2} \oplus \partial\bar{\partial}^*\mathcal{A}^{n-2, n} \\ \oplus \bar{\partial}\partial^*\mathcal{A}^{n, n-2}) + C^\infty(\{\omega^{n-1}\}).$$

When $X = \mathbf{CP}^2$, we recover the result of Proposition 2.

The proof of the following lemma is elementary and is left to the reader.

Lemma C. *Let $\alpha \in T^{p,p}$, with $p \geq 1$. If*

$$\alpha(\xi_1, \dots, \xi_p, \bar{\xi}_1, \dots, \bar{\xi}_p) = 0,$$

for all $\xi_1, \dots, \xi_p \in T'$, then α vanishes.

For $p \geq 1$, we consider the first-order differential operator

$$D = \pi_{p,p} d : \mathcal{A}^{2p-1} \rightarrow \mathcal{A}^{p,p}.$$

If $\alpha \in \mathcal{A}^{2p-1}$, we have

$$D\alpha = \bar{\partial}\pi_{p,p-1}\alpha + \partial\pi_{p-1,p}\alpha,$$

and

$$\partial\bar{\partial}D = 0;$$

moreover, the integral of $D\alpha$ vanishes over an arbitrary closed complex submanifold of X of dimension p .

We now suppose that $X = \mathbf{CP}^n$. From the preceding lemma and the description of the tangent spaces of the linearly imbedded complex projective p -planes in $\mathbf{CP}^n$ given by [1, p. 79], we obtain:

Lemma D. *Let $X = \mathbf{CP}^n$ and α be an element of $\mathcal{A}^{k,k}$. The following assertions are equivalent:*

- (i) $\partial\bar{\partial}\alpha = 0$;
- (ii) *for every complex projective $(k+1)$ -plane Y of $\mathbf{CP}^n$, the restriction $\alpha|_Y$ of α to Y satisfies $\partial\bar{\partial}(\alpha|_Y) = 0$.*

The following result is a generalization of Theorem 1.

Theorem. *Let $X = \mathbf{CP}^n$ and $\alpha \in \mathcal{A}^{k,k}$, with $n \geq 2$ and $1 \leq k \leq n-1$. Then the following statements are equivalent:*

- (i) *The integral of α over every complex projective k -plane of $\mathbf{CP}^n$ vanishes;*
- (ii) *There exists an element β of $\mathcal{A}^{2k-1}$ such that $D\beta = \alpha$.*

Proof. It suffices to prove (i) $\Rightarrow$ (ii). Assume that (i) holds. Let Y be a complex projective $(k+1)$ -plane of $\mathbf{CP}^n$ and ω_Y be its Kähler form. According to Lemma B, we may write

$$\alpha|_Y = f\omega_Y^k + D\varphi,$$

where f is a complex-valued function and β is a form of degree $2k-1$ on Y . Condition (i) implies that the integral of f vanishes over every complex projective k -plane of Y . Thus, by Proposition 3, (i), f vanishes, and so $\partial\bar{\partial}(\alpha|_Y)=0$. From Lemmas A and D, we infer that

$$\alpha = c\omega^k + D\beta,$$

where $c \in \mathbb{C}$ and $\beta \in \mathcal{A}^{2k-1}$. By condition (i), the constant c must be equal to zero.

References (continued)

8. Tsukamoto, C.: Infinitesimal Blaschke conjectures on projective spaces. Ann. Sci. Éc. Norm. Super., IV. Ser. **14**, 339–356 (1981)

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