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Errata to the Paper: On the Heat Equation and the Index Theorem

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The joint paper of the above title which appeared in Inventiones math. **19**, 279–330 (1973), though correct in principle, contained some technical errors which we shall here explain and rectify. Our thanks are due to D. Epstein, Y. Colin de Verdière and A. Vasquez whose computations and queries alerted us to our errors.

1. The Notion of Regularity

The main error occurs on page 306 where it is implicitly assumed that the coefficients of the two operators A^*A and AA^* (associated to the signature operator A) are polynomial functions in the g_{ij}, their derivatives and $(\det g)^{-1}$. As we shall show later this is not quite true – the coefficients also involve $\sqrt{\det g}$ and the inverses of the principal minors of the matrix g_{ij}. Thus the form ω in (5.1) is *not* a regular invariant of the metric in the sense of § 2, and so the Gilkey Theorem as formulated on p. 284 does not apply.

To correct this we shall widen the notion of regularity (so as to include, in particular, the form ω above) and then check that our proof of Gilkey's Theorem still holds in this wider context.

In § 2 regularity was only defined for *invariants* of a Riemann structure g (i.e. satisfying the naturality or invariance property (2.3)). It will perhaps make for greater clarity if we introduce our new notion of regularity for *any* function of g, independently of the invariance property. We shall say that $f(g)$ is a regular function of g if, in any coordinate system, we have

$$f(g)(x)=\sum_{\alpha} a_\alpha(x, g(x))\, m_\alpha \qquad \text{(finite sum)}$$

where $a_\alpha(x, y)$ are C^∞ functions and m_α denotes a monomial in the partial derivatives of $g(x)$. Here $g(x)$ stands of course for the classical components $g_{ij}(x)$ relative to the basis dx^i given by the coordinates $(x_1, \ldots, x_n)$. Clearly regularity is a local property and it has only to be checked in one coordinate system. The essential difference between this definition and that of p. 282–284 is that we now allow C^∞ dependence on g and do not insist on polynomial dependence on g and g^{-1}. Another less significant difference is that we now allow the coefficients a_α to depend also on x. If f is both regular *and* invariant then this dependence is illusory – in fact translation invariance alone shows the a_α must be independent of x. For a differential form regularity is defined in terms of regularity of its components relative to the usual basis $dx^{i_1} \wedge \cdots \wedge dx^{i_r}$.

2. Proof of the Gilkey Theorem

The proof of the Gilkey Theorem given in §2 is a "point-wise" proof using geodesic coordinates. For this reason the C^∞ dependence on g introduced in our new definition of regularity is quite innocuous. In more detail let $\omega(g)$ be a regular form-valued invariant of g in our new sense. Then in $\mathbb{R}^n$ each component $\omega_\beta(g)$ is given by an expression:

$$\omega_\beta(g)[x]=\sum a_\alpha^\beta(x, g(x))m_\alpha$$

which we call the universal polynomial of ω. To evaluate $\omega(g)$ at a point p of a given Riemannian manifold M we choose a geodesic coordinate system centered at p and interpret it as a map of an ε-ball about 0 in $\mathbb{R}^n$

$$f\colon \mathbb{R}_\varepsilon^n \to M$$

sending 0 to p. The invariance property of ω implies that $f^*(\omega(g)_p)=\omega(f^*g)_0$. Applying our universal polynomial to the right-hand side now yields

$$f^*(\omega_\beta(g)_p)=\sum_\alpha a_\alpha^\beta(0, f^*g|_0)\, m_\alpha(f^*g)_0$$

and as the coordinates are geodesic for f^*g at 0 we see that $f^*g|_0$ is the unit matrix. Furthermore we may now apply Proposition (2.1) to $f^*(g)$ so that the monomials $m_\alpha(f^*g)_0$ are given by polynomials in the curvature of f^*g and its covariant derivatives (at 0). In this way we arrive at formula (2.17) and the proof now proceeds as before.

3. The Signature Operator

The operators A^*A and AA^* of p. 306 are just the restrictions of the Hodge Laplacian Δ to the subspaces $\Omega_\pm$ of Ω (the space of all forms). Certainly Δ, relative to the usual basis of the $dx^{k_1}\wedge\cdots\wedge dx^{k_r}$, has coefficients which are polynomial in g, its derivatives and $(\det g)^{-1}$. However this basis is not compatible with the decomposition $\Omega=\Omega_+\oplus\Omega_-$ (determined by the eigenspaces of $*$). This difficulty is a consequence of the fact that $\Lambda^k(T^*)$ is associated to the principal tangent bundle via a representation of $GL(n,\mathbb{R})$, whereas $\Omega_\pm$ are associated only to the principal $SO(n)$-bundle (determined by g) via a representation of $SO(n)$ which is *not the restriction of a $GL(n,\mathbb{R})$ representation.*

The upshot is that we have to resort to an *adhoc framing* of $\Omega_\pm$, which can be constructed as follows. Let $\phi^1,\ldots,\phi^n$ be an orthonormal frame of T^* obtained from the $dx^1,\ldots,dx^n$ by applying the Gramm-Schmidt procedure. In terms of these ϕ's the $*$ operator and the corresponding τ operator defined by

$$\tau\alpha=i^{p(p-1)+l}*\alpha \qquad \text{for } \alpha\in\Omega^p,$$

is especially simple. Indeed if $\phi^K=\phi^{k_1}\wedge\cdots\wedge\phi^{k_r}$ is an exterior monomial then

$$\tau\phi^K=\sigma(K)\cdot\phi^L$$

where L denotes the complementary monomial and $\sigma(K)$ is ± 1, when $n/2=l$ is even, and $\pm i$ when l is odd. It follows that if Φ_n denotes the subspace of Ω generated by $\phi^1,\ldots,\phi^{n-1}$ then $\tau\Phi_n\subset\phi^n\wedge\Phi_n$, so that in particular

$$\tau\Phi_n\cap\Phi_n=0.$$

We may therefore frame Ω_+, with the forms $(\phi^K + \tau\phi^K)/\sqrt{2} = \phi_+^K$ where ϕ^K does not involve ϕ^n, and similarly frame Ω_- by $\phi_-^K = (\phi^K - \tau\phi^K)/\sqrt{2}$. Furthermore the ϕ_+^K, ϕ_-^K together give rise to an orthonormal framing of Ω.

Now the ϕ's are related to the dx's by a triangular matrix

$$\phi = T\,dx$$

whose coefficients are C^∞ functions of the g_{ij} but are not just polynomials in the g_{ij} and $\det g^{-1}$. Indeed here square roots of $\det g$ and inverses of principal minors of g will appear.

In any case relative to the frame ϕ_+^K, ϕ_-^K the operators d and d^* will have regular coefficients (in our new sense) and therefore $(d+d^*)^2$ also. But in this frame the operators A^*A and AA^* just correspond to the "diagonal" parts of this matrix operator and hence still have regular coefficients. Moreover their leading terms are (in any base) the scalar operator

$$-\sum g^{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

Since regular functions of g are closed under multiplication and under differentiation by C^∞ vector fields the Seeley formula (4.40) applied to AA^* and A^*A shows that their heat expansion coefficients are regular functions of g. The form ω appearing in (5.1) is therefore a regular invariant of g and so we can apply the Gilkey Theorem and proceed as before.

4. Other Operators

For the generalized signature operators A_ξ of §6 the argument is quite analogous. The definition of regularity in §3 is widened in a similar manner by allowing polynomials in the variables in (3.4) to have coefficients depending on g and (for functions not necessarily invariant) on x. The generalized Gilkey Theorem (Theorem II on p. 290) is still true and can be applied to the form $\omega(g, \xi)$ in (6.1) as before.

The Dirac operator B on p. 314–315 presents essentially the same features as the signature operator. To write its coefficients out explicitly we must first choose an orthonormal base of the two Spin bundles E^+ and E^-. Since the Spin representations are not representations of $GL(n, \mathbb{R})$ a local coordinate system on M does not automatically give rise to such a base, so we must again use the Gramm-Schmidt process to orthogonalize the dx^i. The coefficients of B are then regular functions of g as before while the leading terms in BB^* and B^*B are scalar and given by $-\sum g^{ij} \dfrac{d^2}{\partial x_i \partial x_j}$.

5. Corollary on Page 303

A second error occurs in the last part of the Corollary on p. 303. The statement that in the quadratic case $\mu_k(A)$ is a polynomial is incorrect and should be modified by replacing $\mu_k(A)$ with $(\det a)^{\frac{1}{2}}\,\mu_k(A)$. The error crept in through a wrong sign on p. 305 where, after the change of variable $\xi = g^{-1}\xi'$, we wrote down

$$\hat{f}(g) = (\det g)^{-1} \int_{S^{n-1}} p(g^{-1}\xi)\,\omega(\xi)$$

instead of the correct expression:

$$\hat{f}(g) = |(\det g)|^{-1} \int_{S^{n-1}} p(g^{-1}\xi)\,\omega(\xi)$$

It follows that it is $|\det g|\,\hat{f}(g)$ which is in the coordinate ring of $GL(n, \mathbb{R})$ rather than $\hat{f}(g)$.

This alteration does not now affect the rest of the paper in view of our widened definition of regularity. Incidentally the necessity for the factor $(\det a)^{\frac{1}{2}}$ is at once seen by considering the case $A = d + d^*$: $\Omega^{\text{even}} \to \Omega^{\text{odd}}$ which leads to the Gauss-Bonnet formula. For the signature operator the square root is eventually cancelled out by another factor $(\det a)^{\frac{1}{2}}$ occurring in the coefficients of A (as observed in (3) above), which explains why the Hirzebruch formula for the signature is rational in the g_{ij}.

6. Appendix II

There is an unfortunate confusion of notation on p. 328 which affects the precise recurrence formula (a 11) but does not vitiate the main conclusion. Precisely the lower line of the formula

$$\begin{aligned} r \cdot \mathscr{R} \cdot \frac{1}{r} \cdot \mathscr{R}\theta^i &= -i(\mathscr{R})\,d\theta^i_j x^j \\ &= -2R^i_{jkl}\,x^j x^k\,dx^l \end{aligned}$$

is wrong if R^i_{jkl} is to have its standard meaning, that is, if the curvature matrix of g relative to the frame $\partial/\partial x^i$ is to be given by

$$\tfrac{1}{2} R^i_{jkl}\,dx^k\,dx^l.$$

The correct expression is obtained by replacing the $-2R^i_{jkl}$ of our paper with

$$a^i_\alpha\, b^\beta_j\, R^\alpha_{\beta kl},$$

where b is the inverse matrix to a. Indeed the a's and b's correct for the switch of frames: $\theta^i \to dx^i$, while the minus sign corrects for the switch $dx^i \to \dfrac{\partial}{\partial x^i}$ and the 2 is cancelled by 1/2 above.

Correcting (a 10) and (a 11) correspondingly we obtain the recursion (a 11)

$$(n^2 + n)\,\hat{a}^i_l[n] = x^j x^k \{\widehat{R^\alpha_{\beta jkl}\, a^i_\alpha\, b^\beta_j}\}[n-2]$$

which still serves to determine the $\hat{a}$'s in terms of the $\hat{R}$'s.

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