Erratum to On the Infinite Product Construction of Non-Singular Transformations of a Measure Space

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The statement that appears in [3] under Lemma (2.3) is incorrect and part (c) of the statement of Theorem (2.4) has to be deleted¹. [3] is erroneous since it can happen that one is confronted with $\lambda\{0\}=1$ (λ as on p. 147 of [3]). Therefore condition (c) is only sufficient. The theorem that is given in this note replaces (2.4). (The remainder of the paper is not affected. Here are some errata: p. 153 read T_q instead of T , pp. 154, 155 in (3.2) and (3.3) and their proofs read R instead of N , p. 160, in (33) read $+s(i)$ instead of $-s(i)$, and in (36) read the exponent $-\frac{1}{2}$ as -1 .) The terminology and notation is that of [3].

Theorem. *The following assertions on $\mathfrak{p}$ are equivalent.*

- (a) $\mathcal{G}_{\mathfrak{p}}$ does not admit an invariant measure equivalent to $\mu_{\mathfrak{p}}$.
- (b) $\mathcal{G}_{\mathfrak{p}}$ has property A.
- (c) There exists a $\beta > 0$, $\beta < \frac{1}{2}$, and sequences $c(i) \in \mathbb{R}$, $b(i) \geq 0$, $i \in \mathbb{N}$, such that

$$\sup_{i \geq 1} b(i) = \infty, \quad (1)$$

$$\mu_{\mathfrak{p}} \{x \in X : |c(i) + \sum_{1 \leq j \leq i} a_{\mathfrak{p},j}(x)| \leq b(i)\} \geq 1 - 2\beta, \quad (2)$$

$$\mu_{\mathfrak{p}} \{x \in X : c(i) + \sum_{1 \leq j \leq i} a_{\mathfrak{p},j}(x) \geq b(i)\} \geq \beta, \quad (3)$$

$$\mu_{\mathfrak{p}} \{x \in X : c(i) + \sum_{1 \leq j \leq i} a_{\mathfrak{p},j}(x) \leq -b(i)\} \geq \beta. \quad (4)$$

Proof. We prove first that (a) implies (c). For this purpose let us assume that for no $\beta > 0$, $\beta < \frac{1}{2}$, there are sequences $c(i) \in \mathbb{R}$, $b(i) \geq 0$, $i \in \mathbb{N}$, with the properties (1), (2), (3) and (4), and let us proceed to show that $\mathcal{G}_{\mathfrak{p}}$

¹ My thanks are due to Christian Grillenberger and Lee Jones for pointing this out to me.

admits then an invariant measure. Recall that for every real valued μ -measurable function a on a probability space $(X, \mathcal{B}, \mu)$ one has with

$$t_0 = \sup \{t \in \mathbb{R} : \mu \{x \in X : a(x) \geq t\} \geq \alpha\}, \quad 0 < \alpha < 1,$$

that

$$\mu \{x \in X : a(x) \leq t_0\} \geq 1 - \alpha.$$

Hence if we set for $i, m \in \mathbb{N}$

$$t_+(i, m) = \sup \{t \in \mathbb{R} : \mu_p \{x \in X : \sum_{1 \leq j \leq i} a_{p,j}(x) \geq t\} \geq 2^{-m-3}\},$$

$$t_-(i, m) = \inf \{t \in \mathbb{R} : \mu_p \{x \in X : \sum_{1 \leq j \leq i} a_{p,j}(x) \leq t\} \geq 2^{-m-3}\},$$

$$q(i, m) = -\frac{1}{2}(t_+(i, m) + t_-(i, m)),$$

$$r(i, m) = \frac{1}{2}(t_+(i, m) - t_-(i, m)),$$

we have

$$\mu_p \{x \in X : q(i, m) + \sum_{1 \leq j \leq i} a_{p,j}(x) \geq r(i, m)\} \geq 2^{-m-3},$$

$$\mu_p \{x \in X : q(i, m) + \sum_{1 \leq j \leq i} a_{p,j}(x) \leq -r(i, m)\} \geq 2^{-m-3}$$

and

$$\mu_p \{x \in X : |q(i, m) + \sum_{1 \leq j \leq i} a_{p,j}(x)| \leq r(i, m)\} \geq 1 - 2^{-m-2}.$$

Since it is assumed that for no $\beta = 2^{-m-2}$, $m \in \mathbb{N}$, sequences $c(i) \in \mathbb{R}$, $b(i) \geq 0$, $i \in \mathbb{N}$, with the properties (1), (2), (3) and (4) exist, there are $R(m) \geq 0$, $m \in \mathbb{N}$, such that

$$r(i, m) \leq R(m), \quad i, m \in \mathbb{N},$$

and we find that

$$\mu_p \{x \in X : |q(i, 1) + \sum_{1 \leq j \leq i} a_{p,j}(x)| \leq R(1) + 2R(m)\} \geq 1 - 2^{-m-2}, \quad i, m \in \mathbb{N}.$$

As a consequence there is a sequence $(i(l))_{l \in \mathbb{N}}$ such that, as l tends to infinity, the random variables

$$q(i(l), 1) + \sum_{1 \leq j \leq i(l)} a_{p,j}$$

converge in distribution, the limit measure being a probability measure. Setting inductively

$$d(1) = q(i(1), 1), \quad d(l) = q(i(l), 1) - \sum_{1 \leq l' < l} d(l'), \quad l > 1,$$

we find that the sums

$$\sum_{1 \leq l \leq L} (d(l) + \sum_{i(l-1) < j \leq i(l)} a_{p,j}), \quad L \in \mathbb{N}, \quad (i(0) = 0)$$

of independent random variables converge in distribution as L tends to infinity with the limit measure a probability measure, hence they converge a.e. ([1], Proposition 8.36, 176). Setting

$$f(x) = \sum_{l \in \mathbb{N}} (d(l) + \sum_{i(l-1) < j \leq i(l)} a_{p,j}(x)), \quad \text{f.a.a. } x \in X,$$

we find that

$$\nu(A) = \int_A e^{-f(x)} d\mu_p(x), \quad A \subset X,$$

defines an invariant measure ν for $\mathcal{G}_p$.

We assume now that (c) holds and we proceed to show that $\mathcal{G}_p$ has property A. We denote

$$\xi = \min(\beta, 1 - 2\beta).$$

Using (1) we choose a sequence $(i(k))_{k \in \mathbb{N}}$ such that

$$\lim_{k \rightarrow \infty} b(i(k)) = \infty \quad (5)$$

and such that, as k tends to infinity, the random variables

$$b(i(k))^{-1} (c(i(k)) + \sum_{1 \leq j \leq i(k)} a_{p,j})$$

converge in distribution with a limit measure to be denoted by λ . By (2)

$$\lambda([-1, 1]) \geq \xi.$$

We choose a ρ , $|\rho| \leq 1$, such that

$$\lambda(\square\rho - \tfrac{1}{2}, \rho + \tfrac{1}{2}\square) \geq \tfrac{\xi}{3}. \quad (6)$$

Let now $A \subset X$, and let $N > 0$. There exists an $I \in \mathbb{N}$ together with an $u \in \prod_{1 \leq i \leq I} \Omega_i$ such that with

$$\begin{aligned} B &= A \cap Z_u, \\ \mu_p(B) &> \left(1 - \frac{\xi}{32}\right) \mu_p(Z_u). \end{aligned} \quad (7)$$

We have from (5) that the random variables

$$b(i(k))^{-1} (c(i(k)) + \sum_{I < j \leq i(k)} a_{p,j})$$

converge also in distribution as k tends to infinity and the limit measure is again λ . Therefore we can, by means of (5) and (6), select a $k \in \mathbb{N}$ such that

$$b(i(k)) > e^{N+1}, \quad (8)$$

and that

$$b(i(k)) > -4 \sum_{1 \leq j \leq l} \min_{\omega \in \Omega_j} \log p_j(\omega), \quad (9)$$

and such that with

$$\begin{aligned} \Xi &= \left\{ v \in \prod_{l < j \leq i(k)} \Omega_j : (\rho - \tfrac{1}{2}) b(i(k)) \right. \\ &\quad \left. \leq c(i(k)) - \sum_{l < j \leq i(k)} \log p_j(v_j) \leq (\rho + \tfrac{1}{2}) b(i(k)) \right\}, \end{aligned}$$

one has

$$\mu_p(Z_\Xi) > \frac{\xi}{4}. \quad (10)$$

We set

$$\begin{aligned} \Gamma_- &= \left\{ v \in \prod_{l < j \leq i(k)} \Omega_j : c(i(k)) - \sum_{l < j \leq i(k)} \log p_i(v_j) \leq -\tfrac{3}{4} b(i(k)) \right\}, \\ \Gamma_+ &= \left\{ v \in \prod_{l < j \leq i(k)} \Omega_j : c(i(k)) - \sum_{l < j \leq i(k)} \log p_j(v_j) \geq \tfrac{3}{4} b(i(k)) \right\}, \end{aligned}$$

and we obtain from (3), (4) and (9) that

$$\mu_p(Z_{\Gamma_-}) \geq \xi, \quad \mu_p(Z_{\Gamma_+}) \geq \xi. \quad (11)$$

Let further

$$\Xi_0 = \{x \in \Xi : \mu_p(B \cap Z_v) > \tfrac{3}{4} \mu_p(Z_u) \mu_p(Z_v)\}.$$

We claim that

$$\mu_p(Z_{\Xi_0}) > \tfrac{1}{2} \mu_p(Z_\Xi), \quad (12)$$

and in particular Ξ_0 is not empty. Indeed, were this not so, then it would follow from (10) that

$$\begin{aligned} \mu_p(Z_u - B) &\geq \tfrac{1}{4} \mu_p(Z_u) \mu_p(Z_{\Xi - \Xi_0}) \\ &\geq \tfrac{1}{8} \mu_p(Z_u) \mu_p(Z_\Xi) \\ &> \tfrac{\xi}{32} \mu_p(Z_u), \end{aligned}$$

a contradiction to (7). Employing (11) one proves in the same way the existence of a

$$v^- \in \Gamma_- \quad (13)$$

and of a

$$v^+ \in \Gamma_+$$

such that

$$\mu_p(B \cap Z_{v^-}) > \tfrac{3}{4} \mu_p(Z_u) \mu_p(Z_{v^-}), \quad (14)$$

and

$$\mu_p(B \cap Z_{v^+}) > \tfrac{3}{4} \mu_p(Z_u) \mu_p(Z_{v^+}).$$

We assume now that

$$\rho \geq 0, \quad (15)$$

and we set

$$s = \log \left| c(i(k)) - \sum_{I < j \leq i(k)} \log p_j(v_j^-) \right|.$$

We have by (13)

$$e^s \geq \frac{3}{4} b(i(k)), \quad (16)$$

and hence by (8)

$$s > N. \quad (17)$$

For $v \in \Xi_0$ let $S_v \in [\mathcal{G}_p]$ be such that

$$\begin{aligned} S_v: Z_u \cap Z_{v-} &\rightarrow Z_u \cap Z_v, \\ (S_v x)_j &= x_j, \quad j > i(K). \end{aligned}$$

We have from (14)

$$\mu_p(B \cap Z_v \cap S_v(B \cap Z_{v-})) > \frac{1}{2} \mu_p(Z_u) \mu_p(Z_v), \quad v \in \Xi_0. \quad (18)$$

Our intent is to prove that

$$B \cap Z_v \cap S_v(B \cap Z_{v-}) \subset K_{\mu_p, T_p}(B, s, 2), \quad v \in \Xi_0. \quad (19)$$

To achieve this we show that

$$\left| \log \left(\sum_{I < j \leq i(K)} (\log p_j(v_j^-) - \log p_j(v_j)) \right) - s \right| < 2, \quad (20)$$

and to demonstrate (20) we observe first that by (16)

$$\begin{aligned} &\sum_{I < j \leq i(K)} (\log p_j(v_j^-) - \log p_j(v_j)) \\ &= c(i(K)) - \sum_{I < j \leq i(K)} \log p_j(v_j) + e^s \\ &\leq (\rho + \frac{1}{2}) b(i(K)) + e^s \leq \frac{4}{3} (\rho + \frac{1}{2}) e^s + e^s \leq e^{s+2}, \quad v \in \Xi_0. \end{aligned}$$

And by our assumption (15) and by (16) we have that

$$\sum_{I < j \leq i(K)} (\log p_j(v_j^-) - \log p_j(v_j)) \geq e^s - (\rho + \frac{1}{2}) b(i(K)) \geq e^s - \frac{1}{2} e^s > e^{s-2},$$

which proves (20). We conclude now from (10), (12), (18) and (19) that

$$\mu_p(K_{\mu_p, T_p}(B, s, 2)) > \frac{\xi}{16} \mu_p(B),$$

and (17) shows that property A is established. If $\rho \leq 0$ then one proceeds in the same way with v^+ taking the place of v^- . Q.e.d.

After this correction note had been submitted it was learned that in the meantime Hamachi [2] has proved that T_p admits an invariant measure if and only if there are constants c_i such that $\sum_{i=1}^{\infty} (a_{p,i} - c_i)$ converges μ_p -a.e. Compare here also Størmer's condition [4].

References

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