ADDENDUM to v-adic Zeta Functions, L-series, and Measures for Function Fields

David Goss

Department of Mathematics, University of California, Berkeley, CA 94720, USA

In [1] we presented a theory of L -series for function fields. In particular, we interpolated certain special values to obtain one-variable “ v -adic” L -series. We show here how our techniques may be slightly altered to obtain much more powerful “two-variable” v -adic functions and measures.

We will follow the notation given in [1].

Section 1. Let $\mathfrak{p} \in \text{Spec}(A)$ be a prime associated to the place v of k . Let $\beta \in A - \mathfrak{p}$. Let H be a nonzero ideal and $\alpha \in A$. Let f be a $\bar{k}$ -valued function on A/H . Finally, let $i \in \mathbb{N}^+$. In the spirit of Sect. 6 of [1] we have:

1.1. Definition. We formally set

$$\zeta_v(\alpha, H, (\beta^{-1}, -i)) = \sum_{j=0}^{\infty} \beta^j \left(\sum_{\substack{n \equiv \alpha(H) \\ D(n)=j \\ (n, \mathfrak{p})=1}} n^i \right)$$

and

$$L_v(f, (\beta^{-1}, -i)) = \sum_{j=0}^{\infty} \beta^j \left(\sum_{\substack{D(n)=j \\ (n, \mathfrak{p})=1}} f(n) n^i \right).$$

By 4.1 of [1] the terms in parenthesis vanish for $j \gg 0$. Thus the series are finite and define elements of $\bar{k}$.

We remark that if $\mathfrak{p} = (f)$ then

$$\zeta_v(0, H, (\beta^{-1}, -i)) = (1 - f^x) \zeta(0, H, (\pi^i \beta^{-1}, -i))$$

where $x = (\beta \pi^{-i}, i) \in S_{\sigma}$ and ζ is defined in Sect. 3 of [1]. There is a similar statement for L_v when f is a character. If $\beta = 1$, our series reduce to those of Sect. 6 of [1].

Let $\overline{A_v}$ and S_v be as in Sect. 6 of [1].

1.2. Theorem. 1. $\zeta_v(\alpha, H, (\beta^{-1}, -i))$ interpolates to a continuous two-variable function from $A_v^* X S_v \rightarrow A_v$

2. $L_v(f, (\beta^{-1}, -i))$ interpolates to a continuous function from $A_v^* X S_v \rightarrow \overline{A_v}$.

Sketch of Proof. Let α be the degree of p and let $i_1, i_2 \in \mathbf{N}$ with $i_1 \equiv i_2 ((r^\alpha - 1)p^m)$. It is trivial that for $\lambda \in A - p$, $\lambda^{i_1} \equiv \lambda^{i_2} (p^m)$.

Now let β_1^{-1} , be v -adically close to β_2^{-1} , say $\beta_1^{-1} - \beta_2^{-1} \in p^n$. As the elements in parenthesis in 1.1 vanish for $j \gg 0$, we may think of our special values as being given by sums involving the same elements of A . Thus,

$$\zeta_v(\alpha, H, (\beta_1^{-1}, -i_1)) - \zeta_v(\alpha, H, (\beta_2^{-1}, -i_2))$$

is divisible by at least $p^{(m+n)}$. The result now follows easily.

It is easy to see by the Riemann-Roch Theorem that if $i \in S_v$ is fixed then the resulting map $A_v^* \rightarrow A_v$ is analytic. Also, if $\beta = 1$, we recover the construction of Sect. 6 of [1].

Section 2. We now obtain our functions by integrating over a *continuously varying measure*. In the spirit of Sect. 7 of [1], we have:

2.1. Definition. Let $j \in \mathbf{N}$ and $\beta \in A_v^*$. We set

$$\mu_{j,\beta}(\alpha + (H' p^m)) = \sum_{i=0}^{\infty} \beta^i \left(\sum_{\substack{n \equiv \alpha(H' p^m) \\ D(n)=i}} n^j \right).$$

It is immediate that $\mu_{j,\beta}$ defines a v -adic measure on A_H .

It is easy to see that $\mu_{0,\beta}$ has the following interpretation: Let f be the monic of least degree in $H' p^m$.

Then,

$$\mu_{0,\beta}(\alpha + (H' p^m)) = \begin{cases} 0 & \text{if } D(\alpha) > D(f) \\ \beta^{D(\alpha)} + \beta^{D(f)} & \text{if } D(\alpha) < D(f), \alpha \text{ monic} \\ \beta^{D(f)} & \text{if } D(\alpha) < D(f), \alpha \text{ non-monic} \end{cases}$$

With this definition all the analogous statements to those of Sect. 7, [1] are true. For instance:

2.1. Theorem. 1. $\int_{(HA_H \cap U)} G(x)^i d\mu_{0,\beta} = \zeta_v(0, H, (\beta^{-1}, -i))$ for all $i \in S_v$.

2. $\int_{(HA_H \cap U)} f(x) G(x)^i d\mu_{0,\beta} = L_v(f, (\beta^{-1}, -i))$ for all $i \in S_v$.

Sketch of Proof. By continuity, we need only show the results for $i \in \mathbf{N}$.

Let $\bigcup (\alpha + H' p^m)$ be a decomposition of $A_H \cap U$ with α chosen of least degree in its class. As an approximation to our integral in 1, we have

$$\sum_{\alpha} \alpha^i \mu_{0,\beta}(\alpha + H' p^m)$$

Let $d_m = D(\mathfrak{p}^m H')$ be the degree of the lowest monic in $\mathfrak{p}^m H'$. By our interpretation of $\mu_{0,\beta}$ the sum becomes

$$\sum_{j < d_m} \sum_{\substack{\alpha \text{ monic} \\ D(\alpha) = j \\ (\alpha, \mathfrak{p}) = 1}} \alpha^i (\beta^i + \beta^{d_m}) + \sum_{j < d_m} \sum_{\substack{(\alpha, \mathfrak{p}) = 1 \\ \alpha \text{ non-monic} \\ D(\alpha) = j}} \alpha^i \beta^{d_m}$$

Now multiply out and factor β^{d_m} . As $m \rightarrow \infty$, the term multiplied by β^{d_m} is 0, by 7.4 of [1]. We are left with precisely $\zeta_v(0, H, (\beta^{-1}, -i))$.

2 follows similarly.

Reference

1. Goss, D.: *v*-adic Zeta Functions, L-Series and Measures for Function Fields, *Inventiones math.* **55**, 107–116 (1979)

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