Addendum to : A Reduction Theorem for Cohomology Groups of Very Strongly q -Convex Kähler Manifolds

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We are going to show an immediate but important consequence of the following theorem which is the main result in [5].

Theorem 1 (cf. Theorem (3.7)). *Let X be a noncompact complex manifold of dimension n with a C^∞ exhaustive plurisubharmonic function φ , and let $X_c = \{x; \varphi(x) < c\}$. Assume that X has a Kähler metric ω and that there is a compact subset K outside which the Levi form of φ has at least $n - q + 1$ positive eigenvalues. Then, for any positive integers s and t with $s + t \geq n + q$, the cohomology groups $H^{s+t}(X_c, \mathbb{C})$ and $H^s(X_c, \Omega^t)$ are finite dimensional and canonically isomorphic to the spaces of harmonic forms with respect to the metric $\omega_c := \omega + \partial\bar{\partial} \log(c - \varphi)$, provided that ∂X_c is smooth and that $X_c \supset K$.*

Since Hodge's star operator sends harmonic forms to harmonic ones, from Poincaré-Serre's duality theorem we obtain the following

Corollary 2. *Under the above situation for X_c , the cohomology groups with compact supports $H_0^{s+t}(X_c, \mathbb{C})$ and $H_0^s(X_c, \Omega^t)$ are canonically isomorphic to the spaces of harmonic forms with respect to ω_c , if $s + t \leq n - q$.*

Let L_c be the multiplication by ω_c . Since L_c sends harmonic forms to harmonic forms, by a standard argument we have the following

Proposition 3. *Under the above situation for X_c , L_c^{n-s-t} gives rise to isomorphisms from $H_0^{s+t}(X_c, \mathbb{C})$ and $H_0^s(X_c, \Omega^t)$ to $H^{2n-s-t}(X_c, \mathbb{C})$ and $H^{n-s}(X_c, \Omega^{n-t})$.*

If $k \geq n + q$, then $H^k(X, \mathbb{C}) \cong H^k(X_c, \mathbb{C})$ by Morse theory. Hence, taking Poincaré duality into account, we have

Corollary 4. *The natural inclusion maps*

$$j^k: H^k(X, \mathbb{C}) \rightarrow H^k(X, \mathbb{C})$$

are surjective for $k \geq n + q$ and injective for $k \leq n - q$.

On the other hand, if $s+t \geq n+q$, then $H^s(X, \Omega^t) \cong H^s(X_c, \Omega^t)$ by Andreotti-Grauert-Hörmander's theorem (cf. [1] or [4]). Hence we have

Corollary 5. *The natural inclusion maps*

$$j^{s,t}: H_0^s(X, \Omega^t) \rightarrow H^s(X, \Omega^t)$$

are surjective for $s+t \geq n+q$.

The dual of Corollary 5 is somewhat delicate:

Corollary 5'. *Let $s+t \leq n-q$. If moreover $H^s(X, \Omega^t)$ is separated, then $j^{s,t}$ is injective.*

We note that the above Corollaries 4 through 5' are valid for any complex manifold Y for which there exists a proper holomorphic map $\eta: X \rightarrow Y$ which is biholomorphic outside a compact subset of X . Now we can prove the following

Theorem 6. *Let X be a connected noncompact complex manifold of dimension n with a C^∞ exhaustive plurisubharmonic function φ . Assume that the Levi form of φ has at least $n-q+1$ positive eigenvalues outside a compact subset K of X and that there is a Kähler manifold $\tilde{X}$ and a proper holomorphic map $\pi: \tilde{X} \rightarrow X$ which is biholomorphic outside a compact subset of X . Then the followings hold:*

i) *the restriction map $H^k(X, \mathbb{C}) \rightarrow \varinjlim H^k(X - X_d, \mathbb{C})$ is surjective if $k \leq n-q-1$ and bijective if $k \leq \min(1, n-q-1)$,*

i') *the restriction map $H^s(X, \Omega^t) \rightarrow \varinjlim H^s(X - X_d, \Omega^t)$ is surjective if $s+t \leq n-q-1$, provided that $H^{s+1}(X, \Omega^t)$ is separated,*

ii) *the restriction map $H^0(X, \Omega^t) \rightarrow H^0(X - K, \Omega^t)$ is bijective, if $t \leq n-q-1$ and $H^1(X, \Omega^t)$ is separated,*

iii) *the restriction maps $H^t(X, \mathcal{O}) \rightarrow \varinjlim H^t(X - X_d, \mathcal{O})$ are isomorphisms for $t \leq n-q-1$.*

Corollary 7. *Let X be a strongly pseudoconvex domain with smooth boundary in a complex manifold of dimension $n \geq 3$. Then there are canonical epimorphisms*

$$\begin{aligned} H^k(X, \mathbb{C}) &\twoheadrightarrow H^k(\partial X, \mathbb{C}) & \text{for } k \leq n-2, \\ H^s(X, \Omega^t) &\twoheadrightarrow H^{t,s}(\partial X) & \text{for } s+t \leq n-2, \end{aligned}$$

and isomorphisms

$$\begin{aligned} H^k(X, \mathbb{C}) &\xrightarrow{\sim} H^k(\partial X, \mathbb{C}) & \text{for } k=0, 1, \\ H^0(X, \Omega^t) &\xrightarrow{\sim} H^{t,0}(\partial X) & \text{for } t \leq n-2, \\ H^t(X, \mathcal{O}) &\xrightarrow{\sim} H^{0,t}(\partial X) & \text{for } t \leq n-2. \end{aligned}$$

Here $H^{t,s}(\partial X) := \varinjlim_{X \supset K \text{ compact}} H^s(X - K, \Omega^t)$.

Proof. In view of Corollary 4 and Corollary 5', the surjectivities in i) and i') follows from the exact sequences:

$$H^k(X, \mathbb{C}) \rightarrow \varinjlim H^k(X - X_d, \mathbb{C}) \rightarrow H_0^{k+1}(X, \mathbb{C}) \rightarrow H^{k+1}(X, \mathbb{C}),$$

$$H^s(X, \Omega^t) \rightarrow \varinjlim H^s(X - X_d, \Omega^t) \rightarrow H_0^{s+1}(X, \Omega^t) \rightarrow H^{s+1}(X, \Omega^t).$$

The proof of the bijectivity is as follows: If $q \leq n-1$, then $H_0^1(\tilde{X}, \mathcal{O})=0$ by Grauert-Riemenschneider's vanishing theorem (cf. [2]). So $H_0^1(X, \mathcal{O})=0$, too. Since $H_0^0(X, \mathcal{O})=0$ and $H_0^0(X, \Omega^1)=0$, by the Hodge-to-de Rham spectral sequence we have $H_0^k(X, \mathbb{C})$ for $k=0, 1$. Hence, in view of the above exact sequence the restriction map $H^k(X, \mathbb{C}) \rightarrow \varinjlim H^k(X - X_d, \mathbb{C})$ is injective for $k=0, 1$, provided that $q \leq n-1$. Hence combining this with the surjectivity above we have the required bijectivity. ii) is a consequence of i) and the identity theorem. iii) follows from the above exact sequence and Grauert-Riemenschneider's vanishing theorem. In virtue of Hironaka's desingularization theorem (cf. [3]) Corollary 7 follows from Theorem 6 immediately. Q.E.D.

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Note Added in Proof

Under the situation of Corollary 7, if moreover X is a sufficiently small neighbourhood of the maximal compact analytic subset, then the restriction map $H^{n-1}(X, \mathbb{C}) \rightarrow H^{n-1}(\partial X, \mathbb{C})$ is also surjective. The proof is completely different from that of the present note and will appear elsewhere.