Erratum

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On the convergence of multistep methods for nonlinear stiff differential equations

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In the way it is stated, Theorem 4 of [4] on the convergence of multistep methods for a class of semilinear parabolic equations does not apply to Navier-Stokes equations as the discussion preceding that theorem erroneously suggests: The local Lipschitz continuity of the nonlinearity is not fulfilled in the way required there. The following rather straightforward generalization of Theorem 4 does however apply. See Fujita and Kato [1], or Goldstein [2], pp. 168ff., for a verification of the abstract conditions below for the case of Navier-Stokes equations. We also refer to Pazy [5], Ch. 6.3, and references therein, for an analysis of semilinear parabolic equations under similar assumptions, and to Henry [3] for further examples and theory.

On a Banach space where $\|\cdot\|$ denotes the norm as well as the induced operator norm, we consider the initial value problem

$$(1) \quad y' + Ay = g(t, y), \quad y(0) = y_0.$$

Here $-A$ is the generator of an analytic semigroup, with spectrum to the left of the origin:

$$(2) \quad \|(s + A)^{-1}\| \leq \frac{M}{1 + |s|} \quad \text{for } |\arg s| \leq \pi - \phi, \quad \text{with some } \phi < \frac{\pi}{2}.$$

For the nonlinearity g , it is assumed that there exist real numbers α and $\gamma < 1$ such that the following local Lipschitz condition is satisfied for every $r > 0$:

$$(3) \quad \|g(t, u) - g(t, v)\|_{\alpha - \gamma} \leq L(r) \cdot \|u - v\|_{\alpha} \quad \text{for } \|u\|_{\alpha} + \|v\|_{\alpha} \leq r, \quad 0 \leq t \leq T,$$

where $\|\cdot\|_{\alpha}$ denotes the norm $\|v\|_{\alpha} = \|A^{\alpha}v\|$, defined for $v \in D(A^{\alpha})$, the domain of A^{α} . We note that the Stokes operator with Dirichlet boundary conditions satisfies (2) on the divergence-free subspace of $L^2(\Omega)^d$ (with a smooth bounded domain Ω in $\mathbf{R}^d$, $d = 2$ or 3), and the nonlinearity in the Navier-Stokes equations

satisfies (3) for any $\gamma > \frac{1}{2}$ in the case of two space dimensions, and for any $\gamma > \frac{3}{4}$ in the three-dimensional case, for both $\alpha = \gamma$ and $\alpha = \frac{1}{2}$ [1].

We consider time discretization of (1) by a linear multistep method with step size h , yielding discrete approximations y_n , $n \geq k$, for given starting values $y_0, \dots, y_{k-1}$. It is assumed that the multistep method is $A(\theta)$ -stable with $\theta > \phi$ of (2), stable in a neighborhood of infinity, strongly zero-stable, and of order p . These are conditions (2.4) in [4], which are satisfied in particular for BDF methods. The numerical solution is compared to the solution of a perturbed equation

$$(4) \quad \eta' + A\eta = g(t, \eta) + \sigma(t).$$

Then there is the following error bound in case that (4) has on an interval $[0, T]$ a solution which is sufficiently smooth in time.

Theorem 4' (Convergence of multistep methods for semilinear parabolic equations). *Consider an $A(\theta)$ -stable k -step method of order p satisfying the conditions stated above, applied to Eqs. (1)–(3). Then the error is bounded in the norm of $D(A^\alpha)$: $\|v\|_\alpha = \|A^\alpha v\|$ by*

$$(5) \quad \|y_n - \eta(t_n)\|_\alpha \leq C \cdot \left(\max_{0 \leq j \leq k-1} \|y_j - \eta(t_j)\|_\alpha + h^p \int_0^{t_n} \|\eta^{(p+1)}(t)\|_\alpha dt \right. \\ \left. + \|A^{-1}\sigma(0)\|_\alpha + \int_0^{t_n} \|A^{-1}\sigma'(t)\|_\alpha dt \right).$$

This estimate holds for $t_n = nh \leq T$ provided that the expression on the right-hand side is bounded by a sufficiently small constant c . The constants C and c depend only on the constants in (2) and (3), on T and on $\max_{0 \leq t \leq T} \|\eta(t)\|_\alpha$. They are otherwise

independent of A , and independent of n and h with $nh \leq T$. The term involving σ in (5) can be replaced by $\max_{0 \leq t \leq t_n} \|\sigma(t)\|$.

The proof follows closely that of Theorem 4 in [4]. In particular, it starts from the representation (cf. formula (4.13) in [4])

$$y_n = h \sum_{j=k}^n e(n-j, -hA) g(t_j, y_j) + p_n$$

where again $e(n, -hA)$ are the coefficients of the power series

$$h \sum_{n=0}^{\infty} e(n, -hA) \zeta^n = (\delta(\zeta)/h + A)^{-1}, \quad |\zeta| < 1,$$

with $\delta(\zeta)$ denoting the quotient of the generating polynomials of the multistep method. The bound for the resolvent

$$\|A^\gamma(s + A)^{-1}\| \leq \text{Const} \cdot |s|^{\gamma-1}, \quad |\arg s| \leq \pi - \phi$$

implies via Lemma 7 of [4] (with $\mu = 1 - \gamma$) the estimate

$$\|A^\gamma e(n, -hA)\| \leq \text{Const} \cdot [(n+1)h]^{-\gamma}, \quad 0 \leq nh \leq T.$$

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