Corrections to "Higher Ramification Groups for Rank One Valuations"

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Mr. J. L. Chabert, has been kind enough to point out some mistakes in my paper "Higher ramification groups for rank one valuations", appeared in this journal, volume 173, pages 253–259 (1967).

Statements (b), (d) and Proposition 1 are incorrects and counter-examples are being provided by Mr. Chabert, in his note «Sur une publication de P. Ribenboim», in the same issue.

We use the notations of our paper, to which the reader may refer.

(d) holds for all ideals $\tilde{J}$ such that $\tilde{J} \neq \tilde{J}\tilde{P}$, as shown in Zariski and Samuel "Commutative Algebra", volume II, page 79. But we may also prove:

(d') If v is a non-discrete valuation, $\tilde{J} = \tilde{J}\tilde{P}$, but $\tilde{J}$ is not a ramification ideal, then $\mathcal{R}(\tilde{J}) = \mathcal{V}(\tilde{J})$.

Proof. In fact, the inclusion $\mathcal{V}(\tilde{J}) \subseteq \mathcal{R}(\tilde{J})$ is clear. Conversely, let $\sigma \in \mathcal{R}(\tilde{J})$ and $\alpha \in \mathbb{R}$, $\alpha > 0$ be such that $\tilde{J} = \{x \in \tilde{A} \mid \tilde{v}(x) > \alpha\}$ ($\tilde{J}$ must be of this type, since $\tilde{J} = \tilde{J}\tilde{P}$). By hypothesis, there exists an ideal $\tilde{J}'$, properly contained in $\tilde{J}$, such $\mathcal{R}(\tilde{J}') = \mathcal{R}(\tilde{J})$. Let $\beta = \inf\{\tilde{v}(x) \mid x \in \tilde{J}'\}$, so $\beta > \alpha$. We choose a real number δ such that $0 < \delta < \beta - \alpha$. Let $x \in \tilde{K}$, $x \neq 0$. Since the value group of v is dense, there exists an element $a \in K$, $a \neq 0$ such that

$$-\tilde{v}(x) < v(a) < -\tilde{v}(x) + \delta,$$

hence $0 < \tilde{v}(ax) < \delta$, so $ax \in \tilde{A}$. It follows that

$$\tilde{v}\left(\frac{\sigma x}{x} - 1\right) = \tilde{v}\left(\frac{\sigma(ax)}{ax} - 1\right) = \tilde{v}(\sigma(ax) - ax) - \tilde{v}(ax) \geq \beta - \delta > \alpha,$$

because $\sigma \in \mathcal{R}(\tilde{J}) = \mathcal{R}(\tilde{J}')$. Thus $\sigma \in \mathcal{V}(\tilde{J})$.

Now we shall establish a replacement to Proposition 1, which was false.

Proposition 1'. Let $[\tilde{K} : K] = ef$, where e is the ramification index and f is the inertial degree of $\tilde{v}$. Then every large ramification ideal is principal.

Proof. Let $t_1, \dots, t_e \in \tilde{A}$ be elements such that $\tilde{v}(t_1), \dots, \tilde{v}(t_e)$ are representatives of the e cosets of $\tilde{v}(\tilde{K})$ with respect to the subgroup $v(K)$. Let $u_1, \dots, u_f \in \tilde{A}$ be elements whose residue classes form a basis of the K/v -space $\tilde{K}/\tilde{v}$. As it is well-known, the elements $t_i u_j$ ($i = 1, \dots, e; j = 1, \dots, f$) are linearly independent

over K and if $a_{ij} \in K$ then

$$\tilde{v}\left(\sum_{i,j} a_{ij} t_i u_j\right) = \min_{i,j} \{v(a_{ij} t_i)\}.$$

Let $\tilde{J}$ be a large ramification ideal and $\alpha = \inf\{\tilde{v}(x) \mid x \in \tilde{J}\}$. We assume that $\tilde{J}$ is not principal, hence $\tilde{v}(x) > \alpha$ for every $x \in \tilde{J}$.

Let us consider the finite set of real numbers

$$\tilde{v}\left(\frac{\sigma t_i}{t_i} - 1\right), \quad \tilde{v}\left(\frac{\sigma u_j}{u_j} - 1\right), \quad \text{for all } \sigma \in \mathcal{V}(\tilde{J}),$$

$i = 1, \dots, e$ and $j = 1, \dots, f$. Then there exists $m > 1$ such that every such number is larger than $\alpha + \frac{1}{m}$.

Since $[\tilde{K} : K] = ef$ the elements $t_i u_j$ ($i = 1, \dots, e; j = 1, \dots, f$) generate the K -space $\tilde{K}$. Thus, if $x \in \tilde{K}$, $x \neq 0$, we may write

$$x = \sum_{i,j} a_{ij} t_i u_j, \quad \text{with } a_{ij} \in K, \quad a_{ij} \neq 0.$$

It follows that if $\sigma \in \mathcal{V}(\tilde{J})$ then

$$\frac{\sigma x}{x} - 1 = \sum_{i,j} \left(\frac{\sigma(a_{ij} t_i u_j)}{a_{ij} t_i u_j} - 1 \right) \cdot \frac{a_{ij} t_i u_j}{x}.$$

But $\tilde{v}(x) \leq v(a_{ij} t_i u_j)$ and $\sigma(a_{ij}) = a_{ij}$, therefore

$$\begin{aligned} \frac{\sigma(a_{ij} t_i u_j)}{a_{ij} t_i u_j} - 1 &= \frac{\sigma(t_i u_j)}{t_i u_j} - 1 \\ &= \left(\frac{\sigma(t_i)}{t_i} - 1 \right) \left(\frac{\sigma(u_j)}{u_j} - 1 \right) + \left(\frac{\sigma(t_i)}{t_i} - 1 \right) + \left(\frac{\sigma(u_j)}{u_j} - 1 \right). \end{aligned}$$

Combining these relations, we deduce that

$$\tilde{v}\left(\frac{\sigma x}{x} - 1\right) > \alpha + \frac{1}{m} \quad \text{for every } x \in K',$$

$\sigma \in \mathcal{V}(\tilde{J})$.

Hence, if $\tilde{J}' = \left\{ x \in \tilde{A} \mid \tilde{v}(x) > \alpha + \frac{1}{m} \right\}$ we have $\mathcal{V}(\tilde{J}) \subseteq \mathcal{V}(\tilde{J}') \subseteq \mathcal{V}(\tilde{J})$, concluding that $\tilde{J}$ couldn't be a large ramification ideal.

Let us note that Proposition 1' holds when (K, v) is a maximal valued field (see [3], page 231), or more generally when the valuation $\tilde{v}$ has no defect over the valued field (K, v) (see [3], page 236); in particular, this is the case when K/v has characteristic not dividing the degree $[K : K]$. However, the above proposition does not hold in general, as we see considering $K = \bigcup_{\mu=1}^{\infty} \mathbb{Q}(\sqrt[2\mu]{2})$, $\tilde{K} = K(\sqrt[3]{3})$

and the valuation $\tilde{v}$ extending the 2-adic valuation of $\mathbb{Q}$; the maximal ideal of $\tilde{v}$ is a large ramification ideal which is not principal.

Bibliography

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3. — *Théorie des valuations*. Presses de l'Université de Montréal, 1964.

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