Corrected Reprint of
Berichtigung zu
„A Contribution to the Nonlinear Theory
of the Trade Cycle“

By

Michio Morishima, Osaka University, Osaka, Japan

Infolge bedauerlicher drucktechnischer Fehler ist es im obigen Artikel (erschienen in Heft 1–2 dieses Jahrganges, S. 165 ff.) zu sinnstörenden Zeilenverschiebungen auf den S. 168, 169 und 170 gekommen. Diese veranlassen Redaktion und Verlag zum (*beiliegenden*) *Neudruck* der Seiten 167–170, die wir gegen die fehlerhaften alten Seiten auszutauschen bitten.

Außerdem ist auf S. 173 in der 4. Zeile des Abschnittes VI (Oscillation and Groth) zu lesen statt *derivate*: *derivative*; sowie in der 15. Zeile desselben Abschnittes statt *de derivative*: *the derivative*.

Regrettable typographical mistakes in the above article (published in No. 1–2 of this volume, pp. 165) have caused a seriously misleading disarrangement of several lines on pp. 168, 169 and 170. These errata induce editors and publisher to reprint the pages 167–170, by which *reprint (attached!)* they ask to replace the faulty pages.

Furthermore, on p. 173, in line 4 of part VI (Oscillation and Growth) read „*derivative*“ for „*derivate*“ and in line 15 of the same part read „*the derivative*“ for „*de derivative*“.

rate of change in income, $\dot{Y}$, over some middle range but passes to complete inflexibility at either extreme, as is shown in Figure 2⁷. Then we have

$$I = \nu (Q - K) \quad (\nu > 0) \quad (1)$$

$$\dot{Q} = \varphi (\dot{Y}). \quad (2)$$

Differentiating (1) with respect to t and taking (2) into account, we obtain

$$\dot{K} + \theta \dot{I} = \varphi (\dot{Y}),$$

where $\theta = 1/\nu$. Since $\dot{K} = I$, the left-hand side of the above expression is the first two terms of the Taylor expansion of $I(t + \theta)$. Therefore we may say

$$I_{t+\theta} \approx \varphi (\dot{Y}_t). \quad (3)$$

This expression is nothing else but the Hicks-Goodwin investment function which assumes the nonlinear accelerator and the lag, θ , between investment decisions and the corresponding outlays⁸. Thus we may regard (1) and (2) as an alternative formulation of the Hicks-Goodwin type of investment function (3).

So far we are concerned with the case of no innovations. Now let L be capital stock required by innovations. Since L has no negative effect on investment decisions, the Kaldor type of investment function may be reformulated as⁹:

$$I = \psi (\dot{Y}) - \mu (K - L);$$

and since the total ideal stock of capital now consists of the induced part, Q , and the autonomous part, L , the Hicks-Goodwin type of investment function may be rewritten as¹⁰:

$$I = \nu (Q + L - K), \quad \dot{Q} = \varphi (\dot{Y}).$$

It is assumed in this paper that the rate of investment is a linear combination of the Kaldor-Yasui and the Hicks-Goodwin investment functions; then we get

$$I = a [\psi (\dot{Y}) - \mu (K - L)] + b [\nu (Q + L - K)], \quad (4)$$

where a and b are constant and positive.

Next we assume that consumption at time t is a linear function of the level of income at time $t - \tau$; accordingly we have

$$C_t = \alpha Y_{t-\tau} + \beta (t),$$

⁷ See Goodwin: The Nonlinear Accelerator and the Persistence of Business Cycles, pp. 7-9.

⁸ Goodwin, *op. cit.*, pp. 11-12.

⁹ Cf. Goodwin: Econometrics in Business-Cycle Analysis, in A. H. Hansen's Business Cycles and National Income, 1951, pp. 458-59.

¹⁰ See Goodwin: Secular and Cyclical Aspects of the Multiplier and the Accelerator, pp. 118-121.

where $\beta(t)$ is the historically given upward drift of the consumption function. Expanding $Y_{t-\tau}$ in a Taylor series and dropping all but the first two terms of the expansion, we obtain

$$C = \alpha Y - \varepsilon \dot{Y} + \beta(t), \quad (5)$$

where $\varepsilon = \alpha \tau$.

III. The Fundamental Equation of the Trade Cycle Theory

The fundamental equation of nonlinear theory of trade cycle will be deduced as follows. Let $l(t)$ be autonomous or innovational investment; of course $\dot{l}(t)$ is the first derivative of L . Since $Y = C + I + l(t)$, we get from (4) and (5)

$$Y = \alpha Y - \varepsilon \dot{Y} + a \psi(Y) + b \nu Q - (a\mu + b\nu)(K - L) + \beta(t) + l(t).$$

Differentiating this with respect to t and taking $\dot{Q} = \psi(\dot{Y})$ into account, we find

$$(1 - \alpha) \dot{Y} + \varepsilon \ddot{Y} - a \psi'(Y) \dot{Y} - b \nu \varphi(\dot{Y}) + (a\mu + b\nu)(\dot{K} - \dot{L}) = \beta'(t) + l'(t).$$

Because of $\dot{K} = I + l(t) = Y - C$ and $\dot{L} = l(t)$, this turns into

$$\begin{aligned} \varepsilon \ddot{Y} + \left[a \left\{ \varepsilon \mu + (1 - \alpha) - \psi'(Y) \right\} + b \left\{ \varepsilon \nu + \frac{(1 - \alpha)(1 - a)}{b} - \nu \frac{\varphi(\dot{Y})}{\dot{Y}} \right\} \right] \dot{Y} + \\ + (a\mu + b\nu) \left\{ (1 - \alpha) Y - \beta(t) - l(t) \right\} = \beta'(t) + l'(t). \end{aligned}$$

For the moment we may take $\beta(t)$ and $l(t)$ to be constants, β^* and l^* ; hence $\beta'(t)$ and $l'(t)$ are zeros. Then we may study deviations from the equilibrium income, $Y^* = (\beta^* + l^*)/(1 - \alpha)$, by substituting $z = Y - Y^*$, which gives¹¹

$$\begin{aligned} \varepsilon \ddot{z} + \left[a \left\{ \varepsilon \mu + (1 - \alpha) - f'(z) \right\} + b \left\{ \varepsilon \nu + \frac{(1 - \alpha)(1 - a)}{b} - \nu \frac{\varphi(\dot{z})}{\dot{z}} \right\} \right] \dot{z} + \\ + (a\mu + b\nu)(1 - \alpha)z = 0, \end{aligned} \quad (6)$$

where $f(z) = \psi(Y) - \psi(Y^*)$. The equation (6) includes as special cases the Kaldor-Yasui and the Hicks-Goodwin equations; namely, if $a = 1$ and $b = 0$, (6) reduces to the former¹²:

¹¹ Dr. Ichimura has tried to integrate the Kaldor and the Hicks-Goodwin theories and obtained a nonlinear differential equation of the third order. He has offered a qualitative description by which one would expect periodic solutions of his equation. In any case, however, a mere qualitative discussion cannot solve the question of existence of solutions for nonlinear problems. Cf. Ichimura, *op. cit.*, pp. 220-3.

¹² Professor Yasui states the Kaldor model in terms of gross investment and gross saving, whereas we use net aggregates only. His equation reads

$$\varepsilon \ddot{z} + [\varepsilon(\mu + \delta) + (1 - \alpha) - f'(z)] \dot{z} + (1 - \alpha)(\mu + \delta)z - \delta f(z) = 0,$$

where δ is the ratio of the replacement investment to the existing capital stock. Putting $\delta = 0$, the above expression reduces to (7).

$$\varepsilon \ddot{z} + \{\varepsilon \mu + (1 - \alpha) - f'(z)\} \dot{z} + (1 - \alpha) \mu z = 0, \quad (7)$$

while, if $a = 0$ and $b = 1$, to the later¹³:

$$\varepsilon \Theta \ddot{z} + \left\{ \varepsilon + (1 - \alpha) \Theta - \frac{\varphi(\dot{z})}{\dot{z}} \right\} \dot{z} + (1 - \alpha) z = 0, \quad (8)$$

where $\Theta = 1/\nu$. These two equations are important and well-known in the fields of physics and engineering; (7) is investigated in detail by van der Pol and (8) by Rayleigh. If $f'(z)$ is greater or less than $\varepsilon \mu + (1 - \alpha)$ according as $[z]$ is small or large, the equation (7) describes relaxation oscillations of the van der Pol type; similarly, $\varphi(\dot{z})/\dot{z}$ is greater or less than $\varepsilon + (1 - \alpha) \Theta$ according as $[\dot{z}]$ is small or large, the equation (8) possesses periodic solution of the Rayleigh type¹⁴.

Now let

$$m = \sqrt{(a \mu + b \nu)(1 - \alpha)/\varepsilon},$$

$$x = m z$$

and

$$t_1 = m t.$$

Since

$$\dot{x} = \frac{dx}{dt_1} = \dot{z}$$

and

$$\ddot{x} = \frac{d^2 x}{dt_1^2} = \frac{\ddot{z}}{m},$$

the equation (6) can be written:

$$\ddot{x} + [a F(x) + b G(\dot{x})] \dot{x} + x = 0, \quad (9)$$

where

$$F(x) = \{\varepsilon \mu + (1 - \alpha) - f'(x/m)\}/(\varepsilon m),$$

$$G(\dot{x}) = \left\{ \varepsilon \nu + \left[\frac{(1 - \alpha)(1 - a)}{b} \right] - \nu \varphi(\dot{x})/\dot{x} \right\}/(\varepsilon m).$$

The equation (9) may be regarded as *the fundamental equation of nonlinear theory of trade cycle*.

¹³ Goodwin: *The Nonlinear Accelerator and the Persistence of Business Cycles*, p. 12.

¹⁴ By differentiating (8) with respect to t , the Rayleigh equation (8) is transformed into the van der Pol equation

$$\varepsilon \Theta \ddot{u} + [\varepsilon + (1 - \alpha) \Theta - \varphi'(u)] \dot{u} + (1 - \alpha) u = 0,$$

where $u = \dot{z}$. Thus the rate of change in income, u , in the Hicks-Goodwin model shows the same kind of oscillations as the level of income, z , in the Kaldor model.

IV. The existence of cyclic solution

Levinson and Smith have proved¹⁵

Theorem 1: *The equation (9) has at least one periodic solution if the following conditions are satisfied:*

- (i) $a F(0) + b G(0) < 0$,
- (ii) *there exists an $M > 0$ such that*

$$a F(x) + b G(\dot{x}) \geq -M, \quad (10)$$
- (iii) *there exists some $x_0 > 0$ such that*

$$a F(x) + b G(\dot{x}) \geq 0 \quad \text{for } |x| \geq x_0,$$
- (iv) *there exists some $x_1 > x_0$ such that*

$$\int_{x_0}^{x_1} [a F(x) + b G(\dot{x})] dx \geq 10 M x_0$$

where $\dot{x} > 0$ is an arbitrary decreasing positive function of x in the integration.

As we shall prove in the next section, we have another theorem which is applicable to other cases: that is,

Theorem 2: *Let the requirements (i) and (ii) of Theorem 1 be satisfied. Moreover let there exist some $x_0 > 0$ and some $\delta_0 > 0$ such that for $x \geq x_0$*

$$a F(x) + b G(\dot{x}) \geq \delta_0. \quad (11)$$

Finally, let there exist some $\dot{x}_1 < 0$ such that for $\dot{x} \leq \dot{x}_1$

$$a F(x) + b G(\dot{x}) \geq 0. \quad (12)$$

Under these conditions (9) has at least one periodic solution.

Let us assume herewith that, as in the Kaldor-Yasui model (7), the marginal propensity to invest, $f'(z)$, is greater or less than $\varepsilon\mu + (1-\alpha)$ according as $[z]$ is small or large; and that, as in the Hicks-Goodwin model (8), the accelerator, $\varphi(\dot{z})/\dot{z}$, is greater or less than $\varepsilon + (1-\alpha)\theta$ according as $[\dot{z}]$ is small or large. Then Theorem 1 is applicable to the equation (9) when b/a is small, while Theorem 2 when it is large¹⁶. If the cases covered by Theorem 1 and those covered by Theorem 2 are distinct, there exists some middle range of b/a over which neither of our theorems is applicable to the equation (9); to the writer's knowledge, there does not seem to be any established theorem which holds when b/a is in that middle range; so that the

¹⁵ N. Levinson and O. K. Smith: A General Equation for Relaxation Oscillations, Duke Mathematical Journal, Vol. 9, 1942, pp. 382-403.

¹⁶ Note that the ratio, b/a , stands for the relative importance of the acceleration principle of the Hicks-Goodwin type in investment decisions.