ERRATA

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On hyponormal Toeplitz operators with polynomial and circulant-type symbols,
 Integral Equations and Operator Theory, 29 (1997), 202-210.

In our paper cited above on the hyponormality of Toeplitz operators T_φ with circulant-type polynomial symbols φ , there is a mistake in the proof of Theorem 2 in those lines where it is asserted that $\log |\varphi|$ is not Lebesgue integrable. If $\log |\varphi|$ indeed were not Lebesgue integrable, then it would be possible to conclude, under the hypothesis on φ in Theorem 2, that $\varphi - q\bar{\varphi} = 0$ for some Blaschke product $q \in H^\infty(\mathbb{T})$. It is the existence of such a function q , rather than the nonintegrability of $\log |\varphi|$, that is needed in the remainder of the proof of Theorem 2. The purpose of this note is to present the lemma below which establishes the fact that $\varphi - q\bar{\varphi} = 0$, for some Blaschke product $q \in H^\infty(\mathbb{T})$, under the hypothesis given in Theorem 2 of [2]. With this lemma in hand, the proof of Theorem 2 in [2] is completed.

Lemma. *Let $a_0, a_1, \dots, a_n$ be fixed complex numbers and let $\alpha \in \mathbb{C}$ be such that either $\alpha = 0$ or $|\alpha| = 1$. Suppose that $\varphi(e^{i\theta}) = \sum_{-m}^N b_n e^{in\theta}$ is a trigonometric polynomial, with $m \leq N$, whose coefficients b_n satisfy the following combinatorial constraints:*

$$(2.1) \quad b_n = \begin{cases} a_n & (0 \leq n \leq N-m) \\ a_n + \alpha a_{n+1} & (N-m+1 \leq n \leq N-1) \\ a_N & (n = N) \end{cases}$$

and

$$b_{-n} = \begin{cases} e^{i\omega}(a_{N-n+1} + \bar{\alpha} a_{N-n}) & (1 \leq n \leq m-1) \\ e^{i\omega} a_{N-m+1} & (n = m). \end{cases}$$

If f denotes the analytic polynomial $f(z) = a_{N-m+1} + a_{N-m+2}z + \dots + a_N z^{m-1}$, then T_φ is hyponormal if and only if there is a Blaschke product $q \in H^\infty(\mathbb{T})$ such that $\varphi - q\bar{\varphi} = 0$.

Proof. In view of Theorem 1 of [2], we will assume without loss of generality that

$$a_0 = \begin{cases} \bar{\alpha} e^{i\omega} a_N & (m < N) \\ \alpha a_1 + \bar{\alpha} e^{i\omega} a_N & (m = N), \end{cases} \quad a_1 = \dots = a_{N-m} = 0.$$

Because φ is a circulant-type symbol,

$$z^m \varphi(z) = f(z) \left(z^N(z + \alpha) + e^{i\omega}(1 + \bar{\alpha}z) \right).$$

So we have

$$\overline{\varphi(z)} = e^{-i\omega} z^{m-N-1} \overline{f(z)} \left(z^N(z + \alpha) + e^{i\omega}(1 + \bar{\alpha}z) \right).$$

Write

$$f(z) = a_N \prod_{j=1}^{m-1} (z - \zeta_j).$$

Since

$$z^{m-1} \overline{f(z)} = \overline{a_N} + \overline{a_{N-1}}z + \cdots + \overline{a_{N-m+1}}z^{m-1},$$

we have that

$$\zeta \text{ is a zero of } f \iff (\bar{\zeta})^{-1} \text{ is a zero of } z^{m-1} \overline{f},$$

so that we can also write

$$z^{m-1} \overline{f(z)} = \overline{a_N} \prod_{j=1}^{m-1} (1 - \bar{\zeta}_j z).$$

On the other hand by Cowen's theorem [1] and Theorem 10 of [3], T_φ is hyponormal if and only if there exists a Blaschke product $q \in \mathcal{E}(\varphi)$, say

$$q(z) = e^{i\theta} z^s \prod_{j=1}^r \frac{z - \beta_j}{1 - \bar{\beta}_j z} \quad (s + r \leq N; \beta_j \neq 0 \text{ for } j = 1, \dots, r).$$

Observe that

$$\varphi - q\bar{\varphi} = \frac{z^N(z + \alpha) + e^{i\omega}(1 + \bar{\alpha}z)}{z^m} a_N g(z),$$

where

$$g(z) = \prod_{j=1}^{m-1} (z - \zeta_j) - e^{i\phi} z^{s+m-N} \prod_{j=1}^r \frac{z - \beta_j}{1 - \bar{\beta}_j z} \prod_{j=1}^{m-1} (1 - \bar{\zeta}_j z) \quad \text{with } e^{i\phi} = e^{i(\theta-\omega)} \frac{\overline{a_N}}{a_N}.$$

Thus $\varphi - q\bar{\varphi} \in H^\infty$ if and only if g should be of the form $g(z) = \sum_{j=m}^\infty c_j z^j$. In this case we claim that $g = 0$. If $s + m < N$ then

$$(†) \quad z^{N-m-s} \prod_{j=1}^{m-1} (z - \zeta_j) - e^{i\phi} \prod_{j=1}^r \frac{z - \beta_j}{1 - \bar{\beta}_j z} \prod_{j=1}^{m-1} (1 - \bar{\zeta}_j z) = \sum_{j=N-s}^\infty c_j z^j.$$

Then substituting $z = 0$ into (†) gives that $\prod_{j=1}^r (-\beta_j) = 0$, which contradicts to our assumption. Therefore we have that $N \leq s + m$. Observe that $g(z) = \sum_{j=m}^{\infty} c_j z^j$ if and only if

$$\begin{aligned}
 (*) \quad & \prod_{j=1}^{m-1} (z - \zeta_j) \prod_{j=1}^r (1 - \overline{\beta_j} z) - e^{i\phi} z^{s+m-N} \prod_{j=1}^r (z - \beta_j) \prod_{j=1}^{m-1} (1 - \overline{\zeta_j} z) \\
 &= \prod_{j=1}^r (1 - \overline{\beta_j} z) \sum_{j=m}^{\infty} c_j z^j \\
 &= \sum_{j=m}^{\infty} d_j z^j \quad \text{for some } d_j \ (j = m, m+1, \dots).
 \end{aligned}$$

But since $s + m - N \leq m$, then substituting $z = 0$ into (*) gives that $\prod_{j=1}^{m-1} (-\zeta_j) = 0$ and hence $\zeta_j = 0$ for some j ($1 \leq j \leq m-1$). Repeating this argument we can see that at least $(s + m - N)$'s ζ_j should be zero, say

$$\zeta_{N-s} = \dots = \zeta_{m-1} = 0.$$

Therefore from (*) we can write

$$\prod_{j=1}^{N-s-1} (z - \zeta_j) \prod_{j=1}^r (1 - \overline{\beta_j} z) - e^{i\phi} \prod_{j=1}^r (z - \beta_j) \prod_{j=1}^{N-s-1} (1 - \overline{\zeta_j} z) = \sum_{j=N-s}^{\infty} d_j z^j.$$

Now put

$$p_1(z) := \prod_{j=1}^{N-s-1} (z - \zeta_j) \prod_{j=1}^r (1 - \overline{\beta_j} z) \quad \text{and} \quad p_2(z) := \prod_{j=1}^r (z - \beta_j) \prod_{j=1}^{N-s-1} (1 - \overline{\zeta_j} z).$$

Thus if we write $p_1(z) = \sum_{j=0}^{N+r-s-1} a_j z^j$ and $p_2(z) = \sum_{j=0}^{N+r-s-1} b_j z^j$ then

$$(**) \quad a_j = \overline{b_{N+r-s-1-j}} \quad \text{for } j = 0, 1, \dots, N+r-s-1.$$

But $p_1(z) - e^{i\phi} p_2(z) = \sum_{j=N-s}^{\infty} d_j z^j$ if and only if

$$(***) \quad a_j = e^{i\phi} b_j \quad \text{for each } j = 0, 1, \dots, N-s-1.$$

Since $s + r \leq N$ and hence $N + r - s - 1 \leq 2(N - s) - 1$, it follows from (**) and (***) that

$$a_j = e^{i\phi} b_j \quad \text{for each } j = N-s, \dots, N+r-s-1.$$

Thus $\varphi - q\overline{\varphi} \in H^\infty$ if and only if $p_1 - e^{i\phi} p_2 = 0$; i.e., $g = 0$. Therefore, the operator T_φ is hyponormal if and only if $\varphi = q\overline{\varphi}$ for some Blaschke product $q \in H^\infty(\mathbb{T})$. $\square$

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