Erratum

On the structure of complete Kähler manifolds with nonnegative curvature near infinity

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We would like to thank Harold Donnelly for pointing out that in the conclusion of Theorem 3.3, the manifold M might not necessarily split into Riemannian product of N with a Riemann surface Σ/D_0 . The correct conclusion is that, for each end e of M , it is the total space of a holomorphic fibration over a complete Riemann surface with boundary Σ with totally geodesic fibres given by N . The Riemann surface is homeomorphic to a half cylinder $S^1 \times \mathbf{R}^+$ and has nonnegative Gaussian curvature. The fibre N is a compact Kähler manifold with nonnegative sectional curvature. Moreover, the metric of M is locally given by the product metric of $\Sigma \times N$. To clarify this point, let us elaborate on the argument.

On page 590, paragraph 3, we have shown that at each end, the level sets of the holomorphic function h must be totally geodesic at the regular values of H . Since the limit of totally geodesic submanifolds is itself a smooth totally geodesic submanifold, we conclude that all the level sets are in fact totally geodesic submanifolds of M . We claim that locally, the metric splits into a product metric. To see this, let us pick an orthonormal frame field $\{e_1, \dots, e_{2m}\}$ of M which has the property that $\{e_1, \dots, e_{2m-2}\}$ are tangent to the level sets of h , $e_{2m-1} = \nabla \beta$, and $e_m = J \nabla \beta$. Since the Busemann function β satisfies $|\nabla \beta| \equiv 1$ and its integral curves are geodesics, we see that $\nabla_{e_{2m-1}} e_{2m-1} = 0$. To establish the splitting locally, it suffices to show that the involutive distribution T spanned by $\{e_1, \dots, e_{2m-2}\}$ is parallel. This is equivalent to proving that $\langle \nabla_X e_i, e_v \rangle = 0$ for all vector X tangent to M , for all $1 \leq i \leq 2m-2$, and for all $2m-1 \leq v \leq 2m$. If $X = e_j$ for some $1 \leq j \leq 2m-2$, this is clearly valid due to the fact that the level sets of h are totally geodesic submanifolds. The term

$$\begin{aligned} \langle \nabla_{e_{2m-1}} e_i, e_{2m-1} \rangle &= -\langle \nabla_{e_{2m-1}} e_{2m-1}, e_i \rangle \\ &= 0 \end{aligned}$$

for all $1 \leq i \leq 2m-2$, by the fact that $e_{2m-1} = \nabla \beta$. The Kähler condition implies that

$$\begin{aligned} \langle \nabla_{e_{2m-1}} e_i, e_{2m} \rangle &= \langle \nabla_{e_{2m-1}} e_i, J e_{2m-1} \rangle \\ &= -\langle \nabla_{e_{2m-1}} J e_i, e_{2m-1} \rangle \\ &= 0. \end{aligned}$$

Observe that since T is the distribution given by the tangent spaces of the level set of a holomorphic function, its orthogonal complement is also an involutive distribution spanned by $\{e_{2m-1}, e_{2m}\}$. Hence,

$$\begin{aligned}\langle \nabla_{e_{2m}} e_i, e_{2m-1} \rangle &= -\langle \nabla_{e_{2m}} e_{2m-1}, e_i \rangle \\ &= -\langle \nabla_{e_{2m-1}} e_{2m}, e_i \rangle \\ &= 0.\end{aligned}$$

Similarly, we conclude that

$$\begin{aligned}\langle \nabla_{e_{2m}} e_i, e_{2m} \rangle &= -\langle \nabla_{e_{2m}} J e_i, e_{2m-1} \rangle \\ &= 0.\end{aligned}$$

This implies that the metric must locally split into a product metric. In particular, the totally geodesic fibres are all isometric. If the end is simply connected, the de Rham decomposition theorem implies that it must be globally a product of N with a Riemannian surface. However, in general, the global splitting only occurs on the universal covering. We would also like to point out that if the level sets of regular values of h have more than one component, when the value approaches a singular value, even though the limiting level set is a totally geodesic submanifold, it could however have multiplicity more than 1. This is indicated by the simple example given by the map $z \mapsto z^k$. Let us now denote the connected totally geodesic submanifolds by N . Clearly N is given by the connected components of level sets of h .

It was established in [L-T1] that the end e is homeomorphic to the product $(\partial C(2) \cap e) \times [0, \infty)$, where $\partial C(2)$ is the level of the Busemann function at value 2. Moreover, if we denote $S(t) = \partial C(t) \cap e$, then $S(t)$ is homeomorphic to $S = S(2)$. The universal covering $\tilde{e}$ of e is clearly homeomorphic to $\tilde{S} \times [0, \infty)$, where $\tilde{S}$ is the universal covering of S , and $\pi_1(e)$ acts on the preimage $\tilde{S}(t)$ of the level sets $S(t)$. Let $\tilde{P}(a)$ be the preimage of the level set $P(a)$ of h , then the preimage $\tilde{N}$ of N in $\tilde{P}(a)$ are totally geodesic submanifolds of $\tilde{e}$ and are contained in some $\tilde{S}(t)$. The global isometric and holomorphic splitting in $\tilde{e}$ is then of the form $\tilde{N}_0 \times \tilde{\Sigma}$ where $\tilde{N}_0$ is a connected component of $\tilde{N}$ and $\tilde{\Sigma}$ is some Riemann surface with boundary. Since the Busemann function is constant on N , it can be lifted to defined on $\tilde{\Sigma}$. Hence Σ must be homeomorphic to $\mathbf{R} \times [0, \infty)$, and each $\tilde{S}(t)$ is isometrically $\tilde{N} \times \mathbf{R}$. Note that the metric of $\mathbf{R}$ might not be smooth and might vary for different values of t . An element of $\pi_1(e)$ must be of the form (f, g) where f is an isometry of $\tilde{N}$ and g is an isometry of $\mathbf{R}$. Since M is a complex manifold, N is a complex submanifold, and the action of $\pi_1(e)$ on S is oriented, the isometry g must be orientation preserving. This implies that g must be a translation. In particular, the set of isometries of $\mathbf{R}$ which induced from $\pi_1(e)$ must be a group generated by some translation g_0 of $\mathbf{R}$. The Riemann surface obtained by $\Sigma = \tilde{\Sigma} / \langle g_0 \rangle$ is homeomorphic to a half cylinder. Moreover, if we denote the isometry group F of $\tilde{N}_0$ given by the set of isometries f which occurs as $(f, id) \in \pi_1(e)$, then e is clearly the total space of the fibration over Σ with totally geodesic fibers given by $N = \tilde{N}_0 / F$. By the assumption on the curvature of M , it is clear that $\tilde{\Sigma}$ has nonnegative Gaussian curvature. In particular, Σ has nonnegative Gaussian curvature.

We would like to point out that the fibration might not be a Riemannian fibration. When the fibration happens to be a Riemannian fibration, it still might not be trivial. This was pointed out by Donnelly by considering a non-trivial S^2 fibre bundle over the flat cylinder.