Erratum

The Quantum Theory of Second Class Constraints: Kinematics

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In the proof of Lemma 3.2, [1], we proved that if for all $U \in \langle \mathcal{U} \rangle$ of the form $U = \sum \lambda_\ell U_\ell$, $U_\ell \in \langle \mathcal{U} \rangle$ we have $\sum \lambda_\ell = 1$, then $\mathbf{1} \notin C^*(\mathcal{U} - \mathbf{1})$. This proof is wrong because not all positive elements $A \in C^*(\mathcal{U})_+$ can be written as $A = B^*B$ with $B = \sum \lambda_\ell U_\ell$, though the set of such elements is of course dense in $C^*(\mathcal{U})_+$. Hence the proven inequality $\omega(B^*B) \geq 0$ is not sufficient to ensure that ω is positive, so it does not follow that ω extends from the $*$ -algebra generated by $\mathcal{U}$ to $C^*(\mathcal{U})$. In fact we know that in general this part of Lemma 3.2 is wrong:

Assertion. *There is no general condition on the $*$ -algebra generated by a set of constraints $\mathcal{U}$ which is equivalent to $\mathbf{1} \in C^*(\mathcal{U} - \mathbf{1})$.*

Proof. We exhibit a $*$ -algebra $\mathcal{K}$ containing a group of unitaries $\mathcal{U}$, and complete it in two different C^* -norms. In one of the resulting C^* -algebras we will have $\mathbf{1} \in C^*(\mathcal{U} - \mathbf{1})$, and in the other $\mathbf{1} \notin C^*(\mathcal{U} - \mathbf{1})$.

Let G be a discrete group acting on a unital C^* -algebra $\mathcal{F}$ with the action $\alpha: G \mapsto \text{Aut } \mathcal{F}$. Construct $\tilde{\mathcal{F}} := M(G_\alpha \times \mathcal{F})$ which contains $\mathcal{F}$ and a faithful unitary representation $U: G \mapsto \tilde{\mathcal{F}}_u$ of G which implements α , i.e. $\alpha_g = \text{Ad } U_g$. So U_G is a group.

Lemma 1. $U_G \subset \tilde{\mathcal{F}}$ is a linearly independent set.

Proof. $(U_g f)(r) := \alpha_g(f(g^{-1}r)) \forall g, r \in G, \forall f \in \ell^1(G, \mathcal{F})$, where $\ell^1(G, \mathcal{F}) := \{f: G \mapsto \mathcal{F} \mid \sum_{g \in G} \|f(g)\| < \infty\}$. Assume U_G is linearly dependent, i.e. $\exists \beta_k \in \mathbb{C} \setminus \{0\}, g_k \in G$ all different and $N < \infty$ such that $\sum_{k=1}^N \beta_k U_{g_k} = 0$. Hence $\forall f \in \ell^1(G, \mathcal{F})$ we have $\sum_{k=1}^N \beta_k \alpha_{g_k}(f(g_k^{-1}r)) = 0$. Choose $f(r) := \mathbf{1} \delta(r, e)$. Then $\sum_{k=1}^N \beta_k \mathbf{1} \delta(g_k^{-1}r, e) = 0 \forall r \in G$, so for $r = g_k$ this implies $\beta_k = 0$, which contradicts our assumption. $\square$

Take $\mathcal{U} = U_G$ for the chosen constraint set, let the $*$ -algebra $\mathcal{K}$ be generated by U_G , hence it is the linear space generated by U_G . Let the C^* -algebra $\mathcal{A}$ be the C^* -

closure of $\mathcal{K}$ in $\tilde{\mathcal{F}}$, then we give below a choice of $\mathcal{F}$, G , and α for which $\mathbf{1} \in C^*(U_G - \mathbf{1})$ in $\mathcal{A}$. First note that if $\mathbf{1} \notin C^*(U_G - \mathbf{1})$, then $\exists \omega \in \mathfrak{S}(\tilde{\mathcal{F}})$ ($\equiv$ the set of states on $\tilde{\mathcal{F}}$), for which $\omega(U_G) = 1$, and since $\mathbf{1} \in \mathcal{F}$, $\omega|_{\mathcal{F}}$ is a nontrivial G -invariant state on $\mathcal{F}$. Choose $\mathcal{F} = C(\mathbf{S}^1)$, and G the discrete group generated in $\text{Aut } \mathcal{F}$ by an irrational rotation of $\mathbf{S}^1$, and by a non-Lebesgue measure preserving homeomorphism of $\mathbf{S}^1$, which always exists.

Lemma 2. *There are no G -invariant states on $\mathcal{F}$, and hence $\mathbf{1} \in C^*(U_G - \mathbf{1})$.*

Proof. By the Riesz representation theorem, to each $\omega \in \mathfrak{S}(\mathcal{F})$ corresponds a unique Borel measure on $\mathbf{S}^1$. By [2, Theorem 5, p. 82], for an irrational rotation the Lebesgue measure is the only invariant Borel measure. Since the other generating element of G does not preserve the Lebesgue measure, there are no invariant measures for G , and hence no G -invariant states on $C(\mathbf{S}^1)$. $\square$

We now construct another C^* -closure $\mathcal{B}$ of $\mathcal{K}$ such that $\mathbf{1} \notin C^*(U_G - \mathbf{1})$ in $\mathcal{B}$. Since by Lemma 1, U_G is linearly independent, each $A \in \mathcal{K}$ has a unique expression $A = \sum_{k=1}^N \lambda_k U_{g_k}$, $\lambda_k \in \mathbb{C} \setminus 0$, $g_k \neq g_j$ if $k \neq j$, $N < \infty$. So define a $*$ -norm $\|A\|_1 := \sum_{k=1}^N |\lambda_k|$ on $\mathcal{K}$ and complete it to a Banach $*$ -algebra $\mathcal{K}_1$. Then let $\mathcal{K}_2$ be the enveloping C^* -algebra of $\mathcal{K}_1$, with C^* -norm $\|\cdot\|_2$, and $\mathcal{B}$ is the $\|\cdot\|_2$ -closure of $\mathcal{K}$ in $\mathcal{K}_2$. Since U_G is linearly independent and generates $\mathcal{K}$, a linear functional on $\mathcal{K}$ is uniquely specified by its values on U_G . Specify ω by $\omega(U_G) = 1$. Consider the set $\mathcal{P} := \{B \in \mathcal{K} | B = A^*A, A \in \mathcal{K}\}$, then for $B \in \mathcal{P}$, $B = A^*A$ with $A = \sum_k \lambda_k U_{g_k}$, we find

$$\omega(B) = \omega \left(\sum_{k,j} \bar{\lambda}_k \lambda_j U_{g_k^{-1}g_j} \right) = \sum_{k,j} \bar{\lambda}_k \lambda_j = \left| \sum_k \lambda_k \right|^2 \geq 0,$$

so ω is positive on $\mathcal{P}$, and

$$\left| \omega \left(\sum_k \lambda_k U_{g_k} \right) \right| \leq \sum_k |\lambda_k| \cdot |\omega(U_{g_k})| = \sum_k |\lambda_k| = \left\| \sum_k \lambda_k U_{g_k} \right\|_1,$$

hence ω is continuous with relation to $\|\cdot\|_1$, hence can be extended as a continuous linear functional to $\mathcal{K}_1$. Now $\mathcal{P}$ is dense in

$$(\mathcal{K}_1)_+ := \{B \in \mathcal{K}_1 | B = A^*A, A \in \mathcal{K}_1\},$$

because if $A \in \mathcal{K}_1$ is the limit of $\{A_n\} \subset \mathcal{K}$, then A^*A is the limit of $\{A_n^*A_n\} \subset \mathcal{P}$ in the norm $\|\cdot\|_1$, by simple manipulations. Hence since ω is positive on $\mathcal{P}$, its extension is positive on $(\mathcal{K}_1)_+$, so ω is a state on the Banach $*$ -algebra $\mathcal{K}_1$, hence by Dixmier [3] 2.7.5 has an extension as a state to the enveloping algebra $\mathcal{K}_2$. But $\omega(U_G) = 1$, hence $\mathbf{1} \notin C^*(U_G - \mathbf{1})$ in $\mathcal{B}$. $\square$

Note. Hence the question of whether a constraint set is first or second class in general depends on the C^* -norm of the field algebra. Clearly there are some algebraic conditions which are sufficient for $\mathbf{1} \in C^*(\mathcal{U} - \mathbf{1})$, e.g. the first part of Lemma 3.2 [1].

The subsequent material in [1] is unaffected by the error in Lemma 3.2 pointed out in this erratum.

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References

1. Grundling, H, Hurst, C.A.: The quantum theory of second class constraints: Kinematics. *Commun. Math. Phys.* **119**, 75 (1988)
2. Cornfeld, I.P., Fomin, S.V., Sinai, Ya.G.: *Ergodic theory*. Berlin, Heidelberg, New York: Springer 1982
3. Dixmier, J.: *C*-algebras*. Amsterdam: North-Holland 1977

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