On a theorem of AUSLANDER and LYNDON: A correction

By

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Dr. D. E. COHEN has pointed out to me that the proof of Lemma 1 in my note [1] is false, and apparently the gap left by the mistake can not be closed completely by elementary arguments — ‘elementary’ in the spirit of that note. More recently I have again made use of just that Lemma 1 in a similar context (cf. [2], Lemma 2.1). The following lines serve the dual purpose to make clear just what gap is left in the note [1], and to point out that the truth of Lemma 2.1 in the note [2] is not affected.

In [1] we were concerned with the following theorem:

If R is a normal subgroup of a non-abelian free group F such that for some integer $n \geq 3$ the n -th term F_n of the lower central series of F is contained in $[R, R]$, then $R = F$.

Lemma 1 served to reduce the proof of the general case to that of a free group of rank two. To this end, we now note first that it suffices to assume that R contains $F' = [F, F]$. For otherwise consider $S = RF'$. The theorem applied to S gives $S = F$; but as the assumptions make F/R nilpotent, $RF' = F$ implies $R = F$.

Now assume the theorem true in every free group of rank two and apply to the given situation a retractive endomorphism ε that maps F onto a free factor of rank two. One obtains: $R\varepsilon = F\varepsilon$ for every such retraction ε . In the ‘proof’ in [1] I stated that $R\varepsilon = R \cap F\varepsilon$ — which is false (but true when R is fully invariant in F); nor is it true that $R\varepsilon = F\varepsilon$ for every such retraction implies $R = F$, as is shown by taking in the free group of countably infinite rank a normal subgroup R such that F/R is the additive group of rationals. We have, however:

Lemma. *If F/R is finitely generated or of finite exponent, then $R\varepsilon = F\varepsilon$ for every retractive endomorphism of F onto a free factor of rank two implies $R = F$.*

For in this case F/R is a direct product of cyclic groups and so, with a suitable choice of the free generators e_i of F , R may be assumed to be generated by powers of the e_i modulo F' ; that is $R = \text{gp}(F', e_i^{\alpha_i})$ where the α_i are non-negative integers. Now use just those retractions mapping F onto the free factor generated by e_i and e_j for all pairs i, j and $R = F$ follows.

This deals with the case of free groups of finite rank; I do not know how the given fact $F_n \leq [R, R]$ may be used in an elementary manner to exclude awkward cases as the one mentioned above when F has infinite rank.

We turn to Lemma 2.1 of the note [2]. Denoting by G_p the least normal subgroup of the group G such that G/G_p is abelian of exponent p , the lemma states:

If R is a normal subgroup of the non-abelian free group F such that $F_p \leq R$ and $[F_p, F] \leq R_p$, then either $R = F$ or $p = 2$ and R is of index two in F .

Again one deduces first from the validity of the theorem in free groups of rank two that for every retraction ε on a free factor of rank two one has either $R\varepsilon = F\varepsilon$ or $p = 2$ and $R\varepsilon$ is of index two in $F\varepsilon$. This time the assumptions ensure that F/R is of exponent p and so the Lemma above does in fact permit to conclude that either $R = F$ or R is of index two in F .

References

- [1] HANNA NEUMANN, On a theorem of Auslander and Lyndon. Arch. Math. **13**, 1–3 (1962).
- [2] HANNA NEUMANN, Note on a theorem by Gaschütz. J. reine angew. Math. **212**, 109–112 (1963).

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