Erratum

Convergence of Diffusion Waves of Solutions for Viscous Conservation Laws

I.-Liang Chern and Tai-Ping Liu

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In our paper the convergence rate should be lower due to a nonlinear interaction term we omitted. The nonlinear term (1.22) when rewritten on the last two lines of p. 511, its i^{th} component should be

$$N(a, b) = \frac{1}{2} \sum_{j \neq i} b_{ijj} a_j^2 + \frac{1}{2} \sum_{j \neq k} b_{ijk} a_j a_k + \frac{1}{2} \sum_{j, k} b_{ijk} b_j b_k + l_i \cdot H(a, b),$$

$$b_{ijk} \equiv l_i (f''(0) \gamma_j \gamma_k).$$

The first term on the right-hand side was missing in the original expression. It creates the interaction of i^{th} characteristic mode with other modes. This contributes to a lower rate of convergence of the solution to the diffusion waves. For instance the rate of L_1 -convergence is around $t^{-1/4}$ instead of $t^{-1/2}$. The correct expression of (1.24) of the main result, Theorem 1.2 in [1] should be

$$\|D^l(u - \theta^*)\|_{L_p}(t) = O(1) \delta t^{-\left(\frac{l}{2} + \frac{3}{4} - \frac{1}{2p} - \sigma\right)}. \quad (1.24)'$$

This rate of convergence is in general optimal. We will present a simple example later to illustrate this. The rate of $t^{-\frac{1}{4}}$ for L_1 -convergence is consistent with the inviscid theory. The same rate was obtained in [4] for convergence of solutions of hyperbolic conservation laws to N -waves. The L_2 -result has also been obtained independently by Kawashima in [2]. The L_1 -result for physical systems which are hyperbolic-parabolic has not been obtained.

To obtain this we follow the same technique as before and use the integration form of (1.19) through parametric methods. The missing term yields

$$\xi_i \equiv \int_0^t \int_{-\infty}^{\infty} G_i(x+y, t-s) \frac{1}{2} \sum_{j \neq i} b_{ijj} (\theta_j^2(y, s))_y dy ds,$$

where

$$G_i(t, t) \equiv \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x - \lambda_i t)^2}{2t}\right).$$

ξ_i satisfied the inhomogeneous heat equation

$$\xi_{it} + \lambda_i \xi_{ix} + \frac{1}{2} \sum_{j \neq i} b_{ijj} (\theta_j^2)_x = \frac{1}{2} \xi_{ixx}, \quad \xi_i(x, 0) = 0.$$

This can be estimated by the technique of hyperbolic waves in [3] as follows. According to Proposition 2.1 of [1],

$$\theta_j(x, t) = \frac{1}{\sqrt{t+1}} \theta_j^* \left(\frac{1}{\sqrt{t+1}} \right) + e_j \equiv \tilde{\theta}_j + e_j, \quad \|e_j\|_{L_p}(t) = O(1) \delta t^{\frac{1}{2p} - \frac{1}{2} - 1}.$$

With this we may decompose ξ_i into $\eta_i + \zeta_i$ with

$$\begin{aligned} \zeta_{it} + \lambda_i \zeta_{ix} + \frac{1}{2} \sum_{j \neq i} b_{ijj} (\tilde{\theta}_j^2)_x &= 0, \quad \zeta_i(x, \infty) = 0, \\ \eta_{it} + \lambda \eta_{ix} &= \frac{1}{2} \eta_{ixx} + \frac{1}{2} \zeta_{ixx} - \sum_{j \neq i} b_{ijj} \left(\tilde{\theta}_j e_j + \frac{1}{2} e_j^2 \right)_x, \quad \eta_i(x, 0) = -\zeta_i(x, 0). \end{aligned}$$

The hyperbolic wave ζ_i was estimated in Sect. 7 of [3] by the characteristic method. We have

$$\|D^l \zeta_i\|_{L_p}(t) = O(1) \delta t^{-\left(\frac{l}{2} + 1 - \frac{1}{2p}\right)}.$$

η_i is estimated by parametric method

$$\eta_i(\cdot, t) = G_i(\cdot, t) * \eta_i(\cdot, 0) + \int_0^t G_i(\cdot, t-s) * \left\{ \frac{1}{2} \zeta_{iyy} - \sum_{j \neq i} b_{ijj} \left(\tilde{\theta}_j e_j + \frac{1}{2} e_j^2 \right)_y \right\} ds.$$

From estimates of ζ_i , e_j , and $\tilde{\theta}_j$, the second term on the right-hand side has the same decay rate as that for ζ_i . Since ξ_i , ζ_i , and η_i all satisfy conservation laws, we have

$$\int_{-\infty}^{\infty} \eta_i(x, 0) dx = 0,$$

and from the hypothesis in [1] and pointwise estimate of ζ in [3], we have

$$\begin{aligned} \eta_i(\cdot, 0) &\in L_1 \cap L^\alpha, \\ L^\alpha &\equiv \left\{ v \mid \|v\|_{L^\alpha} \equiv \int_{-\infty}^{\infty} |x|^\alpha |v(x)| dx < \infty \right\}, \quad 0 \leq \alpha < \frac{1}{2}. \end{aligned}$$

With these, a lemma of Kawashima, [2] and Lemma 3.4 of [1], yields

$$\|D^l [G_i(\cdot, t) * \eta_i(\cdot, 0)]\|_{L_p}(t) = O(1) \delta t^{-\left(\frac{l}{2} + \frac{1}{2} + \frac{\alpha}{2} - \frac{1}{2p}\right)}.$$

This establishes (1.24)'.

We now present a simple example to show that (1.24)' would be optimal if $\sigma = 0$. Consider

$$U_t + (V^2)_x = \frac{1}{2} U_{xx}, \quad V_t + V_x = V_{xx}, \quad U(x, 0) = V(x, 0) = \delta(x).$$

The solution V is the heat kernel with speed one and U differs from the heat kernel by

$$\begin{aligned} d(x, t) &= \int_0^t ds \int_{-\infty}^{\infty} (2\pi(t-s))^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{2(t-s)}} (4\pi s)^{-1} \left(e^{-\frac{(y-s)^2}{4s}} \right)_y dy \\ &= \int_0^t (2\pi t)^{-\frac{1}{2}} (4\pi s)^{-1} \left(e^{-\frac{(x-s)^2}{2t}} \right)_x ds. \end{aligned}$$

Direct calculations yield

$$\|d\|_{L^\infty}(t) \geq |d(0, t)| \geq Ct^{-\frac{3}{4}}, \quad \|d\|_{L^1}(t) \geq \int_0^{\sqrt{t}} |e(x, t) dx| \geq Ct^{-\frac{1}{4}}, \quad t \geq 1,$$

for some positive constant C .

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